CAMBRIDGE

ESSENTIAL

Mathematical Methods 1&2

Fifth edition

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Interactive student CD included

CHAPTER

1

Reviewing Linear Equations

Objectives

- To **solve linear equations** in one unknown.
- To **transpose** and solve formulae.
- To **construct linear equations**.
- To solve **simultaneous** linear equations by substitution and elimination methods.
- To use linear equations to **solve problems**.

1.1 Linear equations

The solutions to many problems may be found by translating the problems into mathematical equations which may then be solved using algebraic techniques. The equation is solved by finding the value or values of the unknowns that would make the statement true.

Consider the equation $2x + 5 = 7$.

If $x = 1$, the true statement

$$2(1) + 5 = 7$$

is obtained.

The solution to the equation therefore is $x = 1$.

In this case there is no other value of x that would give a true statement.

One way of solving equations is to simply substitute numbers in until the correct answer is found. This method of 'guessing' the answer is a very inefficient way of solving the problem. There are a number of standard techniques that can be used for solving linear equations algebraically.

It is often helpful to look at how the equation has been constructed so that the steps necessary to 'undo' the equation can be identified. It is most important that the steps taken to solve the equation are done in the correct order. Essentially it is necessary to do 'the opposite' in 'reverse order'.

Example 1

Solve the equation $3x + 4 = 16$ for x.

Solution

$$\begin{aligned} 3x + 4 &= 16 \\ 3x &= 16 - 4 \quad \text{Subtract 4 from both sides.} \\ &= 12 \\ x &= \frac{12}{3} \quad \text{Divide both sides by 3.} \\ &= 4 \end{aligned}$$

Once a solution has been found it may be checked by substituting the value back into both sides of the original equation to ensure that the left-hand side (LHS) equals the right-hand side (RHS).

LHS $= 3(4) + 4 = 16$
RHS $= 16$
$\therefore$ solution is correct.

Example 2

Equations with the unknown on both sides

Solve $4x + 3 = 3x - 5$.

Solution

$$\begin{aligned} 4x + 3 &= 3x - 5 \\ 4x - 3x + 3 - 3 &= 3x - 3x - 5 - 3 \quad \text{Group all the terms containing the unknown on one side of the equation and the remaining terms on the other.} \\ x &= -8 \end{aligned}$$

Check: when $x = -8$, LHS $= -29$
RHS $= -29$

$\therefore$ solution is correct.

Example 3

Equations containing brackets

Solve $3(2x + 5) = 27$.

Solution

$$\begin{aligned} 3(2x+5) &= 27 \\ 6x+15 &= 27 \qquad \text{First remove the brackets and then} \\ 6x &= 27-15 \qquad \text{use the previous rules.} \\ 6x &= 12 \\ x &= \frac{12}{6} \\ x &= 2 \end{aligned}$$

Check: when $x = 2$, $\quad \text{LHS} = 3(2 \times 2 + 5) = 27$

$\text{RHS} = 27$

$\therefore$ solution is correct.

Example 4

Equations containing fractions

Solve $\frac{x}{5} - 2 = \frac{x}{3}$.

Solution

$$\begin{aligned} \frac{x}{5} - 2 &= \frac{x}{3} \\ \frac{x}{5} \times 15 - 2 \times 15 &= \frac{x}{3} \times 15 \qquad \text{Multiply both sides of the equation by} \\ 3x - 30 &= 5x \qquad \text{the lowest common multiple of 3 and 5.} \\ 3x - 5x &= 30 \\ -2x &= 30 \\ x &= \frac{30}{-2} \\ x &= -15 \end{aligned}$$

Check: $\text{RHS} = -\frac{15}{5} - 2 = -3 - 2 = -5$

$\text{LHS} = -\frac{15}{3} = -5$

$\therefore$ the solution is correct.

Example 5

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Solve $\frac{x-3}{2} - \frac{2x-4}{3} = 5$.

Solution

Remember that the line separating the numerator and the denominator (the vinculum) acts as a bracket.

Multiply by 6, the lowest common denominator.

$$\frac{x-3}{2} \times 6 - \frac{2x-4}{3} \times 6 = 5 \times 6$$
$$3(x-3) - 2(2x-4) = 5 \times 6$$
$$3x - 9 - 4x + 8 = 30$$
$$3x - 4x = 30 + 9 - 8$$
$$-x = 31$$
$$x = \frac{31}{-1}$$
$$= -31$$

Check: $\text{LHS} = \dfrac{-31-3}{2} - \dfrac{2 \times -31 - 4}{3}$

$$= \frac{-34}{2} - \frac{-66}{3} = -17 + 22 = 5$$

$\text{RHS} = 5$

$\therefore$ solution is correct.

An equation for the variable x in which the coefficients of x, including the constants, are pronumerals is known as a **literal equation**.

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Example 6

Literal equation

Solve $ax + b = cx + d$ for x.

Solution

$$ax + b = cx + d$$
$$ax - cx = d - b$$
$$(a-c)x = d - b$$
$$x = \frac{d-b}{a-c}$$

Using a graphics calculator

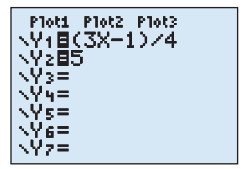
Solving a linear equation can be accomplished using a graphics calculator, either by a suitable graphical representation or by using the **Solver** screen from the **MATH** menu. Students should be able to solve equations both by hand and by using forms of this technology.

Solving an equation

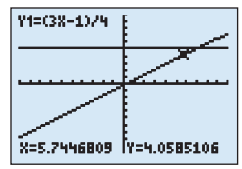
Using the y= and graph windows

For the equation $\frac{3x-1}{4} = 5$

Enter Y1 $= \frac{3x-1}{4}$ and Y2 $= 5$ in the Y= window.

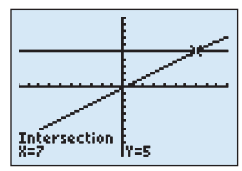


Choose **6: ZStandard** from the ZOOM menu. Press TRACE and take the cursor to a point near the intersection as shown.



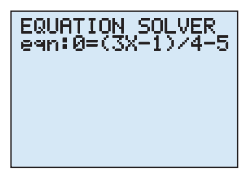
Choose **5: intersect** from the CALC menu and press ENTER three times to obtain the coordinates of the point of intersection.

The intersection point has coordinates (7, 5) and the solution of the equation $\frac{3x-1}{4} = 5$ is $x = 7$.



Using solver

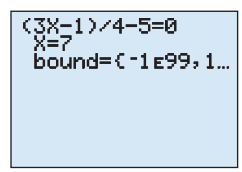
Choose **0: Solver** from the MATH menu. The equation must be entered as $0 = \frac{3x-1}{4} - 5$.



Press ENTER or use the 'down' arrow key to move to the second screen.

Take the cursor to the position to the right of the equals sign and enter a guess, for example X = 5. This is necessary to 'seed' the numerical process the calculator undertakes.

Press ALPHA ENTER to obtain the result shown.



Exercise 1A

1 Solve each of the following equations for x:

a $x + 3 = 6$ **b** $x - 3 = 6$ **c** $3 - x = 2$ **d** $x + 6 = -2$

e $2 - x = -3$ **f** $2x = 4$ **g** $3x = 5$ **h** $-2x = 7$

i $-3x = -7$ **j** $\frac{3x}{4} = 5$ **k** $\frac{-3x}{5} = 2$ **l** $\frac{-5x}{7} = -2$

2 Solve each of the following literal equations for x:

a $x - b = a$ **b** $x + b = a$ **c** $ax = b$ **d** $\frac{x}{a} = b$ **e** $\frac{ax}{b} = c$

Example 1 **3** Solve the following linear equations:

a $y - 4 = 3$ **b** $t + 2 = 7$ **c** $y + 5 = 2$ **d** $x - 9 = 5$

e $2a = 7$ **f** $3a = 14$ **g** $\frac{y}{8} = 6$ **h** $\frac{t}{3} = \frac{1}{2}$

i $2x + 5 = 9$ **j** $5y - 3 = 12$ **k** $3x - 7 = 14$ **l** $14 - 3y = 8$

Example 2 **4** **a** $6x - 4 = 3x$ **b** $x - 5 = 4x + 10$ **c** $3x - 2 = 8 - 2x$

Example 3 **5** **a** $2(y + 6) = 10$ **b** $2y + 6 = 3(y - 4)$ **c** $2(x + 4) = 7x + 2$

Example 4 **d** $5(y - 3) = 2(2y + 4)$ **e** $x - 6 = 2(x - 3)$ **f** $\frac{y + 2}{3} = 4$

Example 5 **g** $\frac{x}{2} + \frac{x}{3} = 10$ **h** $x + 4 = \frac{3}{2}x$ **i** $\frac{7x + 3}{2} = \frac{9x - 8}{4}$

j $\frac{2(1 - 2x)}{3} - 2x = -\frac{2}{5} + \frac{4(2 - 3x)}{3}$ **k** $\frac{4y - 5}{2} - \frac{2y - 1}{6} = y$

Example 6 **6** Solve the following linear literal equations for x:

a $ax + b = 0$ **b** $cx + d = e$ **c** $a(x + b) = c$ **d** $ax + b = cx$

e $\frac{x}{a} + \frac{x}{b} = 1$ **f** $\frac{a}{x} + \frac{b}{x} = 1$ **g** $ax - b = cx - d$ **h** $\frac{ax + c}{b} = d$

7 Solve each of the following for x:

a $0.2x + 6 = 2.4$ **b** $0.6(2.8 - x) = 48.6$ **c** $\frac{2x + 12}{7} = 6.5$

d $0.5x - 4 = 10$ **e** $\frac{1}{4}(x - 10) = 6$ **f** $6.4x + 2 = 3.2 - 4x$

1.2 Constructing linear equations

As stated earlier, many problems can be solved by translating them into mathematical language and using an appropriate mathematical technique to find the solution. By representing the unknown quantity in a problem with a symbol and constructing an equation from the information, the value of the unknown can be found by solving the equation.

Before constructing the equation, each symbol and what it stands for (including the units) should be stated. It is essential to remember that all the elements of the equation must be in units of the same system.

Example 7

A chef uses the following rule for cooking a turkey:
'Allow 30 minutes for each kilogram weight of turkey and then add an extra 15 minutes.'
If the chef forgot to weigh a turkey before cooking it, but knew that it had taken 3 hours to cook, calculate how much it weighed.

Solution

Let the weight of the turkey $= x$ kilograms

Then the time taken $= (30x + 15)$ minutes

$$\therefore \quad 30x + 15 = 180 \quad \text{(3 hours is 180 minutes)}$$
$$x = 5.5$$

The turkey weighed 5.5 kilograms.

Check: $\text{LHS} = 30(5.5) + 15 = 180$
$\text{RHS} = 180$

Example 8

Find the area of a rectangle whose perimeter is 1.08 m, if it is 8 cm longer than it is wide.

Solution

Let length $= l$ cm

Then the width $= (l - 8)$ cm

$$\begin{aligned} \text{Perimeter} &= 2 \times \text{length} + 2 \times \text{width} \\ &= 2l + 2(l - 8) \\ &= 4l - 16 \text{ cm} \end{aligned}$$

$$\text{Perimeter} = 108 \text{ cm}$$

$$\therefore \quad 4l - 16 = 108$$
$$4l = 124$$
$$l = 31 \text{ cm}$$

Therefore the length $= 31$ cm and the width $= 23$ cm.

Check: $\text{LHS} = 4(31) - 16 = 108$
$\text{RHS} = 108$
$\text{Area} = 31 \times 23 = 713 \text{ cm}^2$

Example 9

Adam normally takes 5 hours to travel between Higett and Logett. One day he increases his speed by 4 km/h and finds the journey from Higett to Logett takes half an hour less than the normal time. Find his normal speed.

Solution

Let x km/h be his normal speed.

The distance from Higett to Logett
is $x \times 5 = 5x$ kilometres. (distance = speed × time)

Adam's new speed is $(x + 4)$ km/h.

Hence $(x + 4) \times \frac{9}{2} = 5x$

$$9(x + 4) = 10x$$
$$9x + 36 = 10x$$
$$36 = x$$

His normal speed is 36 km/h.

Exercise 1B

Skillsheet TEACHER

1 For each of the following write an equation using the pronumeral x then solve the equation for x:

- **a** A number plus two is equal to six.
- **b** A number multiplied by three is equal to ten.
- **c** Six is added to a number multiplied by three and the result is twenty-two.
- **d** Five is subtracted from a number multiplied by three and the result is fifteen.
- **e** Three is added to a number. If the result of this is multiplied by six, fifty-six is obtained.
- **f** Five is added to a number and the result divided by four gives twenty-three.

2 \$48 is divided among three students, A, B and C. If B receives three times as much as A and C receives twice as much as A, how much does each receive?

3 The sum of two numbers is 42, and one number is twice the other. Find the two numbers.

Example 7 **4** A chef uses the following rule for cooking on a spit: 'Allow 20 minutes for each kilogram weight and then add an extra 20 minutes.' If the chef forgot to weigh the food before cooking it but knew that it had taken 3 hours to cook, calculate how much it weighed.

Example 8 **5** Find the area of a rectangle whose perimeter is 4.8 m, if it is 0.5 m longer than it is wide.

6 Find three consecutive whole numbers with a sum of 150.

7 Find four consecutive odd numbers with a sum of 80.

8 Two tanks contain equal amounts of water. They are connected by a pipe and 3000 litres of water is pumped from one tank to the other. One tank then contains 6 times as much water as the other. How many litres of water did each tank contain originally?

9 A 120-page book has p lines to a page. If the number of lines were reduced by three on each page the number of pages would need to be increased by 20 to give the same amount of writing space. How many lines were there on the page originally?

Example 9 **10** A rower travels upstream at 6 km per hour and back to the starting place at 10 km per hour. The total journey takes 48 minutes. How far upstream did the rower go?

11 A shopkeeper buys a crate of eggs at \$1.50 per dozen. He buys another crate, containing 3 dozen more than the first crate, at \$2.00 per dozen. He sells them all for \$2.50 a dozen and makes \$15 profit. How many dozens were there in each of the crates?

Example 9 **12** Jess walked for 45 minutes at 3 km/h and then ran for half an hour at x km/h. At the end of that time she was 6 km from the starting point. Find the value of x.

13 A man travels from A to B at 4 km/h and from B to A at 6 km/h. The total journey takes 45 minutes. Find the distance travelled.

14 A boy is 24 years younger than his father. In 2 years time the sum of their ages will be 40. Find the present ages of father and son.

1.3 Simultaneous equations

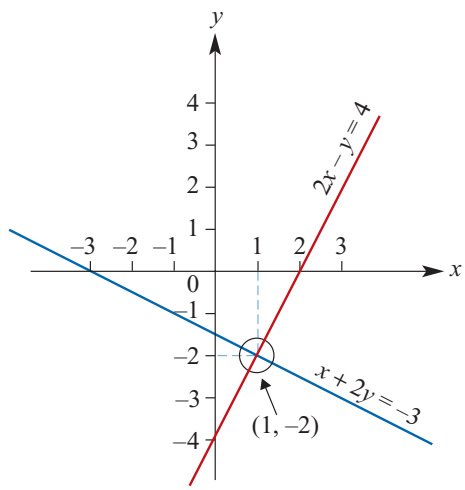
Excel

A linear equation that contains two unknowns, e.g. $2y + 3x = 10$, does not have a single solution. Such an equation actually expresses a relationship between pairs of numbers, x and y, that satisfy the equation. If all the possible pairs of numbers (x, y) that will satisfy the equation are represented graphically, the result is a straight line. Hence the name **linear relation**.

If the graphs of two such equations are drawn on the same set of axes, and they are non-parallel, the lines will intersect at one point only. Hence there is one pair of numbers that will satisfy both equations simultaneously.

Finding the intersection of two straight lines can be done graphically; however the accuracy of the solution will depend on the accuracy of the graphs.

Alternatively this point of intersection may be found algebraically by solving the pair of simultaneous equations.



Three techniques for solving simultaneous equations will be considered.

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Example 10

Solve the equations $2x - y = 4$ and $x + 2y = -3$.

Solution

1 By substitution

$$2x - y = 4 \qquad (1)$$
$$x + 2y = -3 \qquad (2)$$

First express one unknown from either equation in terms of the other unknown.

From equation (2) we get $x = -3 - 2y$.

Then substitute this expression into the other equation.

Equation (1) then becomes $2(-3-2y)-y=4$ (reducing it to one equation in one unknown)

Solving (1) $-6-4y-y=4$

$-5y=10$

$y=-2$

Substituting the value of y into (2) $x+2(-2)=-3$

$x=1$

Check in (1): LHS $=2(1)-(-2)=4$

RHS $=4$

Note: This means that the point (1, –2) is the point of intersection of the graphs of the two linear relations.

2 By elimination

$2x-y=4 \quad (1)$

$x+2y=-3 \quad (2)$

If the coefficient of one of the unknowns in the two equations is the same, we can eliminate that unknown by subtracting one equation from the other. It may be necessary to multiply one of the equations by a constant to make the coefficients of x or y the same for the two equations.

To eliminate x, multiply equation (2) by 2 and subtract the result from equation (1).

Equation (2) becomes $2x+4y=-6 \quad (2')$

Then $2x-y=4 \quad (1)$

$2x+4y=-6 \quad (2')$

Subtracting (1) − (2′): $-5y=10$

$y=-2$

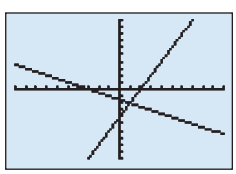
Now substitute for y in equation (1) to find x, and check as in **substitution** method.

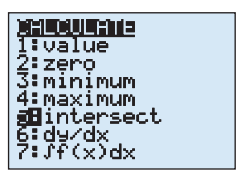
Using a graphics calculator

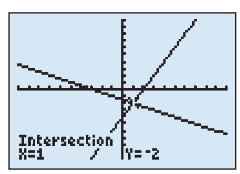
Write each of the equations with y as the subject. Hence:

$$y=-\frac{1}{2}x-\frac{3}{2} \quad \text{and} \quad y=2x-4$$

Enter each of these in the Y= screen and GRAPH as shown in the following screens. Select **5: intersect** from the CALC menu. Use the arrow keys to take the cursor to a point near the intersection. Select the two lines by pressing ENTER after each of the two first prompts. The guess has been established by the placement of the cursor. Press ENTER to obtain coordinates as shown.







Exercise 1C

1 Solve each of the following pairs of simultaneous equations by the substitution method:

a $y = 2x + 1$
$y = 3x + 2$

b $y = 5x - 4$
$y = 3x + 6$

c $y = 2 - 3x$
$y = 5x + 10$

2 Solve each of the following pairs of simultaneous equations by the elimination method:

a $x + y = 6$
$x - y = 10$

b $y - x = 5$
$x + y = 3$

c $x - 2y = 6$
$x + 6y = 10$

Example 10

3 Solve each of the following pairs of simultaneous linear equations by either the substitution or the elimination method:

a $2x - 3y = 7$
$3x + y = 5$

b $2x - 5y = 10$
$4x + 3y = 7$

c $2m - n = 1$
$2n + m = 8$

d $7x - 6y = 20$
$3x + 4y = 2$

e $3s - t = 1$
$5s + 2t = 20$

f $4x - 3y = 1$
$4y - 5x = 2$

g $15x - 4y = 6$
$9x - 2y = 5$

h $2p + 5q = -3$
$7p - 2q = 9$

i $2x - 4y = -12$
$2y + 3x - 2 = 0$

1.4 Constructing and solving simultaneous linear equations

Example 11

The sum of two numbers is 24 and their difference is 96, find the two numbers.

Solution

Let x and y be the two numbers.
Then $x + y = 24$ (1)
and $x - y = 96$ (2)

Add equations (1) and (2).
This gives $2x = 120$
and hence $x = 60$

Substitute in equation (1).
$60 + y = 24$
Hence $y = -36$
The two numbers are 60 and -36.
Check in (2): $60 - (-36) = 96$

Example 12

3 kilograms of jam and 2 kilograms of butter cost \$29 and 6 kilograms of jam and 3 kilograms of butter cost \$54. Find the cost of 1 kilogram of jam and 1 kilogram of butter.

Solution

Let the cost of 1 kg of jam $= x$ dollars and
the cost of 1 kg of butter $= y$ dollars.

Then	$3x + 2y = 29$	(1)
and	$6x + 3y = 54$	(2)
Multiply (1) by 2:	$6x + 4y = 58$	(1′)
Subtract (1′) from (2):	$-y = -4$	
	$y = 4$	
Substituting in (2):	$6x + 3(4) = 54$	
	$6x = 42$	
	$x = 7$	

Jam costs \$7 per kg and butter \$4 per kg.
Check in the original problem:

3 kg of jam $=$ \$21 and 2 kg of butter $=$ \$8
Total $=$ \$29
6 kg of jam $=$ \$42 and 3 kg of butter $=$ \$12
Total $=$ \$54

Exercise 1D

Skillsheet TEACHER

1 Find two numbers whose sum is 138 and whose difference is 88.

2 Find two numbers whose sum is 36 and whose difference is 9.

Example 11

3 Six stools and four chairs cost \$58, while five stools and two chairs cost \$35.

a How much do ten stools and four chairs cost?

b How much do four stools cost? **c** How much does one stool cost?

4 A belt and a wallet cost \$42, while seven belts and four wallets cost \$213.

a How much do four belts and four wallets cost?

b How much do three belts cost? **c** How much does one belt cost?

Use simultaneous equations to solve the following.

5 Find a pair of numbers whose sum is 45 and whose difference is 11.

6 In four years time a mother will be three times as old as her son. Four years ago she was five times as old as her son. Find their present ages.

7 A party was organised for thirty people at which they could have either a hamburger or a pizza. If there were five times as many hamburgers as pizzas calculate the number of each.

8 Two children had 110 marbles between them. After one child had lost half her marbles and the other had lost 20 they had an equal number. How many marbles did each child start with and how many did they finish with?

9 An investor received $1400 interest per annum from a sum of money, with part of it invested at 10% and the remainder at 7% simple interest. This investor found that if she interchanged the amounts she had invested she could increase her return by $90 per annum. Calculate the total amount invested.

Example 12

10 A shopkeeper sold his entire stock of shirts and ties in a sale for $10 000. The shirts were priced at 3 for $100 and the ties $20 each. If he had sold only half the shirts and two-thirds of the ties he would have received $6000. How many of each did he sell in the sale?

11 A tent manufacturer produces two models, the Outback and the Bush Walker. From earlier sales records it is known that 20 per cent more of the Outback model is sold than the Bush Walker. A profit of $200 is made on each Outback sold, but $350 is made on each Bush Walker. If during the next year a profit of $177 000 is planned, how many of each model must be sold?

12 Oz Jeans has factories in Mydney and Selbourne. At the Mydney factory fixed costs are $28 000 per month and the cost of producing each pair of jeans is $30. At the Selbourne factory, fixed costs are $35 200 per month and each pair of jeans costs $24 to produce. During the next month Oz Jeans must manufacture 6000 pairs of jeans. Calculate the production order for each factory, if the total manufacturing costs for each factory are to be the same.

13 A tea wholesaler blends together three types of tea that normally sell for $10, $11 and $12 per kilogram so as to obtain 100 kilograms of tea worth $11.20 per kilogram. If the same amounts of the two higher priced teas are used, calculate how much of each type must be used in the blend.

1.5 Solving linear inequations

An **inequation** is a mathematical statement that contains an inequality symbol rather than an equals sign, for example $2x + 1 < 4$.

To solve linear inequations, proceed exactly as for equations except that:

> **when multiplying or dividing both sides by a negative number, the 'direction' of the inequality symbol is reversed**.

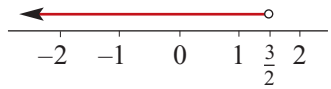
Example 13

Solve the inequation $2x + 1 < 4$.

Solution

$$\begin{aligned} 2x + 1 &< 4 \\ 2x &< 3 \qquad \text{Subtract 1 from both sides.} \\ x &< \frac{3}{2} \qquad \text{Divide both sides by 2.} \end{aligned}$$

The solution to a linear inequation is an infinite set of numbers and may be represented on a number line:



In the number line diagram, the 'endpoint' of the interval is indicated with a closed circle if the point is included and an open circle if it is not.

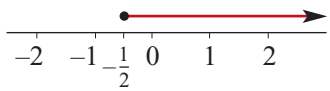
Example 14

Solve the inequation $3 - 2x \leq 4$.

Solution

$$\begin{aligned} 3 - 2x &\leq 4 \\ -2x &\leq 1 \qquad \text{Subtract 3 from both sides.} \\ x &\geq \frac{-1}{2} \qquad \text{Divide both sides by } -2. \\ & \qquad\quad \text{(Note that inequality symbol is reversed.)} \end{aligned}$$

The solution can be represented on a real number line:



Example 15

Solve the inequality $\dfrac{2x+3}{5} > \dfrac{3-4x}{3} + 2$.

Solution

$$\frac{2x+3}{5} > \frac{3-4x}{3} + 2$$

$$\frac{2x+3}{5} - \frac{3-4x}{3} > 2$$

Obtain the common denominator on the left-hand side:

$$\frac{3(2x+3)}{15} - \frac{5(3-4x)}{15} > 2$$

Therefore $3(2x+3) - 5(3-4x) > 30$

and $6x + 9 - 15 + 20x > 30$

Therefore $26x - 6 > 30$

and $x > \dfrac{36}{26}$

$$\therefore x > \frac{18}{13}$$

Exercise 1E

Examples 13, 14

1 Solve each of the following inequalities for x:

a $x + 3 < 4$ **b** $x - 5 > 8$ **c** $2x \geq 6$

d $\frac{x}{3} \leq 4$ **e** $-x \geq 6$ **f** $-2x < -6$

g $6 - 2x > 10$ **h** $\frac{-3x}{4} \leq 6$ **i** $4x - 4 \leq 2$

Example 15

2 Solve for x in each of the following and show the solutions on a real number line:

a $4x + 3 < 11$ **b** $3x + 5 < x + 3$ **c** $\frac{1}{2}(x + 1) - x > 1$

d $\frac{1}{6}(x + 3) \geq 1$ **e** $\frac{2}{3}(2x - 5) < 2$ **f** $\frac{3x - 1}{4} - \frac{2x + 3}{2} < -2$

g $\frac{4x - 3}{2} - \frac{3x - 3}{3} < 3$ **h** $\frac{1 - 7x}{-2} \geq 10$ **i** $\frac{5x - 2}{3} - \frac{2 - x}{3} > -1$

3 **a** For which real numbers x is $2x + 1$ a positive number?

b For which real numbers x is $100 - 50x$ a positive number?

c For which real numbers x is $100 + 20x$ a positive number?

4 In a certain country it cost \$1 to send a letter weighing less than 20 g. A sheet of paper weighs 3 g. Write a suitable inequation and hence state the number of pages that can be sent for \$1.

5 A student scores marks of 66 and 72 on two tests. What is the lowest mark she can obtain on a third test to have an average for the three tests greater than or equal to 75?

1.6 Using and transposing formulae

An equation containing symbols that states a relationship between two or more quantities is called a **formula**. $A = lw$ (Area = length × width) is an example of a formula. The value of A, called the subject of the formula, can be found by substituting in given values of l and w.

Example 16

Find the area of a rectangle with length (l) 10 cm and width (w) 4 cm.

Solution

$A = lw$

$A = 10 \times 4$

$A = 40 \text{ cm}^2$

Sometimes we wish to re-write a formula to make a different symbol the subject of the formula. This process is called **transposing** the formula. The techniques for transposing formulae include those used for solving linear equations detailed in section 1.1.

Example 17

Transpose the formula $v = u + at$, to make a the subject.

Solution

$$v = u + at$$

$v - u = at$ Subtract u from both sides.

$\frac{v-u}{t} = a$ Divide both sides by t.

If we wish to evaluate an unknown that is not the subject of the formula, we can either substitute the values for the given variables and solve the resulting equation or we can first transpose the formula and then substitute the given values.

Example 18

Evaluate p if $2(p + q) - r = z$, and $q = 2$, $r = -3$ and $z = 11$.

Solution

1 Substituting then solving

$$\begin{aligned} 2(p+2) - (-3) &= 11 \\ 2p + 4 + 3 &= 11 \\ 2p &= 4 \\ p &= 2 \end{aligned}$$

2 Transposing then substituting

$$\begin{aligned} 2(p+q) - r &= z \\ 2(p+q) &= z + r \\ p + q &= \frac{z+r}{2} \\ p &= \frac{z+r}{2} - q \\ \therefore p &= \frac{11 + (-3)}{2} - 2 \\ p &= 2 \end{aligned}$$

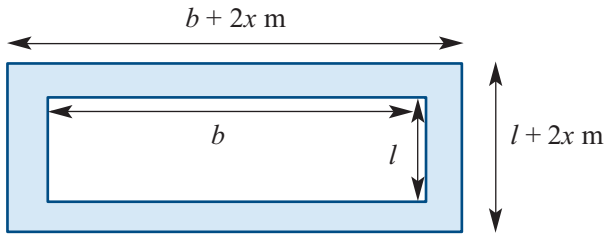
Example 19

A path x metres wide surrounds a rectangular lawn. The lawn is l metres long and b metres wide. The total area of the path is A m^2.

a Find A in terms of l, b and x. **b** Find b in terms of l, A and x.

Solution

a



The area of the path $A = (b + 2x)(l + 2x) - bl$

$= bl + 2xl + 2xb + 4x^2 - bl$

$A = 2xl + 2xb + 4x^2$

b $A - (2xl + 4x^2) = 2xb$

Therefore $b = \dfrac{A - (2xl + 4x^2)}{2x}$

CAS APPENDIX

Example 20

For each of the following make c the subject of the formula.

a $e = \sqrt{3c - 7a}$ **b** $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{c - 2}$

Solution

a $e = \sqrt{3c - 7a}$

Square both sides of the equation.

$e^2 = 3c - 7a$

Therefore $3c = e^2 + 7a$ and $c = \dfrac{e^2 + 7a}{3}$

b $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{c - 2}$

Establish common denominator on left-hand side of the equation.

$\dfrac{b - a}{ab} = \dfrac{1}{c - 2}$

Take the reciprocal of both sides.

$\dfrac{ab}{b - a} = c - 2$

Therefore $c = \dfrac{ab}{b - a} + 2$

Exercise 1F

Skillsheet TEACHER

1 For each of the following find the value of the letter in parentheses:

a $c = ab, a = 6, b = 3$ (c) **b** $r = p + q, p = 12, q = -3$ (r)

c $c = ab, a = 6, c = 18$ (b) **d** $r = p + q, p = 15, r = -3$ (q)

e $c = \sqrt{a}, a = 9$ (c) **f** $c = \sqrt{a}, c = 9$ (a)

g $p = \frac{u}{v}, u = 10, v = 2$ (p) **h** $p = \frac{u}{v}, p = 10, v = 2$ (u)

2 For each of the following construct a formula using the given symbols:

a S, the sum of three numbers a, b and c

b P, the product of two numbers x and y

c the cost, $\$C$, of five CDs which each cost $\$p$

d the total cost, $\$T$, of d chairs which cost $\$p$ each and c tables which cost $\$q$ each

e the time, T, in minutes, of a train journey that takes a hours and b minutes

3 Find the values of the following:

a $E = IR$, when $I = 5$ and $R = 3$ **b** $C = pd$, when $p = 3.14$ and $d = 10$

c $P = \frac{RT}{V}$, when $R = 60$, $T = 150$ and $V = 9$ **d** $I = \frac{E}{R}$, when $E = 240$ and $R = 20$

e $A = \pi rl$, when $\pi = 3.14$, $r = 5$ and $l = 20$ **f** $S = 90(2n - 4)$, when $n = 6$

Example 17

4 For each of the following make the symbol indicated the subject of the formula:

a $PV = c; V$ **b** $F = ma; a$ **c** $I = Prt; P$

d $w = H + Cr; r$ **e** $S = P(1 + rt); t$ **f** $V = \frac{2R}{R - r}; r$

Example 18

5 Find the value of the unknown symbol in each of the following:

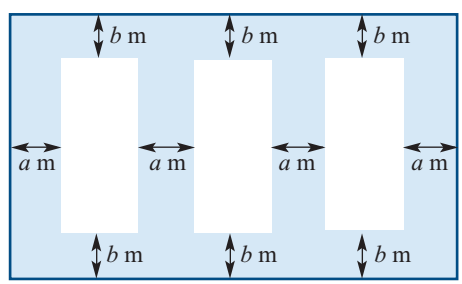
a $D = \frac{T + 2}{P}$, when $D = 10$, $P = 5$

b $A = \frac{1}{2}bh$, when $A = 40$, $h = 10$

c $V = \frac{1}{3}\pi r^2 h$, when $\pi = 3.14$, $V = 100$, $r = 5$

d $A = \frac{1}{2}h(a + b)$, when $A = 50$, $h = 5$, $a = 10$

6 The diagram represents the brick wall of a dwelling with three windows. Each of the windows is h m high and w m wide. Other dimensions are as shown.



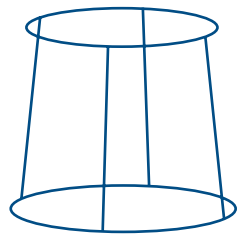
a Find the length of the wall.

b Find the height of the wall.

c Find the total area of the three windows.

d Find the total area of brickwork.

7 A lampshade has a metal frame consisting of two circular hoops of radii p cm and q cm joined by four straight struts of length h cm. The total length of metal is T cm.



a **i** Find an expression for T in terms of p, q and h.

ii Find T when $p = 20$, $q = 24$ and $h = 28$.

b The area of the material covering the frame is A cm^2, where $A = h(p + q)$. Find an expression for p in terms of A, h, q.

8 Find the value of the 'unknown' symbol in each of the following:

a $P = \dfrac{T - M}{D}$, $P = 6$, $T = 8$, $M = 4$

b $H = \dfrac{a}{3} + \dfrac{a}{b}$, $H = 5$ and $a = 6$

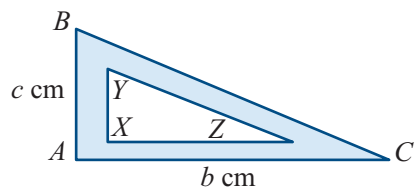
c $a = \dfrac{90(2n - 4)}{n}$, $a = 6$

d $R = \dfrac{r}{a} + \dfrac{r}{3}$, $a = 2$ and $R = 4$

Example 19

9 Right-angled triangles XYZ and ABC are similar.

$$\frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA} = k$$



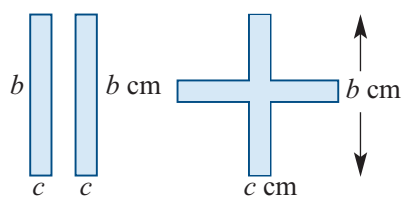
If $AB = c$ cm, $AC = b$ cm, find:

a the area, D cm^2, of the shaded region in terms of c, b and k

b k in terms of D, b and c

c the value of k if $D = 2$, $b = 3$ and $c = 4$

10 Two rectangles each with dimensions c cm $\times$ b cm are used to form a cross as shown. The arms of the cross are all of equal length.



a Find the perimeter, P cm, of the cross in terms of b and c.

b Find the area, A cm^2, of the cross in terms of c and b.

c Find b in terms of A and c.

11 For each of the following make the symbol in brackets the subject of the formula:

Example 20

a $a = \sqrt{a + 2b}$ (b)

b $\dfrac{a + x}{a - x} = \dfrac{b - y}{b + y}$ (x)

c $px = \sqrt{3q - r^2}$ (r)

d $\dfrac{x}{y} = \sqrt{1 - \dfrac{v^2}{u^2}}$ (v)

D&D TEST MC TEST

Review

Chapter summary

- A linear equation is one in which the variable has exponent 1.
- It is often helpful to look at how the equation has been constructed so that the steps necessary to 'undo' the equation can be identified. It is most important that the steps taken to solve the equation are done in the correct order.
- An inequation is a mathematical statement that contains an inequality symbol rather than an equals sign, for example $2x + 1 < 4$.
- To solve linear inequations, proceed exactly as for equations except that, when multiplying or dividing both sides by a negative number, the 'direction' of the inequality symbol is reversed.
- An equation containing symbols that states a relationship between two or more quantities is called a formula. $A = lw$ (area = length × width) is an example of a formula. A is called the subject of the formula.
- If we wish to evaluate an unknown that is not the subject of the formula, we can either substitute the values for the given variables and solve the resulting equation or we can first transpose the formula and then substitute the given values.
- An equation for the variable x in which the coefficients of x, including the constants, are pronumerals is known as a literal equation.

Multiple-choice questions

1 The solution of the linear equation $3x - 7 = 11$ is

A $\frac{4}{3}$ **B** $\frac{11}{3}$ **C** $\frac{-3}{4}$ **D** 6 **E** -6

2 If $\frac{x}{3} + \frac{1}{3} = 2$ then $x =$

A $\frac{1}{3}$ **B** $\frac{2}{3}$ **C** $\frac{7}{3}$ **D** 5 **E** 7

3 The solution of the equation $x - 8 = 3x - 16$ is

A $x = \frac{-8}{3}$ **B** $x = \frac{11}{3}$ **C** $x = 4$ **D** $x = 2$ **E** $x = -2$

4 In algebraic form, 7 is 11 times 2 less than x can be written as

A $7 = 11(x - 2)$ **B** $11x - 2 = 7$ **C** $7 = 11(x + 2)$

D $\frac{11}{x - 2} = 7$ **E** $\frac{7}{11} - 2 = x$

5 The solution of the simultaneous equations $2x - y = 10$ and $x + 2y = 0$ is

A $x = -2$ and $y = 3$ **B** $x = 2$ and $y = -3$ **C** $x = 4$ and $y = -2$

D $x = 6$ and $y = 2$ **E** $x = 1$ and $y = -8$

6 I bought x CDs for \$$a$ and y DVDs for \$$b$. The average price paid, in dollars, is

A $\frac{x+y}{a+b}$ **B** $\frac{a+b}{y+x}$ **C** $\frac{xa+yb}{y+x}$ **D** $\frac{y+x}{xa+yb}$ **E** $\frac{a}{x}+\frac{b}{y}$

7 The solution of the equation $\frac{x+1}{4}-\frac{2x-1}{6}=x$ is

A $x=\frac{8}{5}$ **B** $x=\frac{5}{13}$ **C** $x=1$ **D** $x=-\frac{1}{5}$ **E** $x=-1$

8 The values of z that satisfy the inequation $\frac{72+15z}{3}>4$ are

A $z>4$ **B** $z>-4$ **C** $z=-4$ **D** $z<4$ **E** $z<-4$

9 If $A=\frac{hw+k}{w}$ then

A $w=\frac{k}{A-h}$ **B** $w=\frac{ht+k}{A}$ **C** $w=\frac{A-2k}{2h}$

D $w=\frac{3Ah}{2}-k$ **E** $w=\frac{2}{3}h(A+k)$

10 Bronwyn walks one lap of an oval at 2.5 km/h and then jogs another 8 laps of the oval at 5 km/h. If it takes her 30 minutes in total, how long in metres is each lap?

A 200 m **B** 250 m **C** 300 m **D** 350 m **E** 400 m

Short-answer questions (technology-free)

1 Solve each of the following equations for x:

a $2x+6=8$ **b** $3-2x=6$ **c** $2x+5=3-x$

d $\frac{3-x}{5}=6$ **e** $\frac{x}{3}=4$ **f** $\frac{13x}{4}-1=10$

g $3(2x+1)=5(1-2x)$ **h** $\frac{3x+2}{5}+\frac{3-x}{2}=5$

2 Solve each of the following for t:

a $a-t=b$ **b** $\frac{at+b}{c}=d$ **c** $a(t-c)=d$

d $\frac{a-t}{b-t}=c$ **e** $\frac{at+b}{ct-b}=1$ **f** $\frac{1}{at+c}=d$

3 Solve each of the following inequalities for x:

a $2-3x>0$ **b** $\frac{3-2x}{5}\geq 60$

c $3(58x-24)+10<70$ **d** $\frac{3-2x}{5}-\frac{x-7}{6}\leq 2$

4 Make x the subject of the formula $z=\frac{1}{2}x-3t$. Find x when $z=4$ and $t=-3$.

Review

5 A number d is equal to the square of a number e plus twice a number f.

a Find a formula for d in terms of e and f.

b Make f the subject of the formula.

c Find f when $d = 10$ and $e = 3$.

6 Solve $\frac{b - cx}{a} + \frac{a - cx}{b} + 2 = 0$ for x.

7 Solve $\frac{a}{x + a} + \frac{b}{x - b} = \frac{a + b}{x + c}$ for x.

Extended-response questions

1 The formula for converting degrees Celsius, C, to degrees Fahrenheit, F, is $F = \frac{9}{5}C + 32$.

a Convert 30°F to Celsius.

b Convert 30°C to Fahrenheit.

c If x°C $= x$°F find x.

d If $(x + 10)$°C $= x$°F find x.

e If $2x$°C $= x$°F find the value of x.

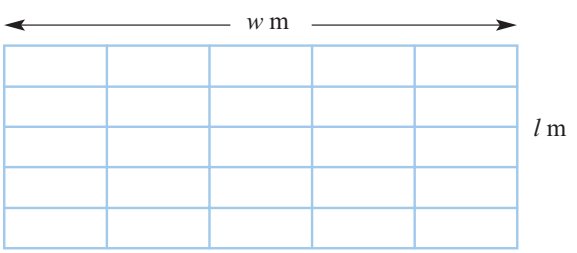
f If k°F $= (-3k)$°C find k.

2 For a spherical mirror of radius r cm, $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$, where u cm is the distance from the mirror to the object and v cm is the distance from the mirror to the image. The magnification is given by $m = \frac{v - r}{r - u}$.

a Find r in terms of v and u from the first formula.

b Find m in terms of v and u only.

3 The diagram shows a section of wire mesh w metres in width and l metres in length.



a Find an expression in terms of w and l for the total length of wire required for the mesh.

b **i** If $w = 3l$, find an expression in terms of w for the total length of wire required.

ii If the length of wire used is 100 m, find the value of w and the value of l.

c The total length of wire, L m, required for another rectangular mesh of dimensions x m by y m, is given by the formula $L = 6x + 8y$.

i Find y in terms of x and L.

ii Find y if $L = 200$ and $x = 4$.

d A third type of mesh can also be used to cover a rectangular region with dimensions x m and y m. In this case, the type of mesh introduced in part c requires 100 m of wire and so $6x + 8y = 100$. This third type of mesh requires 80 m and this gives the equation $3x + 2y = 40$. Find the values of x and y.

4 Tom leaves town A and travels at a constant speed of u km/h towards town B. At the same time Julie leaves town B and travels towards town A at a constant speed of v km/h. Town B is d km from town A.

a How far has each travelled after t hours?

b By considering that the sum of their distances travelled must be d km when they meet, find:

i the time it takes for them to meet

ii their distance from town A when they meet.

c If $u = 30$, $v = 50$ and $d = 100$, find the time it takes for them to meet and their distance from town A.

5 Xiu travels from town A to town B at u km/h and then returns at v km/h. Town A is d km from town B.

a Find the average speed at which Xiu travels for the complete journey, in terms of u and v.

$\left(\text{Remember, average speed} = \dfrac{\text{total distance travelled}}{\text{total time taken}}.\right)$

b If it takes T hours to travel from A to B, find the time taken:

i for the return trip from B to A, in terms of T, u and v

ii the entire trip in terms of T, u and v.

6 A man on a bicycle rides one-third of the way from town A to town B at a speed a km/h and the remainder of the way at $2b$ km/h.

a If the distance between the two towns is 9 km find the time taken to ride from A to B.

If the man had travelled at a uniform rate of $3c$ km/h, he could have ridden from A to B and back again in the same time.

b Show that $\dfrac{2}{c} = \dfrac{1}{a} + \dfrac{1}{b}$.

c **i** Make c the subject of this formula. **ii** Find c, when $a = 10$ and $b = 20$.

7 A man walks 70 km. He walks x km at 8 km/h and y km at 10 km/h.

a Find the length of time he was walking at 8 km/h in terms of x, and the length of time he was walking at 10 km/h in terms of y.

b Find his average speed in terms of x and y.

c If the man walks at 10 km/h for the time he was walking at 8 km/h and at 8 km/h for the time he was walking at 10 km/h, he walks 72 km. Find x and y.

CHAPTER

2

Linear Relations

Objectives

- To calculate the **gradient** of a straight line.
- To interpret and use the **general equation** of a straight line $y = mx + c$.
- To solve **simultaneous linear equations** graphically.
- To calculate the gradient of a straight line as a **tangent**.
- To calculate the product of the gradients of two **perpendicular** straight lines.
- To find the **distance between two points**.
- To find the **midpoint** of a straight line.
- To calculate the angle between two **intersecting** straight lines.
- To **apply** a knowledge of linear relations to solving problems.

A **relation** is defined as a set of ordered pairs in the form (x, y).

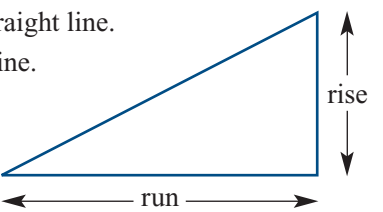
A rule relating the x-value to the y-value of each ordered pair sometimes exists, such as $y = 2x + 1$, and this is a more convenient way of describing the relation.

A relation may also be represented graphically on a set of axes. If the resulting graph is a straight line, then the relation is called a **linear relation**.

2.1 The gradient of a straight line

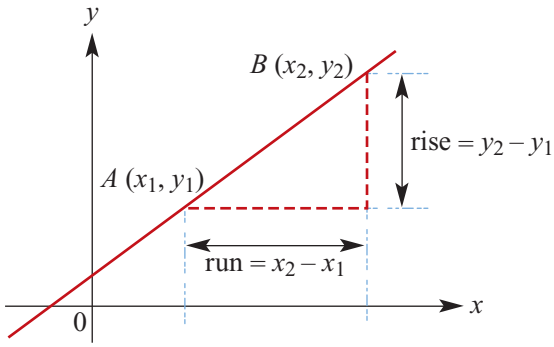
Through any two points it is only possible to draw a single straight line. Therefore a straight line is defined by any two points on the line.

From previous work, you should be familiar with the concept of the **gradient** or slope of a line. The symbol used for gradient is m. This may be defined as:



$$\text{Gradient} = \frac{\text{rise}}{\text{run}}$$

Hence given any two points on the line, (x_1, y_1) and (x_2, y_2), the gradient of the line can be found.



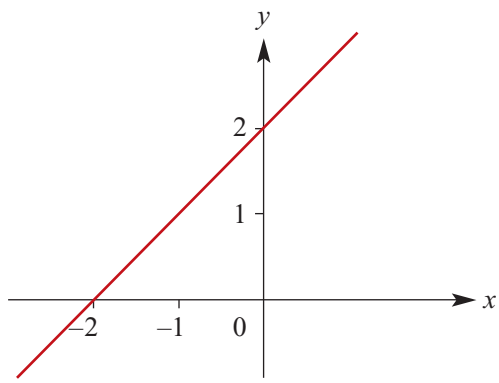
$$\text{Gradient } m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 1

Find the gradient of the given line.

Solution

$$\begin{aligned}
\text{Let } (x_2, y_2) &= (0, 2) \\
\text{and let } (x_1, y_1) &= (-2, 0) \\
\text{Gradient } m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{2 - 0}{0 - (-2)} \\
&= \frac{2}{2} \\
&= 1
\end{aligned}$$

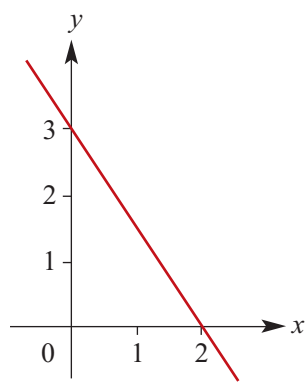


Example 2

Find the gradient of the given line.

Solution

$$\begin{aligned}
\text{Let } (x_1, y_1) &= (0, 3) \\
\text{and } (x_2, y_2) &= (2, 0) \\
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{0 - 3}{2 - 0} \\
&= -\frac{3}{2}
\end{aligned}$$



It should be noted that the gradient of a line that slopes upwards from left to right is **positive**, as illustrated in Example 1, and the gradient of a line that slopes downwards from left to right is **negative**, as illustrated in Example 2.

The gradient of a **horizontal (parallel to the *x*-axis)** line is zero, since $y_2 - y_1 = 0$.

The gradient of a **vertical (parallel to the *y*-axis)** line is undefined, since $x_2 - x_1 = 0$.

Example 3

Find the gradient of the line that passes through the points (1, 6) and (−3, 7).

Solution

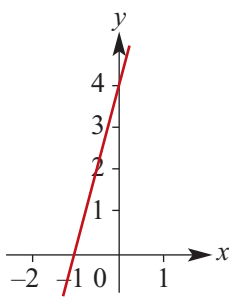
$$\text{The gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 6}{-3 - 1} = \frac{1}{-4} = -\frac{1}{4}$$

Exercise 2A

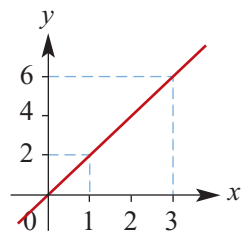
Examples 1, 2

1 Calculate the gradient of each of the following lines:

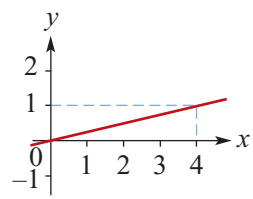
a



b



c



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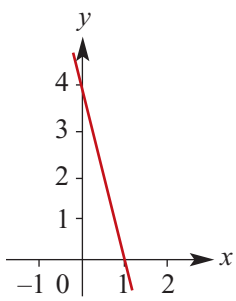

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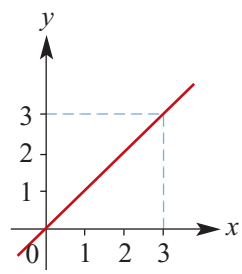
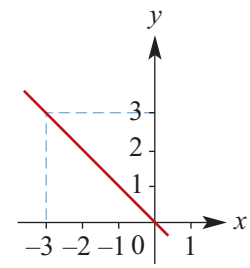
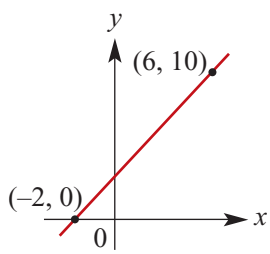


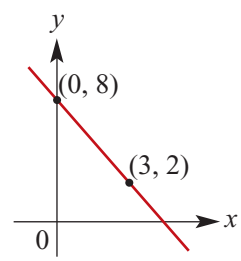
f



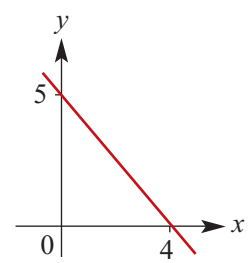
g



h



i



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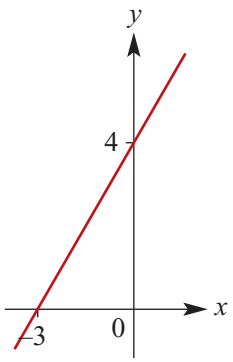

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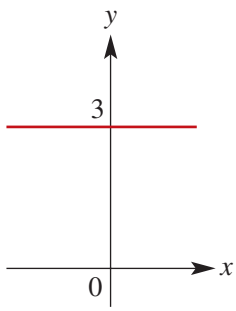


2 Sketch a graph of a line with gradient 1.

3 Sketch a graph of a line with gradient 0 which passes through the point (1, 6).

Example 3

4 For each of the following find the gradient of the line that passes through the two points with the given coordinates:

a (6, 3) (2, 4)	**b** (−3, 4) (1, −6)	**c** (6, 7) (11, −3)
d (5, 8) (6, 0)	**e** (6, 0) (−6, 0)	**f** (0, −6) (−6, 0)
g (3, 9) (4, 16)	**h** (5, 25) (6, 36)	**i** (−5, 25) (−8, 64)
j (1, 1) (10, 100)	**k** (1, 1) (10, 1000)	**l** (5, 125) (4, 64)

5 **a** Find the gradient of the straight line that passes through the points with coordinates $(5a, 2a)$ and $(3a, 6a)$.

b Find the gradient of the straight line that passes through the points with coordinates $(5a, 2a)$ and $(5b, 2b)$.

6 **a** A line has gradient 6 and passes through the points with coordinates $(-1, 6)$ and $(7, a)$. Find the value of a.

b A line has gradient -6 and passes through the points with coordinates $(1, 6)$ and $(b, 7)$. Find the value of b.

7 **a** Find the equation of the line that is parallel to the y-axis and passes through the point with coordinates (4, 7).

b Find the equation of the line that is parallel to the x-axis and passes through the point with coordinates (−4, 11).

2.2 The general equation of a straight line

The general equation of a straight line is $y = mx + c$, where m is the gradient of the line. This form, expressing the relation in terms of y, is called the **gradient form**.

Let $x = 0$, then $y = m(0) + c$
and thus $y = c$

i.e. the y-axis intercept is equal to c.

Example 4

Find the gradient and y-axis intercept of the graph of $y = 3x - 4$.

Solution

The value of m is 3 and the value of c is -4.

Therefore the gradient of the above line is 3 and the y-axis intercept is -4.

If the rule of a straight line is given, the graph can be sketched using the gradient and the y-axis intercept.

Example 5

Sketch the graph of $y = 3x - 1$.

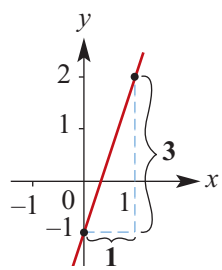
Solution

Gradient $= 3$, i.e. $\dfrac{\text{rise}}{\text{run}} = \dfrac{3}{1}$

y-axis intercept $= -1$

Plot the point $(0, -1)$, the y-axis intercept.

From there move **across 1** (run) and **up 3** (rise) to plot the point $(1, 2)$.



If the equation for the straight line is not written in **gradient form**, to use the above method for sketching a graph, the equation must first be transposed into **gradient form**.

Example 6

Sketch the graph of $3y + 6x = 9$.

Solution

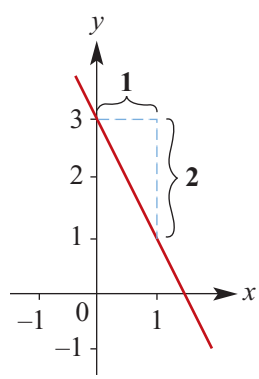
First rearrange the equation into gradient form:

$$3y + 6x = 9$$
$$3y = 9 - 6x \quad \text{(Subtract } 6x \text{ from both sides.)}$$
$$y = \frac{9 - 6x}{3} \quad \text{(Divide both sides by 3.)}$$
$$y = 3 - 2x$$

i.e. $y = -2x + 3$

Therefore $m = -2$

and $c = 3$

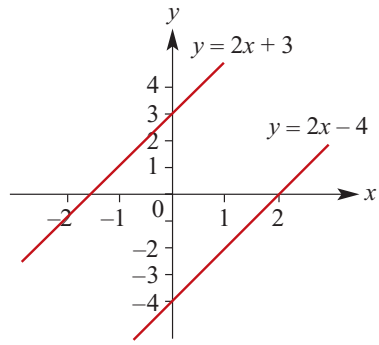


Parallel lines

If the value of m is the same for two rules, then the lines are parallel.

For example, consider the lines with the following rules:

$$y = 2x + 3$$
$$y = 2x - 4$$



Exercise 2B

Example 4

1 Sketch the graphs of each of the following:

a $y = x + 2$ **b** $y = -x + 2$ **c** $y = 2x + 1$ **d** $y = -2x + 1$

Example 5

2 Sketch the graphs of each of the following using the gradient form, $y = mx + c$:

a $x + y = 1$ **b** $x - y = 1$ **c** $y - x = 1$ **d** $-x - y = 1$

3 Sketch the graphs of each of the following using $y = mx + c$:

a $y = x + 3$ **b** $y = 3x + 1$ **c** $y = 4 - \frac{1}{2}x$ **d** $y = 3x - 2$

e $4y + 2x = 12$ **f** $3x + 6y = 12$ **g** $4y - 6x = 24$ **h** $8x - 3y = 24$

4 For which of the following pairs of equations are the corresponding lines parallel to each other? Sketch graphs to show the non-parallel lines.

a $2y = 6x + 4$; $y = 3x + 4$ **b** $x = 4 - y$; $2x + 2y = 6$

c $3y - 2x = 12$; $y + \frac{1}{3} = \frac{2}{3}x$ **d** $4y - 3x = 4$ $3y = 4x - 3$

5 For which of the following do the lines pass through the origin?

a $y + x = 1$ **b** $y + 2x = 2(x + 1)$ **c** $x + y = 0$ **d** $x - y = 1$

6 Give the gradient for each of the lines in Question 5.

2.3 Finding the equation of a straight line

The equation of a straight line may be found if the gradient and y-axis intercept are known.

Excel

Example 7

Find the equation if $m = -3$ and $c = 10$.

Solution

Equation is $y = -3x + 10$.

Example 8

Find the equation of the straight line with gradient -3 which passes through the point with coordinates $(-5, 10)$.

Solution

The general equation of lines with gradient -3 is $y = -3x + c$.
If a point on the line is given the value of c can be determined.
When $x = -5$, $y = 10$
Thus $10 = -3 \times -5 + c$.
Solving for c: $10 - 15 = c$
and therefore $c = -5$. The equation of the line is $y = -3x - 5$.

Example 9

Find the equation of the straight line with y-axis intercept -3 which passes through the point with coordinates $(1, 10)$.

Solution

The general equation of lines with y-axis intercept -3 is $y = mx - 3$.
The line passes through the points with coordinates $(0, -3)$ and $(1, 10)$.
Therefore the gradient $m = \dfrac{10 - -3}{1 - 0} = 13$.
The equation is $y = 13x - 3$.

In general the equation of a straight line can be determined by two 'independent pieces of information'. Two cases are considered below.

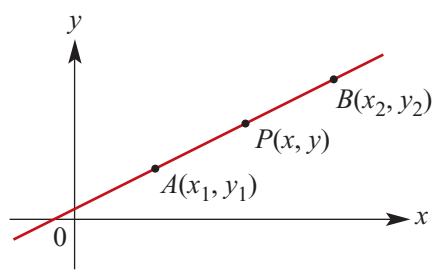
Case 1 Given any two points $A(x_1, y_1)$ and $B(x_2, y_2)$

Using these two points, the gradient of the line AB can be determined:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Using the general point $P(x, y)$, also on the line,

$$m = \frac{y - y_1}{x - x_1}$$



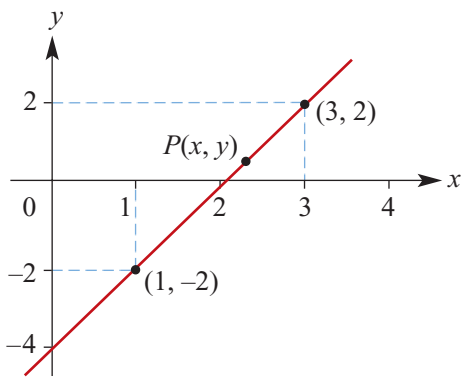
Therefore the equation of the line is $y - y_1 = m(x - x_1)$ where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Example 10

Find the equation of the straight line passing through the points (1, −2) and (3, 2).

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - (-2)}{3 - 1}$$
$$= \frac{4}{2}$$
$$= 2$$
$$\therefore \quad 2 = \frac{y - (-2)}{x - 1}$$
$$2x - 2 = y + 2$$
$$\therefore \quad y = 2x - 4$$



Case 2 Given the gradient m and one other point, $A(x_1, y_1)$

As the gradient m is already known, the rule can be found using $y - y_1 = m(x - x_1)$

Example 11

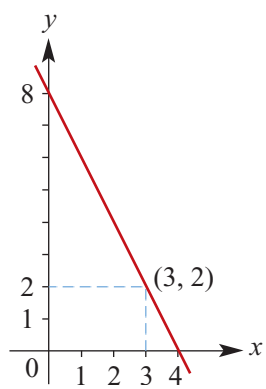
Find the equation of the line that passes through the point (3, 2) and has a gradient of −2.

Solution

$$y - 2 = -2(x - 3)$$
$$y - 2 = -2x + 6$$
$$y = -2x + 8$$
$y = -2x + 8$ is the equation

which could also be expressed as

$$y + 2x - 8 = 0$$



The equation of a straight line can also be found from the graph by reading off two points and using them to find the equation as outlined above.

Example 12

Find the equation of the line shown in the graph.

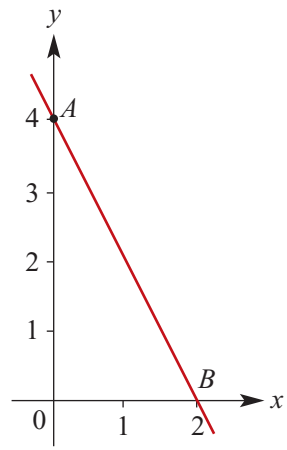
Solution

From the graph it can be seen that the y-axis intercept is (0, 4), i.e. $c = 4$.

As the coordinates of A and B are (0, 4) and (2, 0):

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{0 - 4}{2 - 0}$$
$$= -\frac{4}{2}$$
$$= -2$$

The equation of the line is $y = -2x + 4$.



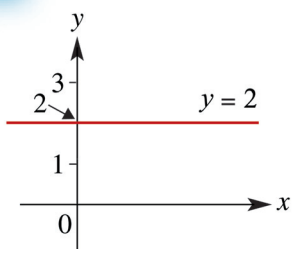
Vertical and horizontal lines

If $m = 0$, then the line is **horizontal** and the equation is simply $\boldsymbol{y = c}$, where c is the y-axis intercept.

If the line is **vertical**, the gradient is undefined and its rule is given as $\boldsymbol{x = a}$, where a is the x-axis intercept.

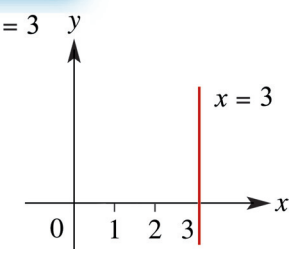
Example 13

Equation $y = 2$



Example 14

Equation $x = 3$



Note that the equation of a vertical line is not in the form $y = mx + c$.

Exercise 2C

Example 7

1 **a** Find the equation of the straight line with gradient 3 and y-axis intercept 5.

b Find the equation of the straight line with gradient -4 and y-axis intercept 6.

c Find the equation of the straight line with gradient 3 and y-axis intercept -4.

Example 8

2 **a** Find the equation of the straight line with gradient 3 and which passes through the point with coordinates (6, 7).

b Find the equation of the straight line with gradient -2 and which passes through the point with coordinates (1, 7).

Example 9

3 For the straight line with y-axis intercept 6 and passing through the point with coordinates (1, 8) find:

a the gradient **b** the equation

4 For the straight line with y-axis intercept 6 and passing through the point with coordinates (1, 4) find:

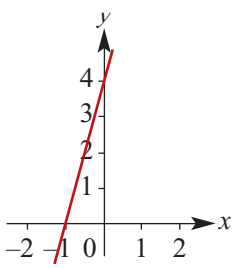
a the gradient **b** the equation

5 Find the equation of the straight line that passes through the point with (1, 6) and has gradient:

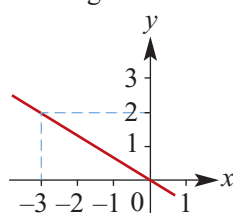
a 2 **b** -2

6 Find the equation of each of the following lines:

a



b



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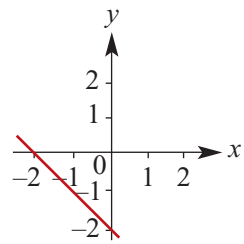

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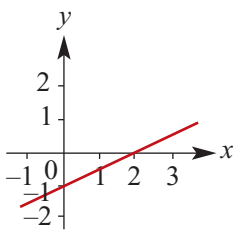

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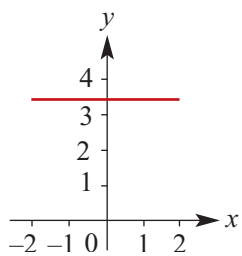

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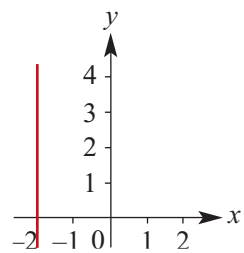


7 Write another equation that would give a parallel line for each of those shown in Question 6. Check your answers by sketching graphs.

8 Find the equations of the following straight lines. (Hint: It may help to sketch the graphs.)

a Gradient $\frac{3}{4}$, passing through $(-6, 5)$ **b** Gradient $-\frac{1}{2}$, passing through $(4, -3)$

c Gradient 0, passing through (0, 3) **d** Gradient 0, passing through $(0, -3)$

9 Write, in the form $y = mx + c$, the equations of lines which have the given gradient and pass through the given point:

a $1\frac{1}{3}$, (0, 3) **b** $-\frac{1}{2}$, (0, 3) **c** -0.7, (1, 6)

d $1\frac{1}{2}$, (4, 3) **e** $-\frac{3}{4}$, (4, 3) **f** -1, (0, 0)

Example 10

10 Find equations defining the lines which pass through the following pairs of points:

a (0, 4), (6, 0) **b** $(-3, 0)$, $(0, -6)$ **c** (0, 4), (4, 2) **d** (2, 6), (5, 3)

11 Find the equations, in the form $y = mx + c$, of the lines which pass through the following pairs of points:

a (0, 4), (3, 6) **b** (1, 0), (4, 2) **c** (−3, 0), (3, 3)
d (−2, 3), (4, 0) **e** (−1.5, 2), (4.5, 8) **f** (−3, 1.75), (4.5, −2)

12 Do the points $P(1, -3)$, $Q(2, 1)$ and $R\left(2\frac{1}{2}, 3\right)$ lie on the same straight line?

13 Find the equations defining each of the three sides of the triangle ABC for which the coordinates of the vertices are $A(-2, -1)$, $B(4, 3)$ and $C(6, 0)$.

2.4 Equation of a straight line in intercept form and sketching graphs

Often we encounter a linear relation that is not expressed in the form $y = mx + c$. An alternative standard notation is

$$ax + by = c$$

This is sometimes referred to as the **intercept form**.

While it is necessary to transpose the equation into gradient form if you wish to find the gradient, it is often convenient to work with linear relations in the intercept form.

Sketching graphs in intercept form

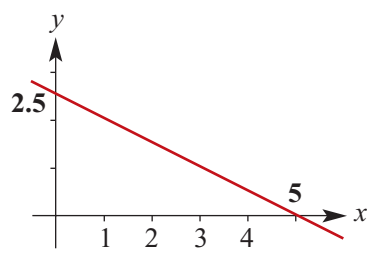
A convenient way to sketch graphs of straight lines is to plot the two axes intercepts.

Example 15

Sketch the graph of $2x + 4y = 10$.

Solution

x-axis intercept ($y = 0$): $2x + 4(0) = 10$
$x = 5$
y-axis intercept ($x = 0$): $2(0) + 4y = 10$
$y = 2.5$



When finding the equation of a straight line, it is also sometimes more convenient to express it in intercept form.

Example 16

Find the equation, in intercept form, of the line passing through the points $A(2, 5)$ and $B(6, 8)$.

Solution

$$m = \frac{8 - 5}{6 - 2} = \frac{3}{4}$$

Therefore, using the gradient and the point $A(2, 5)$, we have the equation:

$$\begin{aligned} y - 5 &= \frac{3}{4}(x - 2) \\ 4(y - 5) &= 3(x - 2) \\ 4y - 20 &= 3x - 6 \\ 4y - 3x &= 14 \\ -3x + 4y &= 14 \end{aligned}$$

Straight lines may be sketched by finding the axes intercepts even when the equation is given in the form $y = mx + c$.

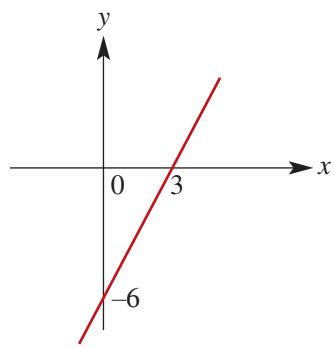
Example 17

CAS APPENDIX

Sketch the graph of $y = 2x - 6$, by first finding the intercepts.

Solution

When $x = 0$, $y = -6$.
Hence the y-axis intercept is -6.
When $y = 0$, $2x - 6 = 0$
$\therefore \quad 2x = 6 \quad$ and $\quad x = 3$
The x-axis intercept is 3.



Exercise 2D

Example 15 **1** For each of the following give the coordinates of the axes intercepts:

a $x + y = 4$ **b** $x - y = 4$ **c** $-x - y = 6$ **d** $y - x = 8$

Example 16 **2** For each of the following find the equation of the straight line graph passing through the points A and B:

a $A(0, 6)$ and $B(3, 0)$ **b** $A(0, -2)$ and $B(4, 0)$
c $A(2, 2)$ and $B(6, 6)$ **d** $A(2, 2)$ and $B(-6, 6)$

Example 17 **3** For each of the following sketch the graph by first finding the axes intercepts:

a $y = x - 1$ **b** $y = x + 2$ **c** $y = 2x - 4$

4 Sketch the graphs of each of the following linear relations:

a $2x - 3y = 12$ **b** $x - 4y = 8$ **c** $-3x + 4y = 24$
d $-5x + 2y = 20$ **e** $4x - 3y = 15$ **f** $7x - 2y = 15$

5 Find the equations of the straight lines passing through the following pairs of points. (Express your answer in intercept form.)

a $(-1, 4), (2, 3)$ **b** $(0, 4), (5, -3)$ **c** $(3, -2), (4, -4)$ **d** $(5, -2), (8, 9)$

6 Transpose from the intercept form to the gradient form and hence state the gradient of each of the following linear relations:

a $2x - y = 9$ **b** $3x + 4y = 10$ **c** $-x - 3y = 6$ **d** $5x - 2y = 4$

7 Sketch the graphs of each of the following by first determining the axes intercepts:

a $y = 2x - 10$ **b** $y = 3x - 9$ **c** $y = 5x + 10$ **d** $y = -2x + 10$

8 A straight line has equation $y = 3x - 4$. The points with coordinates $(0, a)$, $(b, 0)$, $(1, d)$ and $(e, 10)$ lie on the line. Find the values of a, b, d and e.

2.5 Linear models

In many practical situations a linear function can be used.

Example 18

Austcom's rates for local calls from private telephones consist of a quarterly rental fee of \$40 plus 25c for every call. Construct a cost function that describes the quarterly telephone bill.

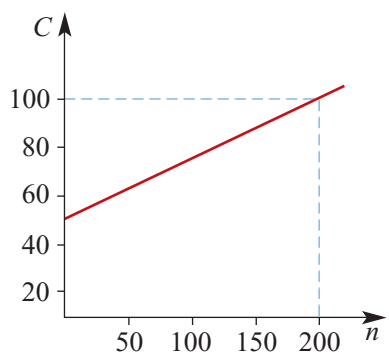
Solution

Let C = cost (\$) of quarterly telephone bill
n = number of calls
then: $C = 0.25n + 40$

As the number of calls is counted in whole numbers only, the domain of this function is $N \cup \{0\}$.

Draw the graph of the function

$C = 0.25n + 40, n \in N \cup \{0\}$



Note: The graph should be a series of discrete points rather than a continuous line because $n \in N \cup \{0\}$. With the scale used it is not practical to show it correctly.

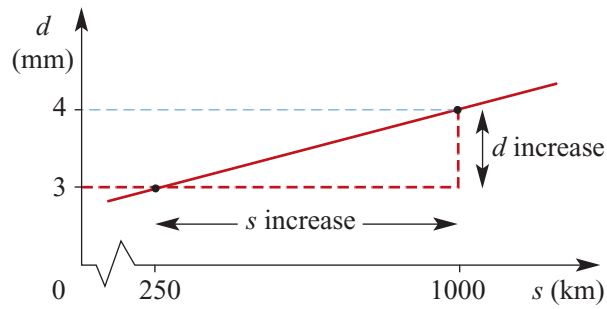
Example 19

The tyres on a racing car had lost 3 mm of tread after completing 250 km of a race and 4 mm of tread after completing 1000 km. Assuming that the loss of tread was proportional to the distance covered, find the total loss of tread, d mm, after s km from the start of the race. What would be the tread loss by the end of a 2000 km race?

Solution

$$\text{Gradient } m = \frac{d \text{ increase}}{s \text{ increase}} = \frac{4-3}{1000-250} = \frac{1}{750}$$

$$\therefore d = \frac{1}{750}s + c$$



When $s = 250$, $d = 3$

$$\therefore \quad 3 = \frac{1}{3} + c$$

$$\therefore \quad c = 2\frac{2}{3}$$

Total loss of tread after s km, $d = \frac{1}{750}s + 2\frac{2}{3}$

When $s = 2000$, $d = \frac{1}{750} \times 2000 + 2\frac{2}{3}$

$$= 2\frac{2}{3} + 2\frac{2}{3}$$

$$= 5\frac{1}{3}$$

$\therefore$ the loss of tread at the end of a 2000 km race is $5\frac{1}{3}$ mm.

An important linear relation is the relation between distance travelled and time taken when an object is travelling with constant speed. If a car travels at 40 km/h, the relationship between distance travelled (s kilometres) and time taken (t hours) is $s = 40t$, $t \geq 0$. The graph of s against t is a straight line graph through the origin. The gradient of the graph is 40.

Example 20

A car starts from point A on a highway 10 kilometres past the Wangaratta post office. The car travels at an average speed of 90 km/h towards picnic stop B, which is 120 kilometres further on from A. Let t hours be the time after the car leaves point A.

a Find an expression for the distance d_1 of the car from the post office at time t hours.

b Find an expression for the distance d_2 of the car from point B at time t hours.

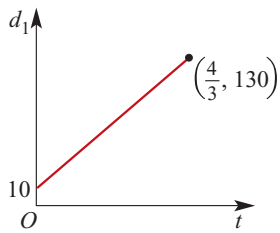
c On separate sets of axes sketch the graphs of d_1 against t and d_2 against t and state the gradient of each graph.

Solution

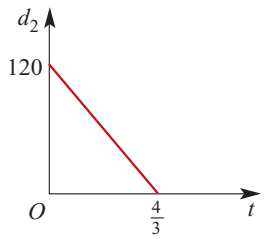
a At time t the distance of the car from the post office is $10 + 90t$ kilometres.

b At time t the distance of the car from B is $120 - 90t$ kilometres.

c



Gradient = 90



Gradient = −90

Exercise 2E

Skillsheet TEACHER

1 **a** A train moves at 50 km/h in a straight line away from town. Give a rule for the distance, d km, from the town at time t hours after leaving the town.

b A train has stopped at a siding 5 km from the town and then moves at 40 km/h in a straight line away from the siding. Give a rule for the distance, d km, from the town at time t hours after leaving the siding.

2 **a** An initially empty container is being filled with water at a rate of 5 litres per minute. Give a rule for the volume, V litres, of water in the container at time t minutes after the filling of the container starts.

b A container contains 10 litres of water. Water is then poured in at a rate of 5 litres per minute. Give a rule for the volume, V litres, of water in the container at time t minutes after the pouring starts.

Example 18

3 The weekly wage, $\$w$, of a vacuum cleaner salesperson consists of a fixed sum of \$350 plus \$20 for each cleaner sold. If n cleaners are sold per week, construct a rule that describes the weekly wage of the salesperson.

4 The reservoir feeding an intravenous drip contains 500 mL of a saline solution. The drip releases the solution into a patient at the rate of 2.5 mL/minute.

a Construct a rule which relates v, the amount of solution left in the reservoir, to time, t minutes.

b State the possible values of t and v. **c** Sketch the graph of the relation.

Example 18

5 The cost ($\$C$) of hiring a taxi consists of two elements, a fixed flagfall and an amount that varies with the number (n) of kilometres travelled. If the flagfall is \$2.60 and the cost per kilometre is \$1.50, determine a rule which gives C in terms of n.

6 A car rental company charges \$85, plus an additional amount of 24c per kilometre.

a Write a rule to determine the total charge $\$C$ for hiring a car and travelling x kilometres.

b What would be the cost to travel 250 kilometres?

Example 20

7 Two towns A and B are 200 km apart. A man leaves town A and drives at a speed of 5 km/h towards town B. Find the distance of the man from town B at time t hours after leaving town A.

8 The following table shows the extension of a spring when weights are attached to it.

x, extension (cm)	0	1	2	3	4	5	6
w, weight (g)	50	50.2	50.4	50.6	50.8	51.0	51.2

a Sketch a graph to show the relationship between x and w.
b Write a rule that describes the graph.
c What will be the extension if $w = 52.5$ g?

9 A printing firm charges \$35 for printing 600 sheets of headed notepaper and \$47 for printing 800 sheets.

a Find a formula, assuming the relationship is linear, for the charge, \$$C$, in terms of number of sheets printed, n.
b How much would they charge for printing 1000 sheets?

10 An electronic bankteller registered \$775 after it had counted 120 notes and \$975 after it had counted 160 notes.

a Find a formula for the sum registered (\$$C$) in terms of the number of notes (n) counted.
b Was there a sum already on the register when counting began?
c If so, how much?

2.6 Problems involving simultaneous linear models

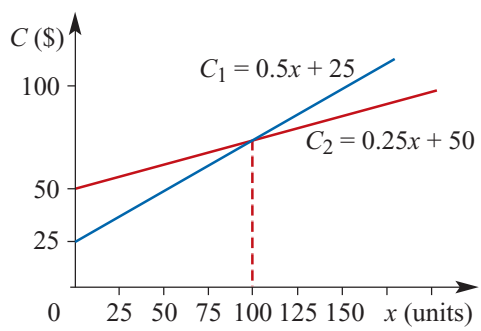
Example 21

There are two possible methods for paying gas bills.
Method A: A fixed charge of \$25 per quarter + 50c per unit of gas used
Method B: A fixed charge of \$50 per quarter + 25c per unit of gas used
Determine the number of units which must be used before method B becomes cheaper than method A.

Solution

Let C_1 = charge in \$ using method A
C_2 = charge in \$ using method B
x = number of units of gas used
Now $C_1 = 25 + 0.5x$
$C_2 = 50 + 0.25x$

It can be seen from the graph that if the number of units exceeds 100 method B is cheaper.



The solution can be obtained by solving simultaneous linear equations:

$$\begin{aligned} C_1 &= C_2 \\ 25 + 0.5x &= 50 + 0.25x \\ 0.25x &= 25 \\ x &= 100 \end{aligned}$$

Example 22

Robyn and Cheryl race over 100 metres. Robyn runs so that it takes a seconds to run 1 metre and Cheryl runs so that it takes b seconds to run 1 metre. Cheryl wins the race by 1 second. The next day they again race over 100 metres but Cheryl gives Robyn a 5 metre start so that Robyn runs 95 metres. Cheryl wins this race by 0.4 second. Find the values of a and b and the speed at which Robyn runs.

Solution

For the first race: time for Robyn − time for Cheryl = 1 s

$100a - 100b = 1 \qquad (1)$

For the second race: time for Robyn − time for Cheryl = 0.4 s

$95a - 100b = 0.4 \qquad (2)$

Subtract (2) from (1). Hence $5a = 0.6$ and $a = 0.12$.

Substitute in (1) to find $b = 0.11$.

Robyn's speed $= \dfrac{1}{0.12} = \dfrac{25}{3}$ metres per second.

Exercise 2F

1 Two bicycle hire companies have different charges. Company A charges $\$C$, according to the rule $C = 10t + 20$, where t is the time in hours for which a bicycle is hired. Company B charges $\$C$, according to the rule $C = 8t + 30$.

a Sketch each of the graphs on the same set of axes.

b Find the time, t, for which the charge of both companies is the same.

2 The distances, d_A km and d_B km, of cyclists A and B travelling along a straight road from a town hall step are given respectively by $d_A = 10t + 15$ and $d_B = 20t + 5$, where t is the time in hours after 1.00 pm.

a Sketch each of the graphs on the one set of axes.

b Find the time in hours at which the two cyclists are at the same distance from the town hall step.

3 A helicopter can be hired for \$210 per day plus a distance charge of \$1.60 per km or, alternatively, at a fixed charge of \$330 per day for an unlimited distance.

a For each of the methods of hiring, find an expression for cost, $\$C$, in terms of x km, the distance travelled.

b On one set of axes, draw the graph of cost versus distance travelled for each of the methods.

Example 21

c Determine for what distances the fixed-charge method is cheaper.

4 Three power boats in a 500 km handicap race leave at 5 hourly intervals. Boat A leaves first and has a speed for the race of 20 km/h. Boat B leaves 5 hours later and travels at an average speed of 25 km/h. Boat C leaves last, 5 hours after B, and completes the race at a speed of 40 km/h.

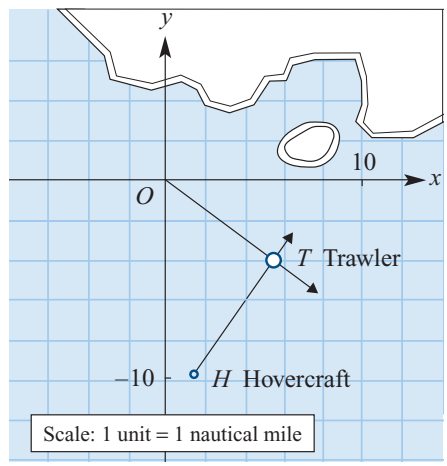
a Draw a graph of each boat's journey on the same set of axes.

b Use your graphs to find the winner of the race.

c Check your answer algebraically.

d Write a short description of what happened to each boat in the race.

5 If the line OT has the equation $y = -\frac{3}{4}x$ and the line HT has the equation $y = \frac{3}{2}x - 12$, determine the point over which both craft would pass.



6 A school wishes to take some of its students on an excursion. If they travel by tram it will cost the school \$2.80 per student. Alternatively, the school can hire a bus at a cost of \$54 for the day plus a charge of \$1 per student.

a For each mode of transport, write an expression for the cost ($\$C$) of transport in terms of the number of students (x).

b On one set of axes, draw the graph of cost, $\$C$, versus number of students, x, for each mode of transport.

Example 21

c Determine for how many students it will be more economical to hire the bus.

7 Anne and Maureen live in towns that are 57 km apart. Anne sets out at 9.00 am one day to ride her bike to Maureen's town at a constant speed of 20 km/h. At the same time Maureen sets out to ride to Anne's town at a constant speed of 18 km/h.

a Write down a rule for the distance, d km, that each of them is from Anne's place at a time t minutes after 9.00 am.

b On the same set of axes, draw graphs of the distance, d km, versus time, t minutes after 9.00 am, for each cyclist.

c Find the time at which they will meet.

d How far has each of them travelled when they meet?

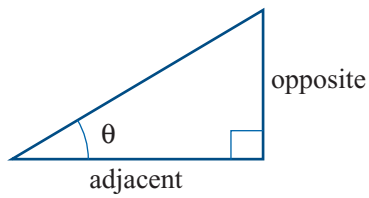
Example 22

8 John and Michael race over 50 metres. John runs so that it takes at a seconds to run 1 metre and Michael runs so that it takes b seconds to run 1 metre. Michael wins the race by 1 second. The next day they again race over 50 metres but Michael gives John a 3 metre start so that John runs 47 metres. Michael wins this race by 0.1 second. Find the values of a and b and the speed at which Michael runs.

2.7 The tangent of the angle of slope and perpendicular lines

From year 10 you will be familiar with the trigonometric ratio

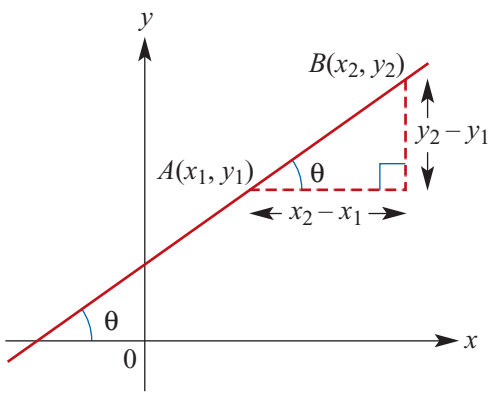
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

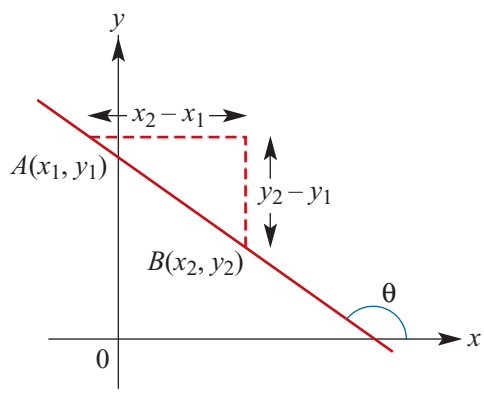


The gradient, m, of a straight line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_2 \neq x_1)$$

It therefore follows that $m = \tan\theta$, where θ is the angle that the line makes with the positive direction of the x-axis.





If m is positive, θ will be an acute angle. If m is negative, θ will be an obtuse angle.

It follows that the value of m in the general equation of a straight line $y = mx + c$ gives the value of the tangent of the angle made by the line and the x-axis.

Example 23

Determine the gradient of the line passing through the given points and the angle the line makes with the positive direction of the x-axis for:

a (3, 2) and (5, 7) **b** (5, −3) and (−1, 5)

Solution

a
$$m = \frac{7-2}{5-3}$$
$$= 2.5$$
Tangent of angle $= 2.5$
$\therefore$ Angle $= 68.20^\circ$

b
$$m = \frac{5-(-3)}{-1-5}$$
$$= -\frac{8}{6}$$
Tangent of angle $= -\frac{4}{3}$
This implies the angle is obtuse
$\therefore$ angle $= 180^\circ - (53.130\ldots)^\circ$
$= 126.87^\circ$

Example 24

Find the magnitude of the angle each of the following make with the positive direction of the x-axis:

a $y = 2x + 3$ **b** $3y = 3x - 6$ **c** $y = -0.3x + 1.5$

Solution

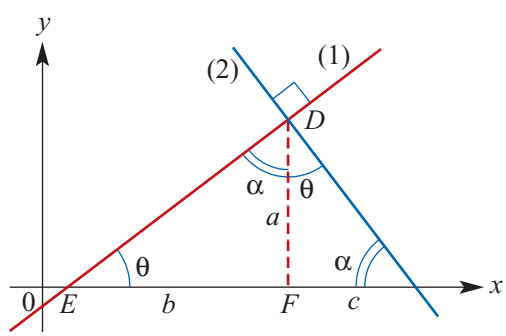
a
$y = 2x + 3$
Gradient $= 2$
$\therefore$ Tangent of angle $= 2$
giving an angle of $63.43^\circ (63^\circ 26')$

b
$3y = 3x - 6$
$y = x - 2$
Gradient $= 1$
$\therefore$ Tangent of angle $= 1$
giving an angle of 45°

c
$y = -0.3x + 1.5$
Gradient $= -0.3$
$\therefore$ Tangent of angle $= -0.3$
giving an angle of $(180 - 16.7)^\circ$ with the positive direction of the x-axis
$= 163.3^\circ (163^\circ 18')$

Perpendicular lines

Multiplying the gradients of two **perpendicular** straight lines leads to a useful result.



Gradient of (1) $= \tan\theta$
$= \frac{a}{b}$
$= m_1$
Gradient of (2) $= m_2$
$= -\frac{a}{c}$
$= -\tan\alpha$
$= -\frac{b}{a}$ (from ΔDEF)
$= -\frac{1}{m_1}$
$\therefore \quad m_1 m_2 = -1$

If two straight lines are perpendicular, the product of their gradients is -1.

Conversely, if the product of the gradients of two lines is -1, then the two lines are perpendicular.

Note: This result holds if one of the two lines is not parallel to an axis.

Example 25

Find the equation of the straight line which passes through (1, 2) and is:

a parallel to the line with equation $2x - y = 4$

b perpendicular to the line with equation $2x - y = 4$.

Solution

The equation $2x - y = 4$ can be rearranged to $y = 2x - 4$. Hence the gradient of the line can be seen to be 2.

a Therefore the gradient of any line parallel to this line is 2.
The equation of the straight line with this gradient and passing through the point with coordinates (1, 2) is

$$y - 2 = 2(x - 1)$$

Therefore $y = 2x$ is the equation of the line which passes through (1, 2) and is parallel to the line with equation $2x - y = 4$.

b The gradient of any line perpendicular to the line with equation $y = 2x - 4$ is $-\frac{1}{2}$.
The equation of the straight line with this gradient and passing through the point with coordinates (1, 2) is

$$y - 2 = -\frac{1}{2}(x - 1)$$

Therefore $2y - 4 = -x + 1$ and equivalently $2y + x = 5$.
Therefore $2y + x = 5$ is the equation of the line which passes through (1, 2) and is perpendicular to the line with equation $2x - y = 4$.

Example 26

The coordinates of the vertices of a triangle ABC are $A(0, -1)$, $B(2, 3)$ and $C\left(3, -2\frac{1}{2}\right)$. Show that the side AB is perpendicular to side AC.

Solution

Note: m_{AB} = the gradient of the line AB.

$$\begin{aligned} m_{AB} &= \frac{3 - (-1)}{2 - 0} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

$$m_{AC} = \frac{-2\frac{1}{2} - (-1)}{3 - 0}$$
$$= \frac{-1\frac{1}{2}}{3}$$
$$= -\frac{1}{2}$$

Since $m_{AB} \times m_{AC} = 2 \times -\frac{1}{2} = -1$, the lines AB and AC are perpendicular to each other.

Exercise 2G

1 Find the angle that the lines joining the given points make with the positive direction of the x-axis:

a $(0, 3), (3, 0)$ **b** $(0, -4), (4, 0)$ **c** $(0, 2), (-4, 0)$ **d** $(0, -5), (-5, 0)$

2 Find the magnitude of the angle made by each of the following with the positive direction of the x-axis:

a $y = x$ **b** $y = -x$ **c** $y = x + 1$
d $x + y = 1$ **e** $y = 2x$ **f** $y = -2x$

Example 23 **3** Find the angle that the lines joining the given points make with the positive direction of the x-axis:

a $(-4, -2), (6, 8)$ **b** $(2, 6), (-2, 4)$ **c** $(-3, 4), (6, 1)$
d $(-4, -3), (2, 4)$ **e** $(3b, a), (3a, b)$ **f** $(c, b), (b, c)$

Example 24 **4** Find the magnitude of the angle made by each of the following with the positive direction of the x-axis:

a $y = 3x + 2$ **b** $2y = -2x + 1$ **c** $2y - 2x = 6$ **d** $3y + x = 7$

5 If the points A, B and C have the coordinates $A(5, 2)$, $B(2, -3)$ and $C(-8, 3)$, show that the triangle ABC is a right-angled triangle.

6 Show that $RSTU$ is a rectangle if the coordinates of the vertices are respectively $R(2, 6)$, $S(6, 4)$, $T(2, -4)$ and $U(-2, -2)$.

Example 25 **7** Find the equation of the straight line which passes through the point $(1, 4)$ and is perpendicular to the line with equation $y = -\frac{1}{2}x + 6$.

8 Find the equation of the straight line which passes through $(4, -2)$ and is:

a parallel to the line with equation $2x - 3y = 4$
b perpendicular to the line with equation $2x - 3y = 4$.

9 Given that the lines $4x - 3y = 10$ and $4x - ly = m$ are perpendicular and intersect at the point (4, 2), find the values of l and m.

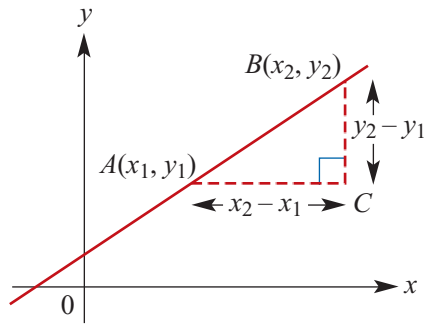
2.8 The distance between two points

The distance between the given points $A(x_1, y_1)$ and $B(x_2, y_2)$ can be found by applying the Theorem of Pythagoras to triangle ABC:

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{aligned}$$

$\therefore$ The distance between the two points A and B is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

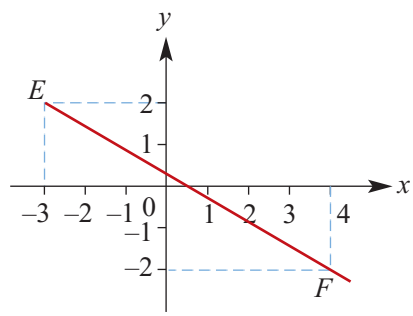


Example 27

Calculate the distance EF if E is $(-3, 2)$ and F is $(4, -2)$.

Solution

$$\begin{aligned} EF &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4 - (-3))^2 + (-2 - 2)^2} \\ &= \sqrt{7^2 + (-4)^2} \\ &= \sqrt{65} \\ &= 8.06 \text{ (to 2 decimal places)} \end{aligned}$$



Exercise 2H

Example 27

1 Find the distance between each of the following (correct to 2 decimal places):

a (3, 6) and (−4, 5)
b (4, 1) and (5, −3)
c (−2, −3) and (−5, −8)
d (6, 4) and (−7, 4)

2 Calculate the perimeter of a triangle with vertices (−3, −4), (1, 5) and (7, −2).

3 There is an off-shore oil drilling platform in Bass Strait situated at $D(0, 6)$, where 1 unit = 5 km. Pipes for this oil drill come ashore at $M(-6, 1)$ and $N(3, -1)$. Assuming the pipelines are straight, which is the shorter DM or DN?

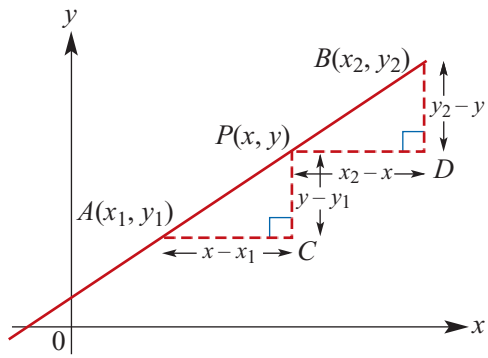
2.9 Midpoint of a line segment

Let $P(x, y)$ be the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$. The triangles APC and PBD are congruent.

$$AC = PD \quad \text{and} \quad PC = BD$$
$$\therefore\ x - x_1 = x_2 - x \qquad y - y_1 = y_2 - y$$
$$2x = x_1 + x_2 \qquad 2y = y_1 + y_2$$
$$x = \frac{x_1 + x_2}{2} \qquad y = \frac{y_1 + y_2}{2}$$

To find the midpoint (x, y)

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$



Example 28

Find the midpoint of the line segment joining $A(2, 6)$ with $B(-3, -4)$.

Solution

Midpoint of line segment AB has coordinates $\left(\frac{2 + -3}{2}, \frac{6 + -4}{2}\right) = \left(-\frac{1}{2}, 1\right)$.

Exercise 21

Example 28

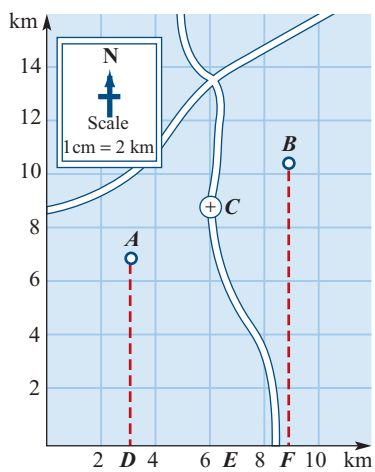
1 Find the coordinates of M, the midpoint of AB, where A and B have the following coordinates:

a $A(2, 12), B(8, 4)$ **b** $A(-3, 5), B(4, -4)$

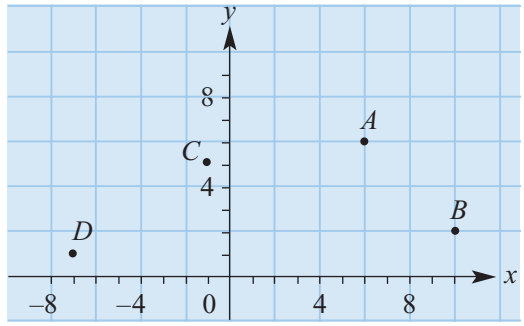
c $A(-1.6, 3.4), B(4.8, -2)$ **d** $A(3.6, -2.8), B(-5, 4.5)$

2 Find the midpoints of each of the sides of a triangle ABC, where A is (1, 1), B is (5, 5) and C is (11, 2).

3 The secretary of a motorcross club wants to organise two meetings on the same weekend. One is a hill climb starting from point $A(3.1, 7.1)$ and the other is a circuit event with the start at $B(8.9, 10.5)$, as shown on the map. Only one ambulance can be provided. The ambulance can be called up by radio, so it is decided to keep it at C, halfway between A and B. What are the coordinates of C?



4 The diagram shows the four points $A(6, 6)$, $B(10, 2)$, $C(-1, 5)$ and $D(-7, 1)$.



a If the midpoint of AB is P and the midpoint of CD is M, calculate the distance PM.

b Does the line joining these midpoints pass through $\left(0, 3\frac{1}{4}\right)$?

5 If M is the midpoint of XY, find the coordinates of Y when X and M have the following values:

a $X(-4, 2)$, $M(0, 3)$ **b** $X(-1, -3)$, $M(0.5, -1.6)$

c $X(6, -3)$, $M(2, 1)$ **d** $X(4, -3)$, $M(0, -3)$

6 Find the coordinates of the midpoint of the line joining $(1, 4)$ and (a, b), in terms of a and b. If $(5, -1)$ is the midpoint find the values of a and b.

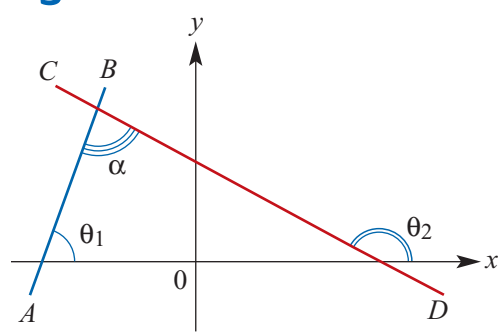
2.10 Angle between intersecting lines

It is possible to use coordinate geometry to find the angle between two intersecting lines.

Let α = angle between the intersecting lines AB and CD.

θ_1 = angle between the positive direction of the x-axis and the line AB.

θ_2 = angle between the positive direction of the x-axis and the line CD.



Then $\theta_1 + \alpha = \theta_2$. (The exterior angle of a triangle equals the sum of the two opposite interior angles.)

$\therefore\ \alpha = \theta_2 - \theta_1$

Example 29

Find the two angles between the intersecting lines $3y + 2x = 6$ and $y = x + 1$. Sketch each line and label the angles.

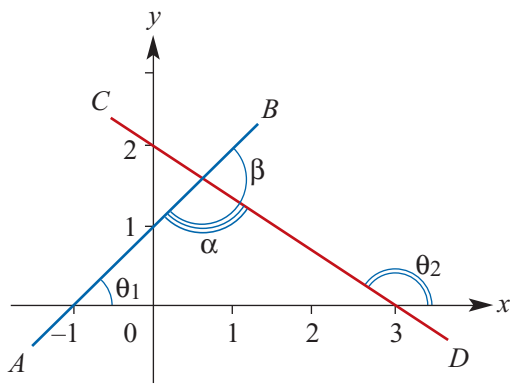
Solution

To find θ_1

$$\begin{aligned} y &= x + 1 \\ \therefore \quad m &= 1 \\ \tan\theta_1 &= 1 \\ \theta_1 &= 45^\circ \end{aligned}$$

To find θ_2

$$\begin{aligned} 3y + 2x &= 6 \\ 3y &= -2x + 6 \\ y &= -\frac{2}{3}x + 2 \\ \therefore \quad m &= -\frac{2}{3} \\ &= -0.6667 \\ \therefore \quad \tan\theta_2 &= -0.6667 \\ \theta_2 &= 180^\circ - 33^\circ 41' \\ &= 146^\circ 19' \\ \therefore \quad \alpha &= \theta_2 - \theta_1 \\ &= 146^\circ 19' - 45^\circ \\ &= 101^\circ 19' \\ \beta &= 180 - 101^\circ 19' \\ &= 78^\circ 41' \end{aligned}$$



Exercise 2J

Example 29

1 Find the acute angle between each of the following pairs of straight lines:

a With gradients 3 and $\frac{3}{4}$

b With gradients –2 and 3

c With gradients $\frac{2}{3}$ and $-\frac{3}{2}$

d With equations $2y = 8x + 10$ and $3x - 6y = 22$

e With equations $4x - 3y = 5$ and $2x - 4y = 9$

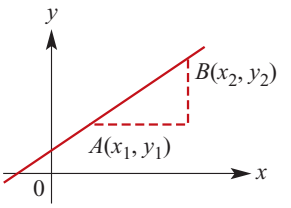
D&D TEST MC TEST

Review

Chapter summary

- Gradient of a straight line joining two points:

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1}$$



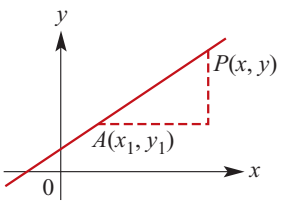
- The general equation of a straight line is

$$y = mx + c$$

where m is the gradient and c is the value of the intercept on the y-axis.

- Equation of a line passing through a given point (x_1, y_1) and having a gradient m is

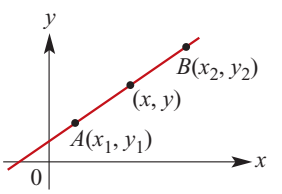
$$y - y_1 = m(x - x_1)$$



- Equation of a line passing through two given points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = m(x - x_1)$$

where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.



- Simultaneous equations can be solved graphically by drawing the graph of each equation on the same set of axes. The point at which the lines intersect, (x, y), gives the x- and y-values that satisfy both equations.
- The tangent of the angle of slope (θ) can be found with

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

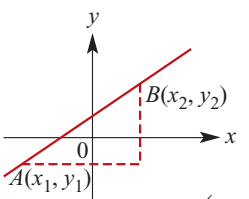
where θ is the angle the line makes with the positive direction of the x-axis.

- If two straight lines are perpendicular to each other the product of their gradients is -1.

$$m_1 m_2 = -1$$

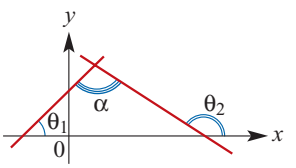
- Distance between 2 points A and B is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



- The midpoint of a straight line joining (x_1, y_1) and (x_2, y_2) is the point $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.
- The angle, α, between intersecting straight lines:

$$\alpha = \theta_2 - \theta_1$$



Multiple-choice questions

1 The gradient of the line passing through the points $(5, -8)$ and $(6, -10)$ is

A -2 **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $-\frac{1}{18}$ **E** $\frac{3}{2}$

2 The gradient of the line passing through points $(4a, 2a)$ and $(9a, -3a)$ is

A a **B** $-5a$ **C** 1 **D** -5 **E** -1

3 The equation of a straight line with gradient 3 and passing through the point $(1, 9)$ is

A $y = x + 9$ **B** $y = 3x + 9$ **C** $y = 3x + 6$

D $y = -\frac{1}{3}x + 1$ **E** $y = -\frac{1}{3}x + 6$

4 A straight line passes through the points $(2, -6)$ and $(-2, -14)$. The equation of the line is

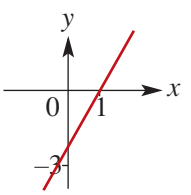
A $y = x - 8$ **B** $y = \frac{1}{2}x - 7$ **C** $y = \frac{1}{2}x - 10$

D $y = 2x - 10$ **E** $y = -\frac{1}{2}x - 8$

5 The line with equation $y = 2x - 6$ passes through the point $(a, 2)$. The value of a is

A 2 **B** 4 **C** 5 **D** -4 **E** -2

6 The relation with graph as shown has rule



A $y = -3x - 3$ **B** $y = -\frac{1}{3}x - 3$

C $y = \frac{1}{3}x - 3$ **D** $y = 3x + 3$

E $y = 3x - 3$

7 The coordinates of the midpoint of AB, where A has coordinates $(4, 12)$ and B has coordinates $(6, 2)$, are

A $(4, 8)$ **B** $(4.5, 8)$ **C** $(5, 8)$ **D** $(5, 7)$ **E** $(1, 5)$

8 If two lines $5x - y + 7 = 0$ and $ax + 2y - 11 = 0$ are parallel then a equals

A -5 **B** 5 **C** -10 **D** 10 **E** $-\frac{1}{2}$

9 $(6, 3)$ is the midpoint of the line joining the points $(-4, y)$ and $(x, -6)$. The value of $x + y$ is

A 0 **B** 16 **C** 20 **D** -10 **E** 28

10 The cost (\$C) of hiring a car is given by the formula $C = 2.5x + 65$, where x is the number of kilometres travelled. A person is charged \$750 for the hire of the car. The number of kilometres travelled was

A 65 **B** 145 **C** 160 **D** 200 **E** 274

Short-answer questions (technology-free)

1 Find the gradients of the lines joining each of the following pairs of points:

a $(4, 3)$ and $(8, 12)$ **b** $(-3, 4)$ and $(8, -6)$ **c** $(2, 1)$ and $(2, 9)$

d $(0, a)$ and $(a, 0)$ **e** $(0, 0)$ and (a, b) **f** $(0, b)$ and $(a, 0)$

Review

2 Find the equation of the straight line of gradient 4 which passes through the point with coordinates:

a (0, 0) **b** (0, 5) **c** (1, 6) **d** (3, 7)

3 **a** The point $(1, a)$ lies on the line with equation $y = 3x - 5$. Find the value of a.

b The point $(b, 15)$ lies on the line with equation $y = 3x - 5$. Find the value of b.

4 Find the equation of the straight line joining the points $(-5, 2)$ and $(3, -4)$.

5 Find the equation of the straight line of gradient $-\frac{2}{3}$, which passes through $(-4, 1)$.

6 The straight line with equation $ax + by = c$ passes through the points $(2, 4)$ and $(-3, 1)$. Find the values of a, b and c.

7 Write down the equation of the straight line that:

a passes through $(5, 11)$ and is parallel to the x-axis

b passes through $(0, -10)$ and is parallel to the line with equation $y = 6x + 3$

c passes through the point $(0, -1)$ and is perpendicular to the line with equation $3x - 2y + 5 = 0$

8 Find the length and the coordinates of the midpoint of the line segment joining each of the following pairs of points:

a $A(1, 2)$ and $B(5, 2)$ **b** $A(-4, -2)$ and $B(3, -7)$ **c** $A(3, 4)$ and $B(7, 1)$

9 Find the equation of a straight line which passes through the point $(2, 3)$ and is inclined at 30° to the positive direction of the x-axis.

10 Find the equation of the straight line which passes through the point $(-2, 3)$ and makes an angle of 135° with the positive direction of the x-axis.

11 Find the angle between the straight line $3x - 2y = 4$ and the line joining the points $(-2, -1)$ and $(4, 1)$.

Extended-response questions

1 The table below shows different shoe sizes and their corresponding lengths in millimetres, to the nearest millimetre.

Shoe size, S	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
Length in mm, l	220	224	229	233	237	241	246	250

Shoe size, S	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
Length in mm, l	254	258	263	267	271	275	279

a Plot a graph to show the approximate linear relationship between S and l.

b Write the formula which describes this linear relationship.

European 'continental' shoe sizes are different and are shown in the new table below.

Continental shoe size, C	33	34	35	36	37	38	39	40	41
Length in mm, l	220	227	233	240	247	253	260	267	273

c Plot a graph to show the approximate linear relationship of C against l.

d Write a rule which describes this linear relationship.

2 The cost of hiring a motor cruiser consists of a down payment of \$100 and a running charge of \$20 per day, or part of a day. The cost of fuel is \$5.50 per day. There is also a charge of \$10 for filling the freshwater tanks and charging the batteries. Food for a cruise of n days costs \$12.50 per day.

a Express C, the total cost in \$, of hiring the cruiser for n days (all costs to be included).

b For how many days can a cruiser be hired if the cost of a cruise is to be no more than \$600?

c A rival company has a fixed rate of \$60 per day. For how many days would it be cheaper to hire from this company?

3 The cost of fitting a new plug and cable for an electric drill is \$$C$, when the length of the cable is x metres and $C = 4.5 + 1.8x$.

a What meaning could be given for the constant term 4.5?

b What could be the meaning of the coefficient 1.8?

c What would be the gradient of the graph of C against x?

d What length of cable would give a total cost of \$24.50?

4 The profit made on a single journey of an Easyride bus tour is \$$P$, when there are x empty seats and $P = 1020 - 24x$.

a What do you think is the meaning of the constant term 1020?

b What is the least number of empty seats which would result in a loss on a single journey?

c Suggest a meaning for the coefficient 24.

5 A quarterly electricity bill shows the following charges:

- For the first 50 kWh (kilowatt hours): 9.10c per kWh
- For the next 150 kWh: 5.80c per kWh
- Thereafter: 3.56c per kWh

a Write down a formula relating cost, \$$C$, to n, the number of kWh of electricity used:

i for the first 50 kWh **ii** for the next 150 kWh **iii** for more than 200 kWh

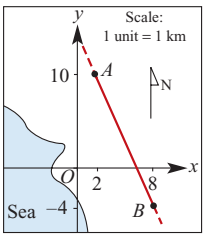
b Draw a graph of C against n. Use the graph, or otherwise, to determine the charges for

i 30 kWh **ii** 90 kWh **iii** 300 kWh

c How much electricity could be used for a cost of \$20?

Review

6 O is the position of the air traffic control tower at an airport. An aircraft travelling in a straight line is identified at $A(2, 10)$ and again at $B(8, -4)$.



a What is the equation that describes the flight path of the aircraft?

b How far south of O is the aircraft when $x = 15$ km?

7 A construction company estimates that for every 1% of air left in concrete as it is being laid the strength of the hardened concrete decreases by 7%. Let x represent the percentage of air in the concrete (by volume), and the strength of the concrete be s units, where $s = 100$ when $x = 0$.

a Write a linear model for s in terms of x. **b** Sketch a graph of s against x.

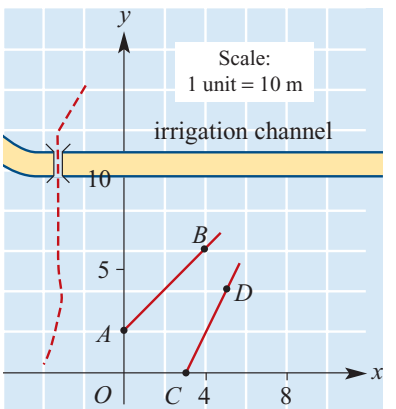
c Calculate how much air can be allowed to remain in the concrete for a strength of at least 95%.

d Estimate how much air the concrete will contain at 0% strength.

e Is the model sensible at 0% strength?

f State the possible values of x.

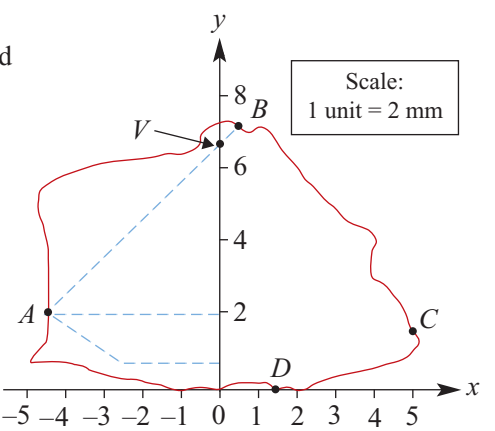
8 The diagram shows a plan view of a paddock over which a cartesian framework has been superimposed. From an observation point O a rabbit has been spotted, first at $A(0, 2)$ and then at $B(4, 6)$. A fox is seen at $C(3, 0)$ and later at $D(5, 4)$.



a Find the equations defining AB and CD.

b Assuming that both the rabbit and the fox were running along straight lines, calculate whether the fox's path would cross the rabbit's track before the irrigation channel.

9 The diagram shows the side view of a rough, uncut diamond fixed in position on a computer-controlled cutting machine. The diamond is held at the points $A(-4.5, 2)$, $B(0.25, 7)$, $C(5, 1.5)$ and $D(1.5, 0)$. The units are in millimetres.

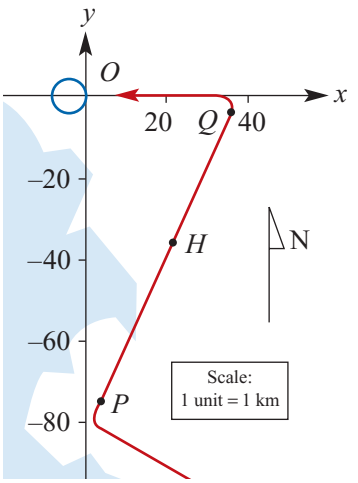


a If a straight cut is made joining A and B, find the y-coordinate of the point V at which the cut will cross the vertical axis.

b Find the equation of the line joining V and C.

c Would the cuts AB and VC be equally inclined to the vertical axis? Explain your answer.

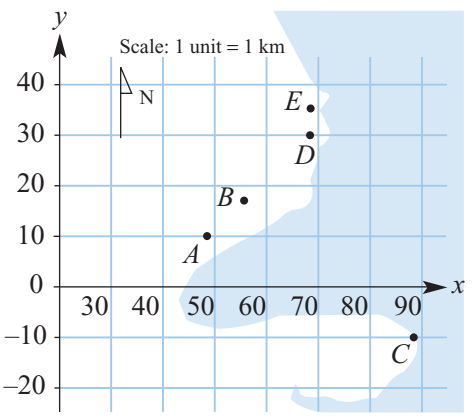
10 A new light beacon is proposed at $P(4, -75)$ for air traffic flying into an airport located at $O(0, 0)$. It is intended that the aircraft should follow a course over beacons at P and $Q(36, -4)$, turning at Q towards the runway at O.



a Would a direct line from P to Q pass directly over a hospital located at $H(20, -36)$?

b If not, state how far east or west of H the aircraft would be when the y-coordinate of an aircraft's flight path is -36.

11 The map shows an area where it is proposed to construct a new airport. It is thought that the main runway of the airport will have one end of its centre line at $A(48, 10)$, but the position of the other end of this line, B, has not been decided. There is a light aircraft airport at $E(68, 35)$ and a radio beacon at $C(88, -10)$.



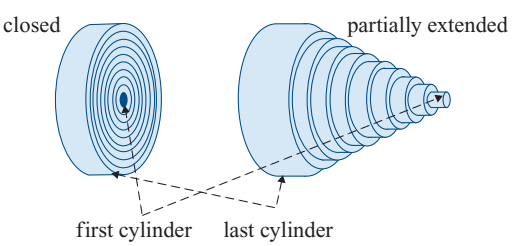
a What is the equation that will define the new runway if aircraft coming in to land from the east must be on the extended central line of the new runway when they are 5 km due south of E?

b If B is to be 8 km to the east of A, what will be its coordinates?

c A marker beacon is to be built at $D(68, 30)$ and it is proposed that several auxiliary beacons should be placed on the line CD. What is the equation defining the line CD?

d If one of the auxiliary beacons is to be placed due east of A, what are the coordinates of its position?

12 A silversmith is making a piece of jewellery consisting of a set of hollow tubular pieces of silver that can slide in and out over each other. Each tube is a circular cylinder made by soldering together the ends of a rectangular strip of silver. The diameters of the cylinders increase in equal steps.



Review

The length of the strip of silver used for the smallest innermost cylinder is 10 mm. The largest is the eighteenth cylinder and requires a strip of silver 61 mm long.

a Find a formula for the length L of the nth strip.

b Draw a graph to enable the lengths of the intermediate strips to be read off.

13 Wheelrite, a small company that manufactures garden wheelbarrows, has overhead expenses of \$30 000 per year. In addition, it costs \$40 to manufacture each wheelbarrow.

a Write a rule which determines the total cost, \$$C$, of manufacturing x wheelbarrows per year.

b If the annual production is 6000 wheelbarrows, what is the overall cost per wheelbarrow?

c How many wheelbarrows must be made so that the overall cost is \$46 per wheelbarrow?

d Wheelrite sells wheelbarrows to retailers for \$80 each. Write a rule which determines the revenue, \$$R$, from the sale of x wheelbarrows to retailers.

e Plot the graphs for C and R against x on the same axes.

f What is the minimum number of wheelbarrows that must be produced for Wheelrite to make a profit each year?

g Write a rule which determines the profit, \$$P$, from the manufacture and sale of x number of wheelbarrows.

14 An electricity supply authority is offering customers a choice of two methods of paying electricity bills. Method 1 involves payment annually and Method 2 involves payment each quarter (that is every three months). The charges for each method are as follows:

Method 1 – per year		**Method 2 – per quarter**	
Fixed charge	\$100	Fixed charge	\$27.50
Price per unit	\$0.08125	Price per unit	\$0.075

a Suppose Customer A used 1560 units of electricity in a year. Calculate which is the cheaper method of payment.

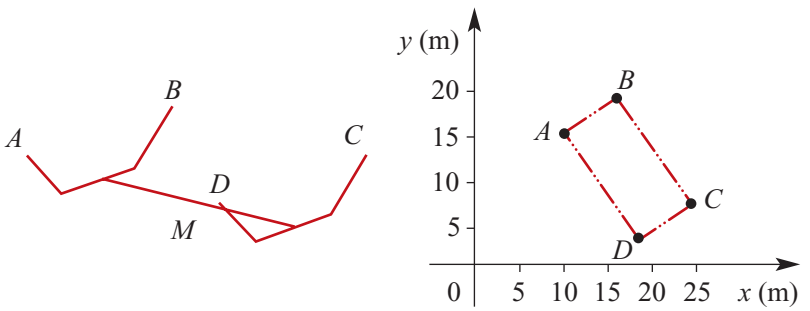
b Copy and then complete the following table:

	Number of units of electricity			
	0	1000	2000	3000
Cost (\$) calculated by Method 1				
Cost (\$) calculated by Method 2				

c Use these values to plot graphs of the costs for each method of paying for electricity. Clearly indicate the approximate number of units of electricity for which the cost is the same for each method of payment.

d If C_1 is the cost by Method 1, C_2 is the cost by Method 2, and x is the number of units of electricity used in a year, write down the two formulae which show the cost of x units calculated by each method. Use these formulae to calculate the exact number of units for which the cost is the same by each method.

15 In a metal fabricating yard which has been flooded by overflow from a local river, a large steel frame (see diagram) has been partly submerged. The ends *A, B, C* and *D* are the only parts visible above the level of the flood water.



The coordinates of the ends relative to an overhead crane are $A(10, 16)$, $B(16, 20)$, $C(24, 8)$ and $D(18, 4)$. The overhead crane moves east–west along its rail, and the distance east from a point $O(0, 0)$ is denoted by x. The crane's hook moves north–south across the frame and the distance to the north of the south rail is denoted by y. Units are in metres. The steel frame is to be raised out of the water by lifting it at the midpoint, M, of its middle section.

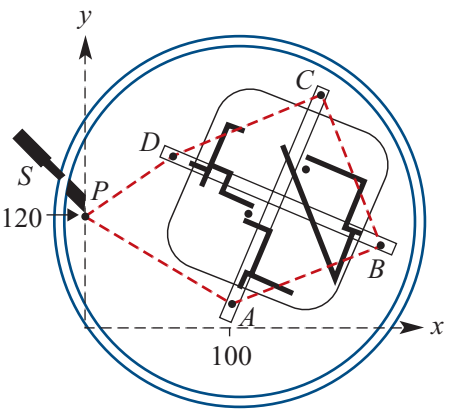
a Find the coordinates, x and y, of the point to which the hook must be moved so that it will be directly above the midpoint, M, of the steel frame.

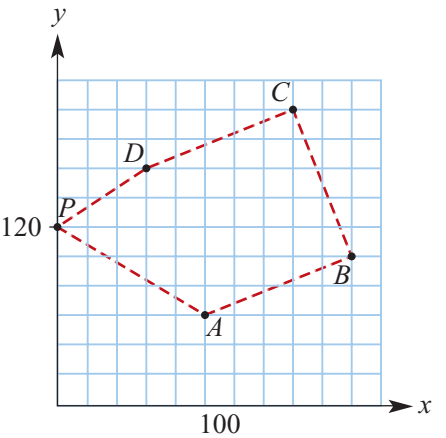
b In order to minimise the risk of the hook slipping, the hook will be moved slowly along a line parallel to AB. Find the equation of the line along which the hook will be moved.

16 The diagram shows part of a micro-electronics circuit, as seen through a magnifying glass; the circuit has been etched onto a chip of plated silica. The four points *A, B, C* and *D* stand away from the chip itself. *A* is (100, 60), *B* is (200, 100), *C* is (160, 200), *D* is (60, 160). Units are in $\frac{1}{25}$ mm.

Review

The unit S is a moveable micro-soldering unit, its tip being at $P(0, 120)$. It is desired to program the tip of the soldering iron, P, to solder wires to the points A, B, C and D, moving along the broken lines as shown in the graph.



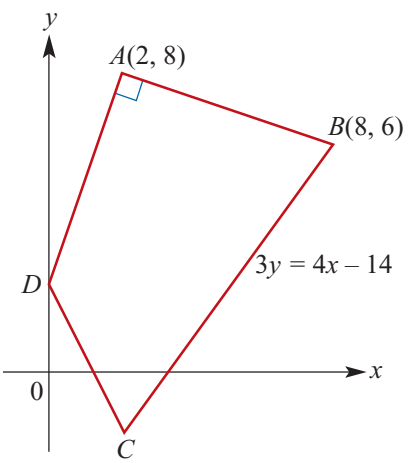


Scale: 1 unit = $\frac{1}{25}$ mm

a Find equations for the lines defining each section of the path along which P must be programmed to move.

b Will any of the turns be through right angles? Explain.

17 The diagram shows a quadrilateral. Angle BAD is a right angle and C lies on the perpendicular bisector of AB. The equation of the line through points B and C is $3y = 4x - 14$.



Find:

a the equation of the line through points A and D

b the coordinates of D

c the equation of the perpendicular bisector of AB

d the coordinates of C

e the area of triangle ADC

f the area of the quadrilateral $ABCD$.

CHAPTER

3

Quadratics

Objectives

- To recognise and sketch the graphs of **quadratic relations**.
- To determine the **maximum** or **minimum** values of a quadratic relation.
- To **solve quadratic equations** by factorising, completing the square and using the general formula.
- To apply the **discriminant** to determine the nature and number of roots of quadratic relations.
- To apply quadratic relations to solving problems.

A **polynomial function** has a rule of the type

$$y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \qquad (n \in N)$$

where $a_0, a_1 \ldots a_n$ are numbers called **coefficients**.

The **degree** of a polynomial is given by the value of n, the highest power of x with a non-zero coefficient.

Examples

i $y = 2x + 3$ is a polynomial of degree 1.

ii $y = 2x^2 + 3x - 2$ is a polynomial of degree 2.

iii $y = -x^3 + 3x^2 + 9x - 7$ is a polynomial of degree 3.

First degree polynomials, otherwise called **linear** relations, have been discussed in Chapter 2.

In this chapter second degree polynomials will be investigated. These are called **quadratics**.

3.1 Expanding and collecting like terms

Finding the x-axis intercepts, if they exist, is a requirement for sketching graphs of quadratics. To do this requires the solution of quadratic equations, and as an introduction to the methods of solving quadratic equations the basic algebraic processes of expansion and factorisation will be reviewed.

Example 1

Simplify $2(x-5)-3(x+5)$, by first expanding.

Solution

$$\begin{aligned} 2(x-5)-3(x+5) &= 2x-10-3x-15 && \text{Expand each bracket.} \\ &= 2x-3x-10-15 && \text{Collect like terms.} \\ &= -x-25 \end{aligned}$$

Example 2

Expand $2x(3x-2)+3x(x-2)$.

Solution

$$\begin{aligned} 2x(3x-2)+3x(x-2) &= 6x^2-4x+3x^2-6x \\ &= 9x^2-10x \end{aligned}$$

For expansions of the type $(a+b)(c+d)$ proceed as follows:

$$\begin{aligned} (a+b)(c+d) &= a(c+d)+b(c+d) \\ &= ac+ad+bc+bd \end{aligned}$$

Example 3

Expand the following:

a $(x+3)(2x-3)$ **b** $(x-3)(2x-2\sqrt{2})$

Solution

a $(x+3)(2x-3)$
$= x(2x-3)+3(2x-3)$
$= 2x^2-3x+6x-9$
$= 2x^2+3x-9$

b $(x-3)(2x-2\sqrt{2})$
$= x(2x-2\sqrt{2})-3(2x-2\sqrt{2})$
$= 2x^2-2\sqrt{2}x-6x+6\sqrt{2}$
$= 2x^2-(2\sqrt{2}+6)x+6\sqrt{2}$

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Example 4

Expand $(2x-1)(3x^2+2x+4)$.

Solution

$$\begin{aligned} (2x-1)(3x^2+2x+4) &= 2x(3x^2+2x+4)-1(3x^2+2x+4) \\ &= 6x^3+4x^2+8x-3x^2-2x-4 \\ &= 6x^3+x^2+6x-4 \end{aligned}$$

Consider the expansion of a perfect square, $(x + a)^2$.

$$\begin{aligned}(x + a)^2 &= (x + a)(x + a)\\ &= x(x + a) + a(x + a)\\ &= x^2 + ax + ax + a^2\\ &= x^2 + 2ax + a^2\end{aligned}$$

Thus the general result can be stated as:

$$(x + a)^2 = x^2 + 2ax + a^2$$

Example 5

Expand $(3x - 2)^2$.

Solution

$$\begin{aligned}(3x - 2)^2 &= (3x)^2 + 2(3x)(-2) + (-2)^2\\ &= 9x^2 - 12x + 4\end{aligned}$$

Consider the expansion of $(x + a)(x - a)$.

$$\begin{aligned}(x + a)(x - a) &= x(x - a) + a(x - a)\\ &= x^2 - ax + ax - a^2\\ &= x^2 - a^2\end{aligned}$$

Thus the expansion of the difference of two squares has been obtained:

$$(x + a)(x - a) = x^2 - a^2$$

Example 6

Expand:

a $(2x - 4)(2x + 4)$ **b** $(x - 2\sqrt{7})(x + 2\sqrt{7})$

Solution

a $$\begin{aligned}(2x - 4)(2x + 4) &= (2x)^2 - (4)^2\\ &= 4x^2 - 16\end{aligned}$$

b $$\begin{aligned}(x - 2\sqrt{7})(x + 2\sqrt{7}) &= x^2 - (2\sqrt{7})^2\\ &= x^2 - 28\end{aligned}$$

Example 7

Expand $(2a - b + c)(2a - b - c)$.

Solution

$$\begin{aligned}(2a - b + c)(2a - b - c) &= ((2a - b) + c)((2a - b) - c) \\ &= (2a - b)^2 - c^2 \\ &= 4a^2 - 4ab + b^2 - c^2\end{aligned}$$

Exercise 3A

1 Expand each of the following:

a $2(x - 4)$ **b** $-2(x - 4)$ **c** $3(2x - 4)$
d $-3(4 - 2x)$ **e** $x(x - 1)$ **f** $2x(x - 5)$

2 Collect like terms in each of the following:

a $2x + 4x + 1$ **b** $2x - 6 + x$ **c** $3x + 1 - 2x$ **d** $-x + 2x - 3 + 4x$

Example 1 **3** Simplify each of the following by expanding and collecting like terms:

a $8(2x - 3) - 2(x + 4)$ **b** $2x(x - 4) - 3x$
c $4(2 - 3x) + 4(6 - x)$ **d** $4 - 3(5 - 2x)$

Example 2 **4** Simplify each of the following by expanding and collecting like terms:

a $2x(x - 4) - 3x$ **b** $2x(x - 5) + x(x - 5)$ **c** $2x(-10 - 3x)$
d $3x(2 - 3x + 2x^2)$ **e** $3x - 2x(2 - x)$ **f** $3(4x - 2) - 6x$

Example 3 **5** Simplify each of the following by expanding and collecting like terms:

a $(3x - 7)(2x + 4)$ **b** $(x - 10)(x - 12)$ **c** $(3x - 1)(12x + 4)$
d $(4x - 5)(2x - 3)$ **e** $(x - \sqrt{3})(x - 2)$ **f** $(2x - \sqrt{5})(x + \sqrt{5})$

Example 5 **6** Simplify each of the following by expanding and collecting like terms:

a $(x - 4)^2$ **b** $(2x - 3)^2$ **c** $(6 - 2x)^2$
d $\left(x - \frac{1}{2}\right)^2$ **e** $(x - \sqrt{5})^2$ **f** $(x - 2\sqrt{3})^2$

7 Simplify each of the following by expanding and collecting like terms:

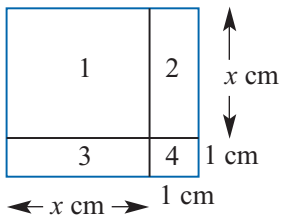
a $(2x - 3)(3x^2 + 2x - 4)$ **b** $(x - 1)(x^2 + x + 1)$
c $(6 - 2x - 3x^2)(4 - 2x)$ **d** $(x - 3)(x + 3)$
e $(2x - 4)(2x + 4)$ **f** $(9x - 11)(9x + 11)$
g $(5x - 3)(x + 2) - (2x - 3)(x + 3)$ **h** $(2x + 3)(3x - 2) - (4x + 2)(4x - 2)$
i $(x - y + z)(x - y - z)$ **j** $(x - y)(a - b)$

8 Find the area of each of the following by:

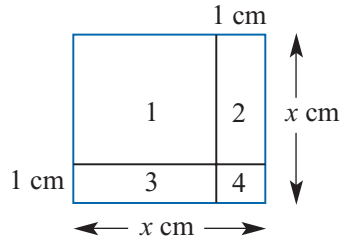
i finding the area of the four 'non overlapping' rectangles, two of which are squares, and adding

ii multiplying length by width of the undivided square (boundary in blue)

a



b



3.2 Factorising

Four different types of factorisation will be considered.

1 Removing the highest common factor (HCF)

Example 8

Factorise $9x^2 + 81x$.

Solution

$$\begin{aligned} 9x^2 + 81x &= 9x \times x + 9x \times 9 \\ &= 9x(x + 9) \end{aligned}$$ removing the HCF

Example 9

Factorise $2a^2 - 8ax^2$.

Solution

$$\begin{aligned} 2a^2 - 8ax^2 &= 2a \times a - 2a \times 4x^2 \\ &= 2a(a - 4x^2) \end{aligned}$$

Example 10

Factorise $7x^2y - 35xy^2$.

Solution

$7x^2y - 35xy^2 = 7xy(x - 5y)$ where $7xy$ is the HCF

2 Grouping of terms

This can be used for expressions containing four terms.

Example 11

Factorise $x^3 + 4x^2 - 3x - 12$.

Solution

The terms in this expression can be grouped as follows:

$$\begin{aligned} x^3 + 4x^2 - 3x - 12 &= (x^3 + 4x^2) - (3x + 12) \\ &= x^2(x + 4) - 3(x + 4) \qquad \text{removing the HCF} \\ &= (x^2 - 3)(x + 4) \end{aligned}$$

3 Difference of two squares (DOTS)

$$x^2 - a^2 = (x + a)(x - a)$$

Example 12

Factorise $3x^2 - 75$.

Solution

$$\begin{aligned} 3x^2 - 75 &= 3(x^2 - 25) \qquad \text{3 is the HCF} \\ &= 3(x + 5)(x - 5) \end{aligned}$$

Example 13

Factorise $9x^2 - 36$.

Solution

This is a difference of two squares:

$$\begin{aligned} 9x^2 - 36 &= 9(x^2 - 4) \\ &= 9(x - 2)(x + 2) \end{aligned}$$

Example 14

Factorise $(x - y)^2 - 16y^2$.

Solution

$$\begin{aligned} (x - y)^2 - 16y^2 &= (x - y)^2 - (4y)^2 \\ &= (x - y + 4y)(x - y - 4y) \\ &= (x + 3y)(x - 5y) \end{aligned}$$

4 Factorising quadratic expressions

Example 15

Factorise $x^2 - 2x - 8$.

Solution

$$x^2 - 2x - 8 = (x + a)(x + b) = x^2 + (a + b)x + ab$$

The values of a and b are such that $ab = -8$

and $a + b = -2$

Values of a and b which satisfy these two conditions are $a = -4$ and $b = 2$

$\therefore\ x^2 - 2x - 8 = (x - 4)(x + 2)$

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Example 16

Factorise $6x^2 - 13x - 15$.

Solution

There are several combinations of factors of $6x^2$ and -15 to consider. Only one combination is correct.

Factors of $6x^2$	Factors of -15	'Cross-products' add to give $-13x$
$6x$	$+5$	$+5x$
x	-3	$-18x$
		$-13x$

$\therefore$ Factors of $6x^2 - 13x - 15$
$= (6x + 5)(x - 3)$

Exercise 3B

1 Factorise each of the following:

a $2x + 4$ **b** $4a - 8$ **c** $6 - 3x$
d $2x - 10$ **e** $18x + 12$ **f** $24 - 16x$

Examples 8, 9

2 Factorise:

a $4x^2 - 2xy$ **b** $8ax + 32xy$ **c** $6ab - 12b$
d $6xy + 14x^2y$ **e** $x^2 + 2x$ **f** $5x^2 - 15x$
g $-4x^2 - 16x$ **h** $7x + 49x^2$ **i** $2x - x^2$
Example 10 **j** $6x^2 - 9x$ **k** $7x^2y - 6y^2x$ **l** $8x^2y^2 + 6y^2x$

Example 11

3 Factorise:

a $x^3 + 5x^2 + x + 5$ **b** $x^2y^2 - x^2 - y^2 + 1$ **c** $ax + ay + bx + by$
d $a^3 - 3a^2 + a - 3$ **e** $x^3 - bx^2 - a^2x + a^2b$

Examples 12, 13

4 Factorise:

a $x^2 - 36$ **b** $4x^2 - 81$ **c** $2x^2 - 98$
Example 14 **d** $3ax^2 - 27a$ **e** $(x - 2)^2 - 16$ **f** $25 - (2 + x)^2$
g $3(x + 1)^2 - 12$ **h** $(x - 2)^2 - (x + 3)^2$

Example 15 **5** Factorise:

a $x^2 - 7x - 18$	**b** $y^2 - 19y + 48$	**c** $3x^2 - 7x + 2$
d $6x^2 + 7x + 2$	**e** $a^2 - 14a + 24$	**f** $a^2 + 18a + 81$
Example 16 **g** $5x^2 + 23x + 12$	**h** $3y^2 - 12y - 36$	**i** $2x^2 - 18x + 28$
j $4x^2 - 36x + 72$	**k** $3x^2 + 15x + 18$	**l** $ax^2 + 7ax + 12a$
m $5x^3 - 16x^2 + 12x$	**n** $48x - 24x^2 + 3x^3$	**o** $(x-1)^2 + 4(x-1) + 3$

3.3 Quadratic equations

In this section the solution of quadratic equations by simple factorisation is considered. There are three steps to solving a quadratic equation by factorisation.

Step 1 Write the equation in the form $ax^2 + bx + c = 0$.

Step 2 Factorise the quadratic expression.

Step 3 Use the result that $ab = 0$ implies $a = 0$ or $b = 0$ (or both) (the null factor theorem).

For example $x^2 - x = 12$

$x^2 - x - 12 = 0$ Step 1

$(x-4)(x+3) = 0$ Step 2

$x - 4 = 0$ or $x + 3 = 0$ Step 3

$\therefore \quad x = 4$ or $x = -3$

Example 17

Solve $x^2 + 11x + 24 = 0$.

Solution

x^2	$+24$	$+11x$
x	$+3$	$+3x$
x	$+8$	$+8x$
		$+11x$

Factorising we obtain

$(x+3)(x+8) = 0$

$\therefore \quad x + 3 = 0$ or $x + 8 = 0$

$\therefore \quad x = -3$ or $x = -8$

i.e. both $x = -8$ and $x = -3$ are solutions of $x^2 + 11x + 24 = 0$

To verify, substitute in the equation.

When $x = -8$ $(-8)^2 + 11(-8) + 24 = 0$

$x = -3$ $(-3)^2 + 11(-3) + 24 = 0$

Example 18

Solve $2x^2 + 5x - 12 = 0$.

Solution

$2x^2$	-12	$+5x$
$2x$	-3	$-3x$
x	$+4$	$+8x$
		$+5x$

Factorising gives

$(2x - 3)(x + 4) = 0$

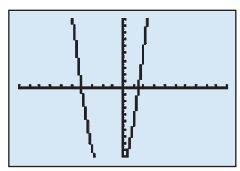
$\therefore \quad 2x - 3 = 0 \quad$ or $\quad x + 4 = 0$

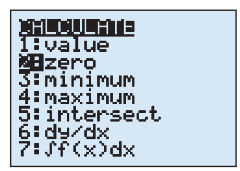
$\therefore \quad x = \frac{3}{2} \quad$ or $\quad x = -4$

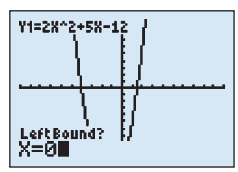
Using a graphics calculator

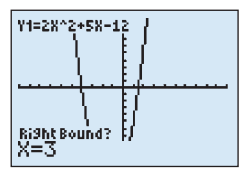
The use of the graphics calculator to solve linear equations was discussed in Chapter 1. Similar techniques can be used to solve quadratic equations. In this case, however, it is advantageous to use **2:zero** from the **CALC** menu. With the graphics calculator exact solutions are not always given, and certainly not if the solutions are not rational.

To solve $2x^2 + 5x - 12 = 0$ first enter $Y1 = 2x^2 + 5x - 12$ in the **Y=** screen. Choose **6:ZStandard** from the **ZOOM** menu to see the graph. It can be seen in this window that there are two values of x which make the value of the expression $2x^2 + 5x - 12$ zero. From the graph they appear to be around $x = 2$ and $x = -4$. The solution to the right will be found. Choose **2:zero** from the **CALC** menu. For the left bound type $x = 0$ (or take the cursor to another suitable point) and press ENTER.

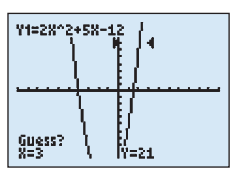


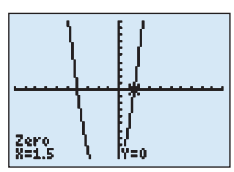




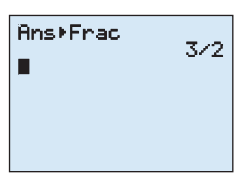


For the right bound choose $x = 3$ and press ENTER. The guess can be 3. Press ENTER twice. To obtain the answer in fractional form press 2ND QUIT to return to the Home screen.





Press 2ND ANS and then choose **1:Frac** from the **MATH** menu. The other solution can be found in a similar manner.



Example 19

The perimeter of a rectangle is 20 cm and its area is 24 cm^2. Calculate the length and width of the rectangle.

Solution

Let x cm be the length of the rectangle and y cm the width.

Then $2(x + y) = 20$ and thus $y = 10 - x$.

The area is 24 cm and therefore $x(10 - x) = 24$.

i.e. $10x - x^2 = 24$

This implies $x^2 - 10x - 24 = 0$

$(x - 6)(x - 4) = 0$

Thus the length is 6 cm or 4 cm. The width is 4 cm or 6 cm.

Exercise 3C

Skillsheet TEACHER

1 Solve each of the following for x:

a $(x - 2)(x - 3) = 0$ **b** $x(2x - 4) = 0$ **c** $(x - 4)(2x - 6) = 0$

d $(3 - x)(x - 4) = 0$ **e** $(2x - 6)(x + 4) = 0$ **f** $2x(x - 1) = 0$

g $(5 - 2x)(6 - x) = 0$ **h** $x^2 = 16$

2 Use a graphics calculator to solve each of the following equations. Give your answer correct to 2 decimal places.

a $x^2 - 4x - 3 = 0$ **b** $2x^2 - 4x - 3 = 0$ **c** $-2x^2 - 4x + 3 = 0$

Example 17

3 Solve for x in each of the following:

a $x^2 - 6x + 8 = 0$ **b** $x^2 - 8x - 33 = 0$ **c** $x(x + 12) = 64$

d $x^2 + 5x - 14 = 0$ **e** $2x^2 + 5x + 3 = 0$ **f** $4x^2 - 8x + 3 = 0$

Example 18

g $x^2 = 5x + 24$ **h** $6x^2 + 13x + 6 = 0$ **i** $2x^2 - x = 6$

j $6x^2 + 15 = 23x$ **k** $2x^2 - 3x - 9 = 0$ **l** $10x^2 - 11x + 3 = 0$

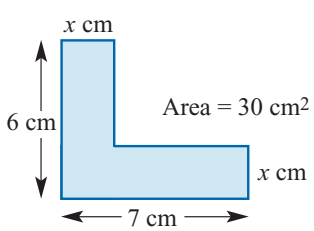
m $12x^2 + x = 6$ **n** $4x^2 + 1 = 4x$ **o** $x(x + 4) = 5$

p $\frac{1}{7}x^2 = \frac{3}{7}x$ **q** $x^2 + 8x = -15$ **r** $5x^2 = 11x - 2$

4 The bending moment, M, of a simple beam used in bridge construction is given by the formula $M = \frac{wl}{2}x - \frac{w}{2}x^2$.

If $l = 13$ m, $w = 16$ kg/m and $M = 288$ kg m, calculate the value of x.

5 Calculate the value of x.



6 The height, h, reached by a projectile after t seconds travelling vertically upwards is given by the formula $h = 70t - 16t^2$. Calculate t if h is 76 metres.

7 A polygon with n sides has $\frac{n(n-3)}{2}$ diagonals. How many sides has a polygon with 65 diagonals?

8 For a particular electric train the tractive 'resistance', R, at speed, v km/h is given by $R = 1.6 + 0.03v + 0.003v^2$. Find v when the tractive resistance is 10.6.

Example 19 **9** The perimeter of a rectangle is 16 cm and its area is 12 cm^2. Calculate the length and width of the rectangle.

Example 19 **10** The altitude of a triangle is 1 cm shorter than the base. If the area of the triangle is 15 cm^2, calculate the altitude.

11 Tickets for a concert are available at two prices. The more expensive ticket is \$30 more than the cheaper one. Find the cost of each type of ticket if a group can buy 10 more of the cheaper tickets than the expensive ones for \$1800.

12 The members of a club hire a bus for \$2100. Seven members withdraw from the club and the remaining members have to pay \$10 more each to cover the cost. How many members originally agreed to go on the bus?

3.4 Graphing quadratics

A quadratic relation is defined by the general rule

$$y = ax^2 + bx + c$$

where a, b and c are constants and $a \neq 0$.

This is called **polynomial** form.

The simplest quadratic relation is $y = x^2$.

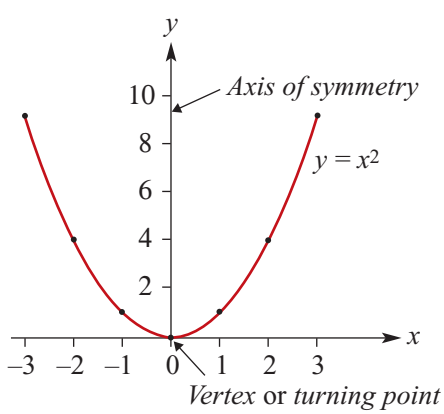
If a table of values is constructed for $y = x^2$ for $-3 \leq x \leq 3$,

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

these points can be plotted and then connected to produce a continuous curve.

Features of the graph of $y = x^2$:

- The graph is called a **parabola**.
- The possible y-values are all positive real numbers and 0. (This is called the **range** of the quadratic and is discussed in a more general context in Chapter 5.)
- It is symmetrical about the y-axis. The line about which the graph is symmetrical is called the **axis of symmetry**.
- The graph has a **vertex** or **turning point** at the origin (0, 0).
- The minimum value of y is 0 and it occurs at the turning point.



By a process called **completing the square** (to be discussed later in this chapter) all quadratics in polynomial form $y = ax^2 + bx + c$ may be transposed into what will be called the **turning point** form:

$$y = a(x - h)^2 + k$$

The effect of changing the values of a, h and k on our basic graph of $y = x^2$ will now be investigated. Graphs of the form $y = a(x - h)^2 + k$ are formed by **transforming** the graph of $y = x^2$. A more formal approach to transformations is undertaken in Chapter 5.

i Changing the value of *a*

First consider graphs of the form $y = ax^2$. In this case both $h = 0$ and $k = 0$. In the basic graph of $y = x^2$, a is equal to 1.

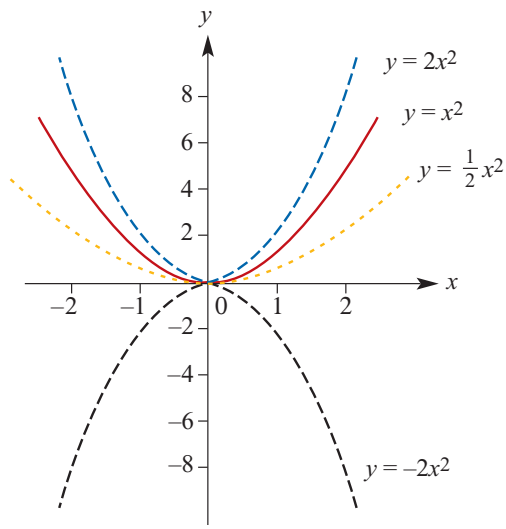
The following graphs are shown on the same set of axes.

$y = x^2$

$y = 2x^2 \qquad (a = 2)$

$y = \frac{1}{2}x^2 \qquad \left(a = \frac{1}{2}\right)$

$y = -2x^2 \qquad (a = -2)$



If $a > 1$ the graph is 'narrower', if $a < 1$ the graph is 'broader'. The transformation which produces the graph of $y = 2x^2$ from the graph of $y = x^2$ is called a **dilation of factor 2 from the *x*-axis**. When a is negative the graph is reflected in the x-axis. The transformation which produces the graph of $y = -x^2$ from the graph of $y = x^2$ is called a **reflection in the *x*-axis**.

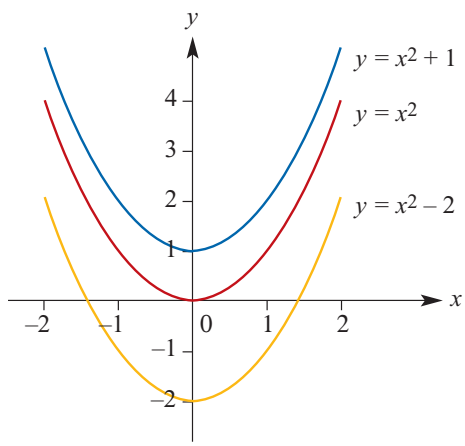
ii Changing the value of *k* ($a = 1$ and $h = 0$)

On this set of axes are the graphs of

$y = x^2$

$y = x^2 - 2 \qquad (k = -2)$

$y = x^2 + 1 \qquad (k = 1)$



As can be seen, changing k moves the basic graph of $y = x^2$ in a vertical direction. When $k = -2$ the graph is **translated** 2 units in the negative direction of the y-axis. The vertex is now $(0, -2)$ and the range is now all real numbers greater than or equal to -2.

When $k = 1$ the graph is **translated** 1 unit in the positive direction of the y-axis. The vertex is now $(0, 1)$ and the range is now all real numbers greater than or equal to 1.

All other features of the graph are unchanged. The axis of symmetry is still the y-axis.

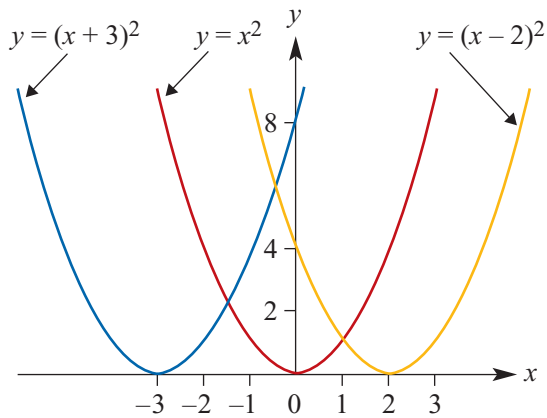
iii Changing the value of h ($a = 1$ and $k = 0$)

On this set of axes are the graphs of

$y = x^2$

$y = (x - 2)^2 \qquad (h = 2)$

$y = (x + 3)^2 \qquad (h = -3)$



As can be seen, changing h moves the graph in a horizontal direction. When $h = 2$ the graph is **translated** 2 units in the positive direction of the x-axis.

The vertex is now (2, 0) and the axis of symmetry is now the line $x = 2$; however, the range is unchanged and is still all non-negative real numbers.

When $h = -3$ the graph is **translated** 3 units in the negative direction of the x-axis. The vertex is now $(-3, 0)$ and the axis of symmetry is now the line $x = -3$; however, again the range is unchanged and is still all non-negative real numbers.

All these effects can be combined and the graph of any quadratic, expressed in the form $y = a(x - h)^2 + k$, can be sketched.

Example 20

Sketch the graph of $y = x^2 - 3$.

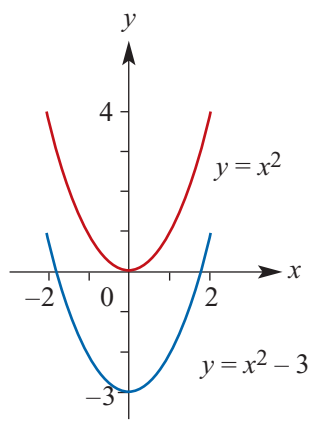
Solution

The graph of $y = x^2 - 3$ is obtained from the graph of $y = x^2$ by a translation of 3 units in the negative direction of the x-axis

The vertex is now at $(0, -3)$.

The axis of symmetry is the line with equation $x = 0$.

The x-axis intercepts are determined by solving the equation



$$x^2 - 3 = 0$$
$$\therefore \quad x^2 = 3$$
$$\therefore \quad x = \pm\sqrt{3}$$

$\therefore$ x-axis intercepts are $\pm\sqrt{3}$.

Example 21

Sketch the graph of $y = -(x + 1)^2$.

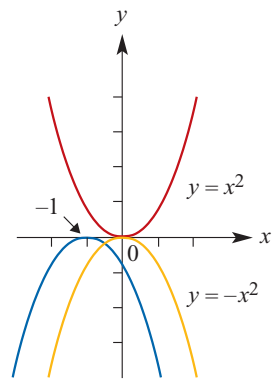
Solution

The graph of $y = -(x + 1)^2$ is obtained from the graph of $y = x^2$ by a reflection in the x-axis followed by a translation of 1 unit in the negative direction of the x-axis.

The vertex is now at $(-1, 0)$.

The axis of symmetry is the line with equation $x = -1$.

The x-axis intercept is $(-1, 0)$.

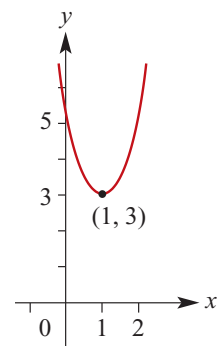


Example 22

Sketch the graph of $y = 2(x - 1)^2 + 3$.

Solution

The graph of $y = 2x^2$ is translated 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.



The vertex has coordinates (1, 3).

The axis of symmetry is the line $x = 1$.

The graph will be narrower than $y = x^2$.

The range will be $y \geq 3$.

To add further detail to our graph, the y-axis intercept and the x-axis intercepts (if any) can be found.

y-axis intercept: Let $x = 0$ $\quad y = 2(0 - 1)^2 + 3$

$y = 5$

x-axis intercept(s): In this example the minimum value of y is 3, so y cannot be 0.

$\therefore$ this graph has no x-axis intercepts.

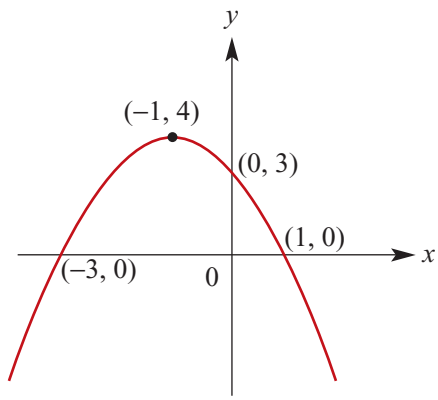
Note: Let $y = 0$ and try to solve for x.

$$0 = 2(x - 1)^2 + 3$$
$$-3 = 2(x - 1)^2$$
$$-\frac{3}{2} = (x - 1)^2$$

As the square root of a negative number is not a real number, this equation has no real solutions.

Example 23

Sketch the graph of $y = -(x + 1)^2 + 4$.



Solution

The vertex has coordinates $(-1, 4)$.

The axis of symmetry is the line $x = -1$.

***y*-axis intercept**:

Let $x = 0$ $\quad y = -(0 + 1)^2 + 4$

$y = 3$

***x*-axis intercepts**:

Let $y = 0$ $\quad 0 = -(x + 1)^2 + 4$

$(x + 1)^2 = 4$

$x + 1 = \pm 2$

$x = \pm 2 - 1$

$\therefore$ x-axis intercepts are $(1, 0)$ and $(-3, 0)$

Exercise 3D

Find: **i** the coordinates of the turning point **ii** the axis of symmetry **iii** the x-axis intercepts (if any)

of each case and use this information to help sketch the following graphs.

Example 20 **a** $y = x^2 - 4$ **b** $y = x^2 + 2$ **c** $y = -x^2 + 3$

Example 21 **d** $y = -2x^2 + 5$ **e** $y = (x - 2)^2$ **f** $y = (x + 3)^2$

Example 22 **g** $y = -(x + 1)^2$ **h** $y = -\frac{1}{2}(x - 4)^2$ **i** $y = (x - 2)^2 - 1$

Example 23 **j** $y = (x - 1)^2 + 2$ **k** $y = (x + 1)^2 - 1$ **l** $y = -(x - 3)^2 + 1$

m $y = (x + 2)^2 - 4$ **n** $y = 2(x + 2)^2 - 18$ **o** $y = -3(x - 4)^2 + 3$

p $y = -\frac{1}{2}(x + 5)^2 - 2$ **q** $y = 3(x + 2)^2 - 12$ **r** $y = -4(x - 2)^2 + 8$

3.5 Completing the square

In order to use the above technique for sketching quadratics, it is necessary for the quadratic to be expressed in **turning point** form.

To transpose a quadratic in **polynomial** form we must **complete the square**.

Consider the perfect square $(x + a)^2$

which when expanded becomes $x^2 + 2ax + a^2$

The last term of the expansion is the square of half the coefficient of the middle term.

Now consider the quadratic $y = x^2 + 2x - 3$

This is not a perfect square. We can however find the 'correct' last term to make this a perfect square.

If the last term is $1 = \left(\frac{1}{2} \times 2\right)^2$, then

$$y = x^2 + 2x + 1$$
$$= (x + 1)^2 \text{ a perfect square.}$$

In order to keep our original quadratic 'intact', we both add and subtract the 'correct' last term. For example:

$$y = x^2 + 2x - 3$$

becomes $y = (x^2 + 2x + 1) - 1 - 3$

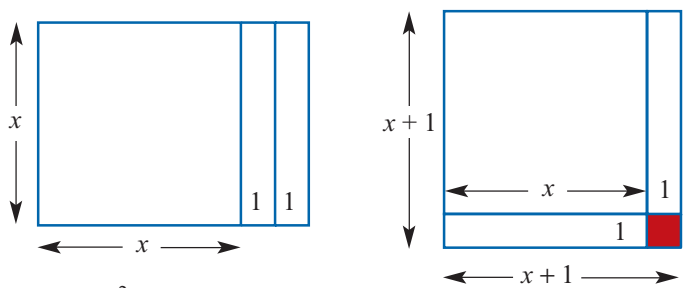
This can now be simplified to

$$y = (x + 1)^2 - 4$$

Hence the **vertex** (turning point) can now be seen to be the point with coordinates $(-1, -4)$.

In the above example the coefficient of x^2 was 1. If the coefficient is not 1, this coefficient must first be 'factored out' before proceeding to complete the square.

Completing the square for $x^2 + 2x$ is represented in the following diagram. The diagram to the left shows $x^2 + 2x$. The small rectangle to the right is moved to the 'base' of the x by x square. The red square of area 1 unit is added. Thus $x^2 + 2x + 1 = (x + 1)^2$.



The process of completing the square can also be used for the solution of equations.

Example 24

Solve each of the following equations for x by first completing the square:

a $x^2 - 3x + 1 = 0$ **b** $x^2 - 2kx + 1 = 0$ **c** $2x^2 - 3x - 1 = 0$

Solution

a $x^2 - 3x + 1 = 0$

Completing the square:

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 + 1 - \left(\frac{3}{2}\right)^2 = 0$$

$$\left(x - \frac{3}{2}\right)^2 - \frac{5}{4} = 0$$

Therefore $$\left(x - \frac{3}{2}\right)^2 = \frac{5}{4}$$

and $$x - \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$$

Hence $$x = \frac{3}{2} \pm \frac{\sqrt{5}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

b $x^2 - 2kx + 1 = 0$

Completing the square:

$x^2 - 2kx + k^2 + 1 - k^2 = 0$

$(x - k)^2 = k^2 - 1$

Therefore $x - k = \pm\sqrt{k^2 - 1}$

And $x = k \pm \sqrt{k^2 - 1}$

Note: If $k = \pm 1$ then $x = \pm 1$.

If $k > 1$ or $k < -1$ then there are two solutions.

If $-1 < k < 1$ then there are no solutions.

c $2x^2 - 3x - 1 = 0$

$2\left(x^2 - \frac{3}{2}x - \frac{1}{2}\right) = 0$

$x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \frac{1}{2} - \left(\frac{3}{4}\right)^2 = 0$ Divide both sides by 2 and then complete the square.

$\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$

Therefore $x - \frac{3}{4} = \pm\frac{\sqrt{17}}{4}$

and $x = \frac{3}{4} \pm \frac{\sqrt{17}}{4} = \frac{3 \pm \sqrt{17}}{4}$

Example 25

Find the coordinates of the vertex by completing the square and hence sketch the graph of $y = -2x^2 + 6x - 8$.

Solution

$y = -2x^2 + 6x - 8$

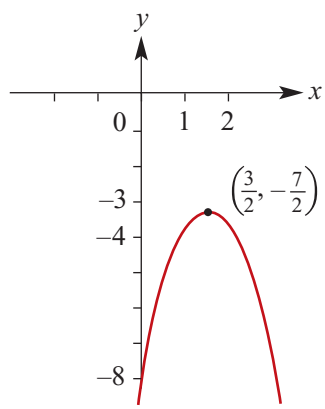
$y = -2(x^2 - 3x + 4)$ -2 is 'factored out'

$y = -2\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4\right)$

$y = -2\left\{\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 4\right\}$

$y = -2\left\{\left(x - \frac{3}{2}\right)^2 + \frac{7}{4}\right\}$

$y = -2\left(x - \frac{3}{2}\right)^2 - \frac{7}{2}$



$\therefore$ The vertex is $\left(\frac{3}{2}, -\frac{7}{2}\right)$.

y-axis intercept $= -8$.

Graph has maximum value of $-\frac{7}{2}$, $\therefore$ there are no x-axis intercepts.

The axis of symmetry is $x = \frac{3}{2}$.

Exercise 3E

1 Expand each of the following:

a $(x - 1)^2$ **b** $(x + 2)^2$ **c** $(x - 3)^2$ **d** $(-x + 3)^2$

e $(-x - 2)^2$ **f** $(x - 5)^2$ **g** $\left(x - \frac{1}{2}\right)^2$ **h** $\left(x - \frac{3}{2}\right)^2$

2 Factorise each of the following:

a $x^2 - 4x + 4$ **b** $x^2 - 12x + 36$ **c** $-x^2 + 4x - 4$

d $2x^2 - 8x + 8$ **e** $-2x^2 + 12x - 18$ **f** $x^2 - x + \frac{1}{4}$

g $x^2 - 3x + \frac{9}{4}$ **h** $x^2 + 5x + \frac{25}{4}$

Example 24 **3** Solve each of the following equations for x by first completing the square:

a $x^2 - 2x - 1 = 0$ **b** $x^2 - 4x - 2 = 0$ **c** $x^2 - 6x + 2 = 0$

d $x^2 - 5x + 2 = 0$ **e** $2x^2 - 4x + 1 = 0$ **f** $3x^2 - 5x - 2 = 0$

g $x^2 + 2x + k = 0$ **h** $kx^2 + 2x + k = 0$ **i** $x^2 - 3kx + 1 = 0$

Example 25 **4** Express each of the following in the form $y = a(x - h)^2 + k$.
Hence state the coordinates of the turning point and sketch the graph in each case.

a $x^2 - 2x + 3$ **b** $x^2 + 4x + 1$ **c** $x^2 - 3x + 1$

d $x^2 - 8x + 12$ **e** $x^2 - x - 2$ **f** $2x^2 + 4x - 2$

g $-x^2 + 4x + 1$ **h** $-2x^2 - 12x - 12$ **i** $3x^2 - 6x + 12$

3.6 Sketching quadratics in polynomial form

It is not essential to convert quadratics to turning point form in order to sketch the graph. We can find the x- and y-axis intercepts and the axis of symmetry from polynomial form by alternative methods and use these details to sketch the graph.

***y*-axis intercept:** Letting $x = 0$ in the general equation $y = ax^2 + bx + c$ we have

$$y = a(0)^2 + b(0) + c$$
$$y = c$$

i.e. the y-axis intercept is always equal to c.

***x*-axis intercepts**: Letting $y = 0$ in the general equation we have

$$0 = ax^2 + bx + c$$

In order to solve such an equation it is necessary to factorise the right-hand side and use the **null factor theorem**.

Once the x-axis intercepts have been found, the turning point can be found by using the symmetry properties of the parabola.

Example 26

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 - 4x$.

Solution

Step 1 $c = 0$, $\therefore$ y-axis intercept is $(0, 0)$

Step 2 Set $y = 0$ and factorise right-hand side of equation:

$$0 = x^2 - 4x$$
$$0 = x(x - 4)$$
$$\therefore x = 0 \text{ or } x = 4$$

x-axis intercepts are $(0, 0)$ and $(4, 0)$

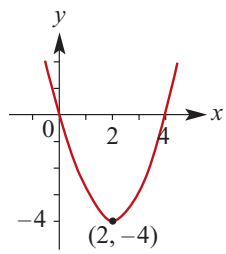
Step 3 Axis of symmetry is the line with equation $x = \dfrac{0 + 4}{2}$

i.e. $x = 2$

When $x = 2$, $y = (2)^2 - 4(2)$

$= -4$

$\therefore$ turning point has coordinates $(2, -4)$.



Example 27

Find the x- and y-axis intercepts and the coordinates of the turning point, and hence sketch the graph of $y = x^2 - 9$.

Solution

Step 1 $c = -9$, $\therefore$ y-axis intercept is $(0, -9)$.

Step 2 Set $y = 0$ and factorise right-hand side:

$$0 = x^2 - 9$$
$$0 = (x + 3)(x - 3)$$
$$\therefore x = -3 \text{ or } x = 3$$

x-axis intercepts are $(-3, 0)$ and $(3, 0)$

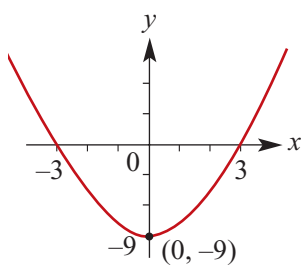
Step 3 Axis of symmetry is the line with equation $x = \dfrac{-3 + 3}{2}$

i.e. $x = 0$

When $x = 0$, $y = (0)^2 - 9$

$= -9$

$\therefore$ turning point has coordinates $(0, -9)$.



Example 28

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 + x - 12$.

Solution

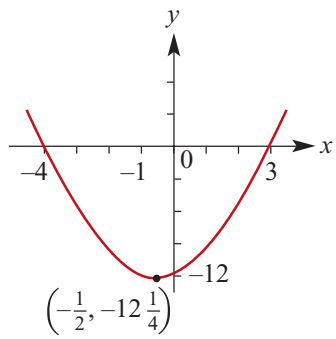
Step 1 $c = -12$, $\therefore$ y-axis intercept is $(0, -12)$

Step 2 Set $y = 0$ and factorise the right-hand side:

$$0 = x^2 + x - 12$$
$$0 = (x + 4)(x - 3)$$

$\therefore x = -4$ or $x = 3$

x-axis intercepts are $(-4, 0)$ and $(3, 0)$



Step 3 Due to the symmetry of the parabola, the axis of symmetry will be the line bisecting the two x-axis intercepts.

$\therefore$ the axis of symmetry is the line with equation $x = \dfrac{-4 + 3}{2} = -\dfrac{1}{2}$.

When $x = -\dfrac{1}{2}$, $y = \left(-\dfrac{1}{2}\right)^2 + \left(-\dfrac{1}{2}\right) - 12$

$$= -12\frac{1}{4}$$

$\therefore$ the turning point has coordinates $\left(-\dfrac{1}{2}, -12\dfrac{1}{4}\right)$.

Exercise 3F

1 **a** A parabola has x-axis intercepts 4 and 10. State the x-coordinate of the vertex.

b A parabola has x-axis intercepts 6 and 8. State the x-coordinate of the vertex.

c A parabola has x-axis intercepts -6 and 8. State the x-coordinate of the vertex.

2 **a** A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at 6. Find the other x-axis intercept.

b A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at -4. Find the other x-axis intercept.

c A parabola has vertex $(-2, 6)$ and one of the x-axis intercepts is at the origin. Find the other x-axis intercept.

3 Sketch each of the following parabolas, clearly showing the axis intercepts and the turning point:

Examples 26, 27

a $y = x^2 - 1$ **b** $y = x^2 + 6x$ **c** $y = 25 - x^2$ **d** $y = x^2 - 4$

e $y = 2x^2 + 3x$ **f** $y = 2x^2 - 4x$ **g** $y = -2x^2 - 3x$ **h** $y = x^2 + 1$

4 Sketch each of the following parabolas, clearly showing the axis intercepts and the turning point:

Example 28

a $y = x^2 + 3x - 10$ **b** $y = x^2 - 5x + 4$ **c** $y = x^2 + 2x - 3$

d $y = x^2 + 4x + 3$ **e** $y = 2x^2 - x - 1$ **f** $y = 6 - x - x^2$

g $y = -x^2 - 5x - 6$ **h** $y = x^2 - 5x - 24$

3.7 The general quadratic formula

Not all quadratics can be factorised by inspection and it is often difficult to find the x-axis intercepts this way. If the general expression for a quadratic in polynomial form is considered, a general formula can be developed by using the 'completing the square' technique. This can be used to solve quadratic equations.

To solve $ax^2 + bx + c = 0$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$ Divide all terms by a.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$ Complete the square by adding and subtracting $\left(\frac{b}{2a}\right)^2$.

$$\therefore \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$\therefore \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore \quad x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: The use of the **quadratic formula** is an alternative method to 'completing the square' for solving quadratic equations, but is probably not as useful as 'completing the square' for curve sketching, which gives the turning point coordinates directly.

It should be noted that the equation of the axis of symmetry can be derived from this general formula.

The axis of symmetry is the line with equation $x = -\frac{b}{2a}$.

Also, from the formula it can be seen that:

if $b^2 - 4ac > 0$ there are two solutions

if $b^2 - 4ac = 0$ there is one solution

if $b^2 - 4ac < 0$ there are no real solutions.

This will be further explored in the next section.

CAS APPENDIX

Example 29

Solve each of the following equations for x by using the quadratic formula:

a $x^2 - x - 1 = 0$ **b** $x^2 - 2kx - 3 = 0$

Solution

a $x^2 - x - 1 = 0$

$a = 1, b = -1$ and $c = -1$

The formula gives

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

b $2x^2 - 2kx - 3 = 0$

$a = 2, b = -2k$ and $c = -3$

The formula gives

$$x = \frac{-(-2k) \pm \sqrt{(-2k)^2 - 4 \times 2 \times (-3)}}{2 \times 2}$$

$$= \frac{2k \pm \sqrt{4k^2 + 24}}{4}$$

$$= \frac{k \pm \sqrt{k^2 + 6}}{2}$$

Example 30

Sketch the graph of $y = -3x^2 - 12x - 7$ by first using the quadratic formula to calculate the x-axis intercepts.

Solution

Since $c = -7$ the y-axis intercept is $(0, -7)$.

Turning point coordinates

Axis of symmetry: $x = -\frac{b}{2a}$

$$= -\frac{(-12)}{2 \times -3}$$

$$= -2$$

When $x = -2$, $y = -3(-2)^2 - 12(-2) - 7$

$$= 5$$

$\therefore$ turning point coordinates are $(-2, 5)$.

x-axis intercepts:

$-3x^2 - 12x - 7 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)}$$

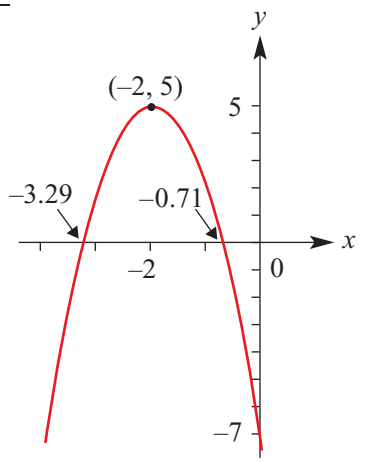
$$= \frac{12 \pm \sqrt{60}}{-6}$$

$$= \frac{12 \pm 2\sqrt{15}}{-6}$$

$$= \frac{6 \pm \sqrt{15}}{-3}$$

$$= \frac{6 \pm 3.87}{-3} \quad \text{(to 2 decimal places)}$$

$= -3.29$ or -0.71



Exercise 3G

1 For each of the following the coefficients a, b and c of a quadratic $y = ax^2 + bx + c$ are given. Find:

i $b^2 - 4ac$ **ii** $\sqrt{b^2 - 4ac}$ in simplest surd form

a $a = 2, b = 4$ and $c = -3$ **b** $a = 1, b = 10$ and $c = 18$

c $a = 1, b = 10$ and $c = -18$ **d** $a = -1, b = 6$ and $c = 15$

e $a = 1, b = 9$ and $c = -27$

2 Simplify each of the following:

a $\dfrac{2 + 2\sqrt{5}}{2}$ **b** $\dfrac{9 - 3\sqrt{5}}{6}$ **c** $\dfrac{5 + 5\sqrt{5}}{10}$ **d** $\dfrac{6 + 12\sqrt{2}}{6}$

Examples 29, 30

3 Solve each of the following for x. Give exact answers.

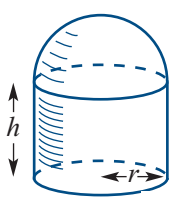
a $x^2 + 6x = 4$ **b** $x^2 - 7x - 3 = 0$ **c** $2x^2 - 5x + 2 = 0$

d $2x^2 + 4x - 7 = 0$ **e** $2x^2 + 8x = 1$ **f** $5x^2 - 10x = 1$

g $-2x^2 + 4x - 1 = 0$ **h** $2x^2 + x = 3$ **i** $2.5x^2 + 3x + 0.3 = 0$

j $-0.6x^2 - 1.3x = 0.1$ **k** $2kx^2 - 4x + k = 0$ **l** $2(1 - k)x^2 - 4kx + k = 0$

4 The surface area, S, of a cylindrical tank with a hemispherical top is given by the formula $S = ar^2 + brh$, where $a = 9.42$ and $b = 6.28$. What is the radius of a tank of height 6 m which has a surface area of 125.6 m^2?



Example 29 **5** Sketch the graphs of the following parabolas. Use the quadratic formula to find the x-axis intercepts (if they exist) and the axis of symmetry and, hence, the turning point.

a $y = x^2 + 5x - 1$ **b** $y = 2x^2 - 3x - 1$ **c** $y = -x^2 - 3x + 1$
d $y + 4 = x^2 + 2x$ **e** $y = 4x^2 + 5x + 1$ **f** $y = -3x^2 + 4x - 2$

3.8 Iteration

Quadratic equations can be solved using a process of **simple iteration**. In this section the process is discussed. Knowledge of sequence notation from General Mathematics is useful but not essential for this topic.

Consider the quadratic equation $x^2 + 3x - 5 = 0$.

This can be rearranged to the equivalent equations

$$x(x + 3) = 5 \quad \text{and} \quad x = \frac{5}{x + 3}$$

Solving the equation $x^2 + 3x - 5 = 0$ is equivalent to solving the simultaneous equations

$$y = x \quad \text{and} \quad y = \frac{5}{x + 3}$$

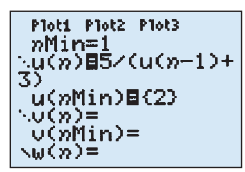
The equation $x = \dfrac{5}{x + 3}$ can be used to form a difference equation (**iterative equation**)

$$x_{n+1} = \frac{5}{x_n + 3} \quad \text{or} \quad x_n = \frac{5}{x_{n-1} + 3}$$

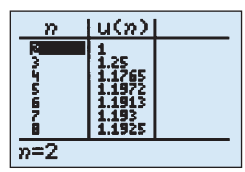
The elements of the sequence are called **iterations**.

Using a graphics calculator

Choose a starting value near the positive solution of $x^2 + 3x - 5 = 0$. Let $x_1 = 2$.



Enter this iterative equation in the graphics calculator as shown. Note that **Seq** has been chosen from the **MODE** menu.



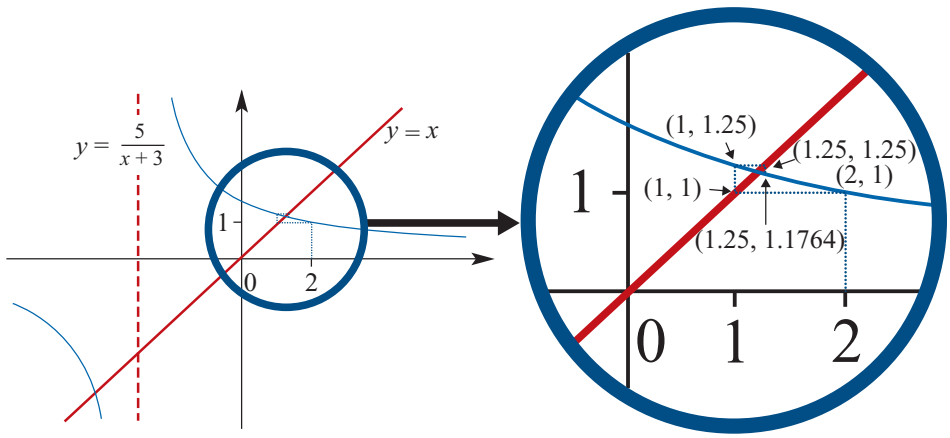
Press 2ND and TABLE to obtain a table of values.

$x_1 = 2, x_2 = 1, x_3 = 1.25, x_4 = 1.1764\ldots,$
$x_{15} = 1.19258241\ldots, x_{16} = 1.19258239\ldots$

The sequence converges to $\dfrac{-3 + \sqrt{29}}{2}$.

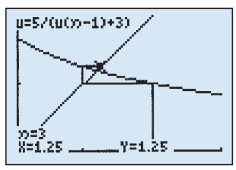
The convergence can be illustrated with a **web** diagram.

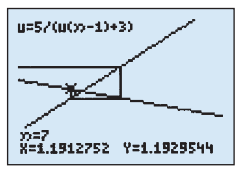
The graphs of $y = \dfrac{5}{x + 3}$ and $y = x$ are sketched on the one set of axes and the 'path' of the sequence is illustrated.



This can be seen on a graphics calculator. The **Seq** mode has been chosen and **Web** selected from the **FORMAT** menu. Set the **WINDOW** with **nMin** $= 1$, **nMax** $= 7$, **Xmin** $= 0$, **Xmax** $= 3$, **Ymin** $= 0$ and **Ymax** $= 2$.

Press TRACE and use the arrow ▶ to move through the sequence. Use **ZBox** from the **ZOOM** menu so that iterations up to $n = 7$ can be seen.





Try other starting points, for example $x_1 = -400$ or $x_1 = 200$.

Convergence is always towards the solution $x = \dfrac{-3 + \sqrt{29}}{2}$.

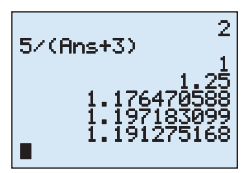
Other arrangements of the equation $x^2 + 3x - 5 = 0$ can be tried to find the other solution. For example:

A $x = \dfrac{-x^2 + 5}{3}$ yields the sequence $x_{n+1} = \dfrac{-(x_n)^2 + 5}{3}$. This converges to $\dfrac{-3 + \sqrt{29}}{2}$ again if $x_1 = 2$ is used for the start. $x_1 = -9$ yields no convergence.

B $x = \dfrac{5}{x} - 3$ yields the sequence $x_{n+1} = \dfrac{5}{x_n} - 3$. This converges to $\dfrac{-3 - \sqrt{29}}{2}$ for the start $x_1 = -9$.

C $x = -\sqrt{5 - 3x}$ yields the sequence $x_{n+1} = -\sqrt{5 - 3x_n}$. This converges to $\dfrac{-3 - \sqrt{29}}{2}$ for the start $x_1 = -10$.

Another method which can be used to generate the sequence is to use **ANS** in the Home screen. Thus for $x_{n+1} = \dfrac{5}{x_n + 3}$ first enter 2 and press ENTER and then enter 5/(ANS + 3). Press ENTER repeatedly to generate the sequence.



Exercise 3H

Solve the following equations by using iteration:

a $x^2 + 5x - 10 = 0$ using $x_{n+1} = \dfrac{10}{x_n + 5}$ **b** $x^2 - 3x - 5 = 0$ using $x_{n+1} = \dfrac{5}{x_n - 3}$

c $x^2 + 2x - 7 = 0$ using $x_{n+1} = \dfrac{7}{x_n + 2}$ **d** $-x^2 - 2x + 5 = 0$ using $x_{n+1} = \dfrac{5}{x_n + 2}$

3.9 The discriminant

Excel

When graphing quadratics it is apparent that the number of x-axis intercepts a parabola may have is:

i zero – the graph is either all above or all below the x-axis

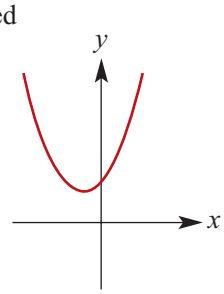
ii one – the graph touches the x-axis; the turning point is the x-axis intercept

or **iii** two – the graph crosses the x-axis.

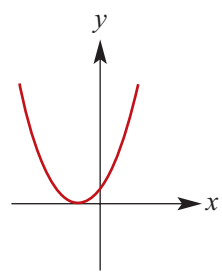
By considering the formula for the general solution to a quadratic equation, $ax^2 + bx + c = 0$, we can establish whether a parabola will have zero, one or two x-axis intercepts.

The expression $b^2 - 4ac$, which is part of the quadratic formula, is called the **discriminant**.

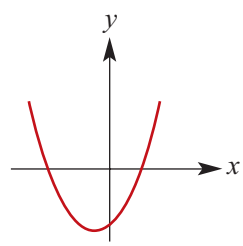
i If the discriminant $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has zero solutions and the corresponding parabola will have no x-axis intercepts.



ii If the discriminant $b^2 - 4ac = 0$, then the equation $ax^2 + bx + c = 0$ has one solution and the corresponding parabola will have one x-axis intercept. (We sometimes say the equation has two coincident solutions.)



iii If the discriminant $b^2 - 4ac > 0$, then the equation $ax^2 + bx + c = 0$ has two solutions and the corresponding parabola will have two x-axis intercepts.



Example 31

Find the discriminant of each of the following quadratics and state whether the graph of each crosses the x-axis, touches the x-axis or does not intersect the x-axis.

a $y = x^2 - 6x + 8$ **b** $y = x^2 - 8x + 16$ **c** $y = 2x^2 - 3x + 4$

Solution

Note: Discriminant is denoted by the symbol $\triangle$.

a Discriminant $\triangle = b^2 - 4ac$

$= (-6)^2 - (4 \times 1 \times 8)$

$= 4$

As $\triangle > 0$ the graph intersects the x-axis at two distinct points, i.e. there are two distinct solutions of the equation $x^2 - 6x + 8 = 0$.

b $\triangle = b^2 - 4ac$

$= (-8)^2 - (4 \times 1 \times 16)$

$= 64 - 64$

$= 0$

As $\triangle = 0$ the graph touches the x-axis, i.e. there is one solution of the equation $x^2 - 8x + 16 = 0$.

c $\triangle = b^2 - 4ac$

$= (-3)^2 - (4 \times 2 \times 4)$

$= -23$

As $\triangle < 0$ the graph does not intersect the x-axis, i.e. there are no real solutions for the equation $2x^2 - 3x + 4 = 0$.

The discriminant can be used to assist in the identification of the particular type of solution for a quadratic equation.

For a, b and c rational:

- If $\triangle = b^2 - 4ac$ is a perfect square, which is not zero, then the quadratic equation has two rational solutions.
- If $\triangle = b^2 - 4ac = 0$ the quadratic equation has one rational solution.
- If $\triangle = b^2 - 4ac$ is not a perfect square then the quadratic equation has two irrational solutions.

Exercise 3I

1 Determine the discriminant of each of the following quadratics:

a $x^2 + 2x - 4$ **b** $x^2 + 2x + 4$ **c** $x^2 + 3x - 4$

d $2x^2 + 3x - 4$ **e** $-2x^2 + 3x + 4$

Example 31

2 Without sketching the graphs of the following quadratics, determine whether they cross or touch the x-axis:

a $y = x^2 - 5x + 2$ **b** $y = -4x^2 + 2x - 1$ **c** $y = x^2 - 6x + 9$
d $y = 8 - 3x - 2x^2$ **e** $y = 3x^2 + 2x + 5$ **f** $y = -x^2 - x - 1$

3 By examining the discriminant, find the number of roots of:

a $x^2 + 8x + 7 = 0$ **b** $3x^2 + 8x + 7 = 0$ **c** $10x^2 - x - 3 = 0$
d $2x^2 + 8x - 7 = 0$ **e** $3x^2 - 8x - 7 = 0$ **f** $10x^2 - x + 3 = 0$

4 By examining the discriminant, state the nature and number of roots for each of the following:

a $9x^2 - 24x + 16 = 0$ **b** $-x^2 - 5x - 6 = 0$ **c** $x^2 - x - 4 = 0$
d $25x^2 - 20x + 4 = 0$ **e** $6x^2 - 3x - 2 = 0$ **f** $x^2 + 3x + 2 = 0$

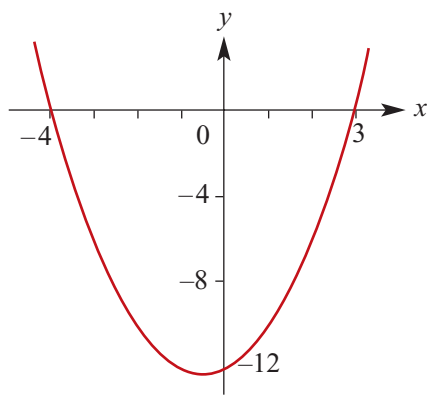
5 Find the discriminant for the equation $4x^2 + (m - 4)x - m = 0$, where m is a rational number and hence show that the equation has rational solution(s).

Exercise 3J contains more exercises involving the discriminant.

3.10 Solving quadratic inequations

Example 32

Solve $x^2 + x - 12 > 0$.



Solution

Step 1 Solve the equation $x^2 + x - 12 = 0$

$(x + 4)(x - 3) = 0$

$\therefore\ x = 3$ or $x = -4$

Step 2 Sketch the graph of the quadratic $y = x^2 + x - 12$.

Step 3 Use the graph to determine the values of x for which $x^2 + x - 12 > 0$.
From the graph it can be seen that $x^2 + x - 12 > 0$ when $x < -4$ or when $x > 3$.
$\therefore \{x : x^2 + x - 12 > 0\} = \{x : x < -4\} \cup \{x : x > 3\}$

Example 33

Find the values of m for which the equation $3x^2 - 2mx + 3 = 0$ has:

a 1 solution **b** no solution **c** 2 distinct solutions

Solution

For the quadratic $3x^2 - 2mx + 3$, $\triangle = 4m^2 - 36$.

a For 1 solution

$$\triangle = 0$$
$$\therefore 4m^2 - 36 = 0$$
$$\therefore m^2 = 9$$
$$m = \pm 3$$

b For no solution

$$\triangle < 0$$
$$\text{i.e.} \quad 4m^2 - 36 < 0$$

From the graph

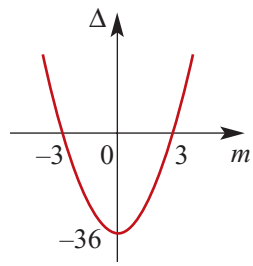
$$-3 < m < 3$$

c For two distinct solutions

$$\triangle > 0$$
$$\text{i.e.} \quad 4m^2 - 36 > 0$$

From the graph it can be seen that

$m > 3$ or $m < -3$



Example 34

Show that the solutions of the equation $3x^2 + (m - 3)x - m$ are rational for all rational values of m.

Solution

$$\begin{aligned}\triangle &= (m-3)^2 - 4 \times 3 \times (-m) \\ &= m^2 - 6m + 9 + 12m \\ &= m^2 + 6m + 9 \\ &= (m+3)^2 \geq 0 \text{ for all } m\end{aligned}$$

Furthermore $\triangle$ is a perfect square for all m.

Exercise 3J

Example 32 **1** Solve each of the following inequalities:

a $x^2 + 2x - 8 \geq 0$ **b** $x^2 - 5x - 24 < 0$ **c** $x^2 - 4x - 12 \leq 0$

d $2x^2 - 3x - 9 > 0$ **e** $6x^2 + 13x < -6$ **f** $-x^2 - 5x - 6 \geq 0$

g $12x^2 + x > 6$ **h** $10x^2 - 11x \leq -3$ **i** $x(x-1) \leq 20$

j $4 + 5p - p^2 \geq 0$ **k** $3 + 2y - y^2 < 0$ **l** $x^2 + 3x \geq -2$

Example 33 **2** Find the values of m for which each of the following equations:

i has no solutions **ii** has one solution **iii** has two distinct solutions

a $x^2 - 4mx + 20 = 0$ **b** $mx^2 - 3mx + 3 = 0$

c $5x^2 - 5mx - m = 0$ **d** $x^2 + 4mx - 4(m-2) = 0$

3 Find the values of p for which the equation $px^2 + 2(p+2)x + p + 7 = 0$ has no real solution.

4 Find the values of p for which the equation $(1-2p)x^2 + 8px - (2+8p) = 0$ has one solution.

5 Find the values of p for which the graph of $y = px^2 + 8x + p - 6$ crosses the x-axis.

6 Show that the equation $(p^2+1)x^2 + 2pqx + q^2 = 0$ has no real solution for any values of p and q $(q \neq 0)$.

Example 34 **7** For m and n rational numbers show that $mx^2 + (2m+n)x + 2n = 0$ has rational solutions.

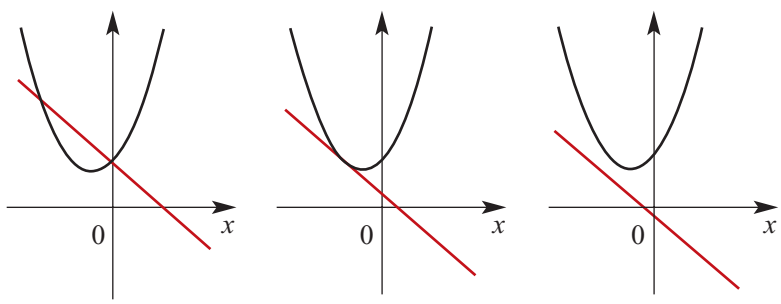
3.11 Solving simultaneous linear and quadratic equations

Excel

As discussed in Section 1.3, when solving simultaneous linear equations we are actually finding the point of intersection of the two linear graphs involved.

If we wish to find the point or points of intersection between a straight line and a parabola we can solve the equations simultaneously.

It should be noted that depending on whether the straight line intersects, touches or does not intersect the parabola we may get two, one or zero points of intersection.



Two points of intersection One point of intersection No point of intersection

If there is one point of intersection between the parabola and the straight line then the line is a **tangent** to the parabola.

As we usually have the quadratic equation written with y as the subject, it is necessary to have the linear equation written with y as the subject (i.e. in gradient form).

Then the linear expression for y can be substituted into the quadratic equation.

CAS APPENDIX

Example 35

Find the points of intersection of the line with the equation $y = -2x + 4$ and the parabola with the equation $y = x^2 - 8x + 12$.

Solution

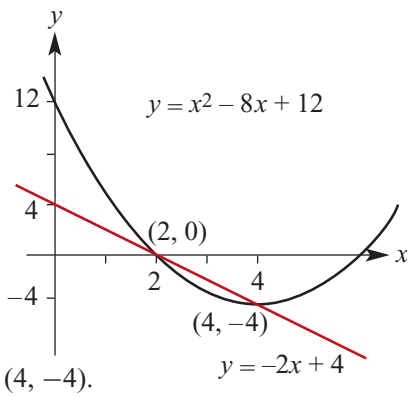
At the point of intersection

$$x^2 - 8x + 12 = -2x + 4$$
$$x^2 - 6x - 8 = 0$$
$$(x - 2)(x - 4) = 0$$

Hence $x = 2$ or $x = 4$

When $x = 2, y = -2(2) + 4 = 0$

$x = 4, y = -2(4) + 4 = -4$



Therefore the points of intersection are (2, 0) and (4, −4).

The result can be shown graphically.

Example 36

Prove that the straight line with the equation $y = 1 - x$ meets the parabola with the equation $y = x^2 - 3x + 2$ once only.

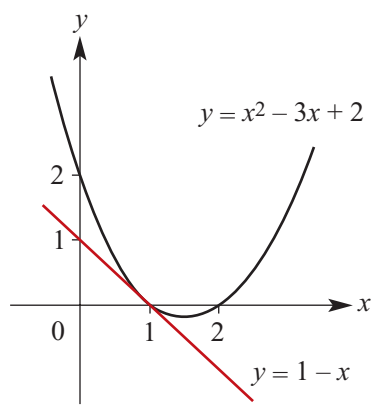
Solution

At the point of intersection:

$$x^2 - 3x + 2 = 1 - x$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)^2 = 0$$

Therefore $x = 1$

When $x = 1, y = 0$

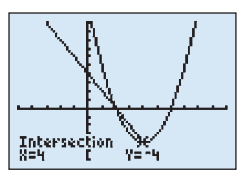


The straight line just touches the parabola.

This can be shown graphically.

Using a graphics calculator

Simultaneous equations involving a linear equation and a quadratic equation can be solved using **5:intersect** from the **CALC** menu. The result for one of the points of intersection of the graphs in Example 35 is shown here.



Exercise 3K

Example 35 **1** Solve each of the following pairs of equations:

a $y = x^2 + 2x - 8$, $y = 2 - x$ **b** $y = x^2 - x - 3$, $y = 4x - 7$ **c** $y = x^2 + x - 5$, $y = -x - 2$

d $y = x^2 + 6x + 6$, $y = 2x + 3$ **e** $y = 6 - x - x^2$, $y = -2x - 2$ **f** $y = x^2 + x + 6$, $y = 6x + 8$

Example 36 **2** Prove that, for the pairs of equations given, the straight line meets the parabola only once:

a $y = x^2 - 6x + 8$, $y = -2x + 4$ **b** $y = x^2 - 2x + 6$, $y = 4x - 3$

c $y = 2x^2 + 11x + 10$, $y = 3x + 2$ **d** $y = x^2 + 7x + 4$, $y = -x - 12$

3 Solve each of the following pairs of equations:

a $y = x^2 - 6x$, $y = 8 + x$ **b** $y = 3x^2 + 9x$, $y = 32 - x$ **c** $y = 5x^2 + 9x$, $y = 12 - 2x$

d $y = -3x^2 + 32x$, $y = 32 - 3x$ **e** $y = 2x^2 - 12$, $y = 3(x - 4)$ **f** $y = 11x^2$, $y = 21 - 6x$

4 **a** Find the value of c such that $y = x + c$ is a tangent to the parabola $y = x^2 - x - 12$. (Hint: Consider the discriminant of the resulting quadratic.)

b **i** Sketch the parabola with equation $y = -2x^2 - 6x + 2$.

ii Find the values of m for which the straight line $y = mx + 6$ is tangent to the parabola. (Hint: Use the discriminant of the resulting quadratic.)

5 **a** Find the value of c such that the line with equation $y = 2x + c$ is tangent to the parabola with equation $y = x^2 + 3x$.

b Find the possible values of c such that the line with equation $y = 2x + c$ twice intersects the parabola with equation $y = x^2 + 3x$.

6 Find the value(s) of a such that the line with equation $y = x$ is tangent to the parabola with equation $y = x^2 + ax + 1$.

7 Find the value of b such that the line with equation $y = -x$ is tangent to the parabola with equation $y = x^2 + x + b$.

8 Find the equation of the straight line(s) which pass through the point $(1, -2)$ and is (are) tangent to the parabola with equation $y = x^2$.

3.12 Determining quadratic rules

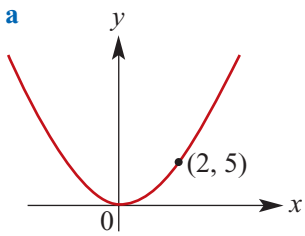
It is possible to find the quadratic rule to fit given points, if it is assumed that the points lie on a suitable parabola.

Example 37

Determine the quadratic rule for each of the following graphs, assuming each is a parabola.

Solution

a



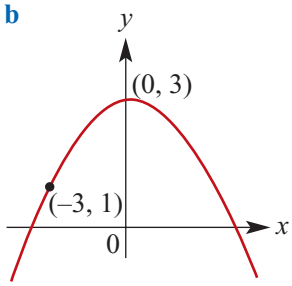
a This is of the form $y = ax^2$

When $x = 2$, $y = 5$, thus $5 = 4a$

Therefore $a = \frac{5}{4}$

and the rule is $y = \frac{5}{4}x^2$

b



b This is of the form $y = ax^2 + c$

For (0, 3) $3 = a(0) + c$

Therefore $c = 3$

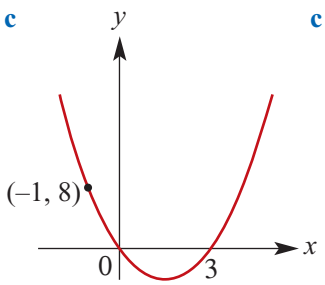
For (−3, 1) $1 = a(-3)^2 + 3$

$1 = 9a + 3$

Therefore $a = -\frac{2}{9}$

and the rule is $y = -\frac{2}{9}x^2 + 3$

c



c This is of the form $y = ax(x - 3)$

For (−1, 8) $8 = -a(-1 - 3)$

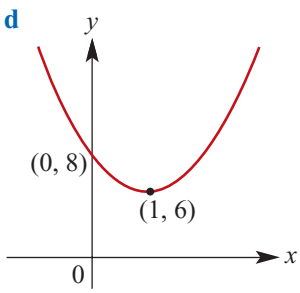
$8 = 4a$

Therefore $a = 2$

and the rule is $y = 2x(x - 3)$

$y = 2x^2 - 6x$

d



d This is of the form $y = k(x - 1)^2 + 6$

When $x = 0$, $y = 8$

$\therefore$ $8 = k + 6$

and $k = 2$

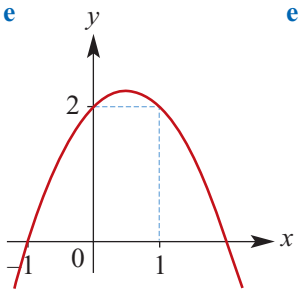
$\therefore$ $y = 2(x - 1)^2 + 6$

and the rule is $y = 2(x^2 - 2x + 1) + 6$

$y = 2x^2 - 4x + 8$

CAS APPENDIX

e



e This is of the form $y = ax^2 + bx + c$

For $(-1, 0)$ $0 = a - b + c$ (1)

For $(0, 2)$ $2 = c$ (2)

For $(1, 2)$ $2 = a + b + c$ (3)

Substitute $c = 2$ in (1) and (3)

$0 = a - b + 2$

$-2 = a - b$ (1a)

$0 = a + b$ (3a)

Substract (3a) from (1a)

$-2 = -2b$

Therefore $b = 1$

Substitute $b = 1$ and $c = 2$ in (1)

Therefore $0 = a - 1 + 2$

$0 = a + 1$

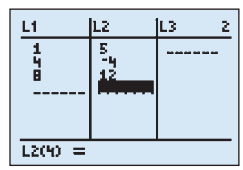
and hence $a = -1$

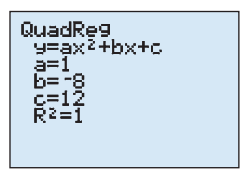
Thus the quadratic rule is $y = -x^2 + x + 2$

Using a graphics calculator

When three arbitrary points are given, **5:QuadReg** from the **CALC** submenu of **STAT** can be used. For example, if it is known that a parabola passes through the points with coordinates (1, 5), (4, −4) and (8, 12), enter these as x- and y-values in lists 1 and 2 respectively. These are accessed by pressing STAT and choosing **1:Edit**. Clear list L1 by taking the cursor over the top of L1 and pressing CLEAR and ENTER.

In the Home screen enter **QuadReg L1, L2** and press ENTER. The result is as shown. The equation of the parabola is $y = x^2 - 8x + 12$.





Skillsheet TEACHER

Exercise 3L

Example 37a

1 A quadratic rule for a particular parabola is of the form $y = ax^2$. The parabola passes through the point with coordinates (2, 8). Find the value of a.

Example 37b

2 A quadratic rule for a particular parabola is of the form $y = ax^2 + c$. The parabola passes through the points with coordinates $(-1, 4)$ and $(0, 8)$. Find the value of a and of c.

Example 37c

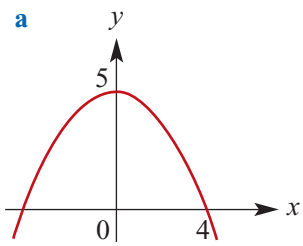
3 A quadratic rule for a particular parabola is of the form $y = ax^2 + bx$. The parabola passes through the points with coordinates $(-1, 4)$ and one of its x-axis intercepts is 6. Find the value of a and of b.

Example 37d

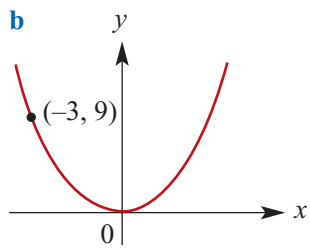
4 A quadratic rule for a particular parabola is of the form $y = a(x - b)^2 + c$. The parabola has vertex $(1, 6)$ and passes through the point with coordinates $(2, 4)$. Find the values of a, b and c.

5 Determine the equation of each of the following parabolas:

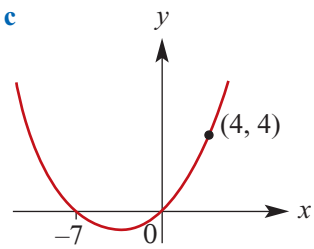
a



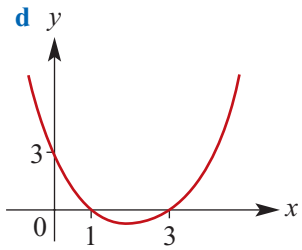
b



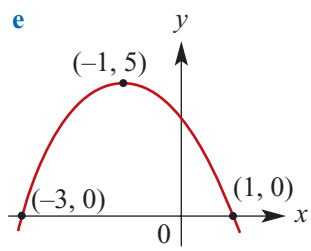
c



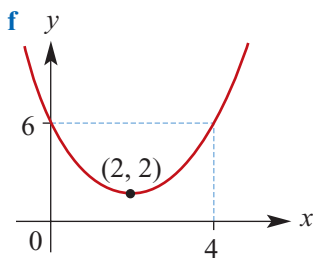
d



e



f

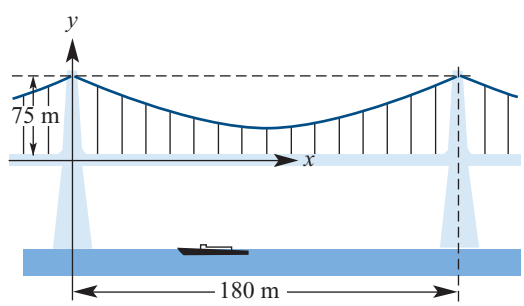


6 A parabola has vertex with coordinates $(-1, 3)$ and passes through the point with coordinates $(3, 8)$. Find the equation for the parabola.

7 A parabola has x-axis intercepts 6 and -3 and passes through the point $(1, 10)$. Find the equation of the parabola.

8 A parabola has vertex with coordinates $(-1, 3)$ and y-axis intercept 4. Find the equation for the parabola.

9 Assuming that the suspension cable shown in the diagram forms a parabola, find the rule which describes its shape. The minimum height of the cable above the roadway is 30 m.

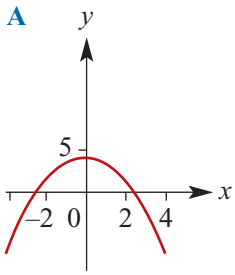


10 Which of the curves could be most nearly defined by each of the following?

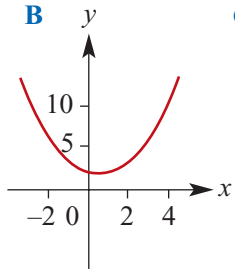
a $y = \frac{1}{3}(x + 4)(8 - x)$ **b** $y = x^2 - x + 2$

c $y = -10 + 2(x - 1)^2$ **d** $y = \frac{1}{2}(9 - x^2)$

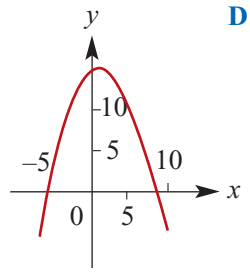
A



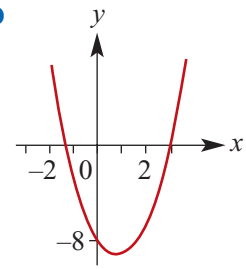
B



C



D

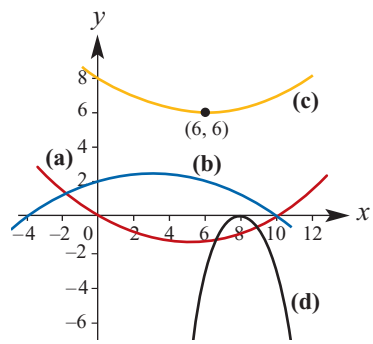


11 A parabola has the same shape as $y = 2x^2$ but its turning point is $(1, -2)$. Write its equation.

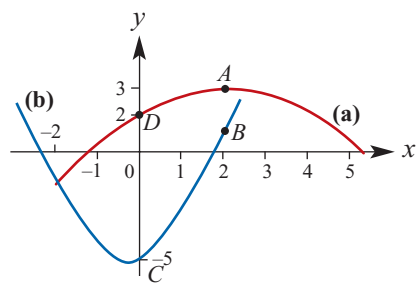
12 A parabola has its vertex at $(1, -2)$ and passes through the point $(3, 2)$. Write its equation.

13 A parabola has its vertex at $(2, 2)$ and passes through $(4, -6)$. Write its equation.

14 Write down four quadratic rules that have graphs similar to those in the diagram.



15 Find quadratic expressions which could represent the two curves in this diagram, given that the coefficient of x is 1 in each case. A is $(2, 3)$, B is $(2, 1)$, C is $(0, -5)$ and D is $(0, 2)$

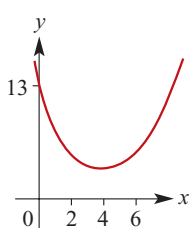


16 The rate of rainfall during a storm t hours after it began was 3 mm per hour when $t = 5$, 6 mm per hour when $t = 9$ and 5 mm per hour when $t = 13$. Assuming that a quadratic model applies, find an expression for the rate, r mm per hour, in terms of t.

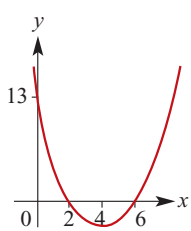
17 **a** Which of the graphs shown below could represent the graph of the equation $y = (x - 4)^2 - 3$?

b Which graph could represent $y = 3 - (x - 4)^2$?

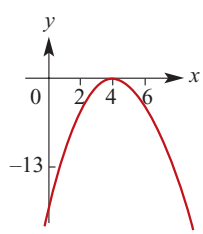
A



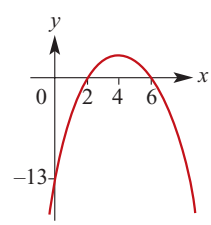
B



C



D

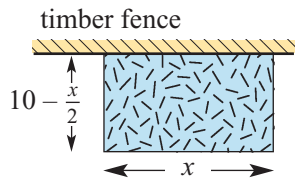


3.13 Quadratic models

Example 38

Jenny wishes to fence off a rectangular vegetable garden in her backyard. She has 20 m of fencing wire which she will use to fence three sides of the garden, with the existing timber fence forming the fourth side. Calculate the maximum area she can enclose.

Solution



Let A = area of the rectangular garden

x = length of the garden

$$\therefore \text{ width} = \frac{20 - x}{2} = 10 - \frac{x}{2}$$

$$\therefore \quad A = x\left(10 - \frac{x}{2}\right)$$

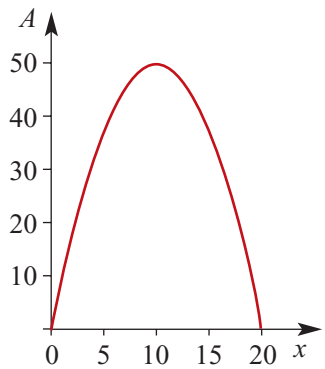
$$= 10x - \frac{x^2}{2}$$

$$= -\frac{1}{2}(x^2 - 20x + 100 - 100)$$

(completing the square)

$$= -\frac{1}{2}(x^2 - 20x + 100) + 50$$

$$= -\frac{1}{2}(x - 10)^2 + 50$$

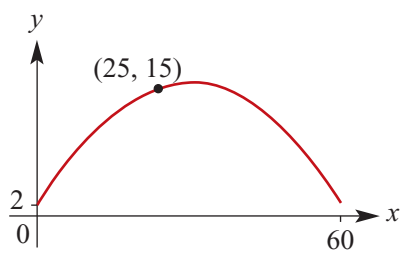


$\therefore$ the maximum area is 50 m^2 when $x = 10$.

The graph of the relation is shown.

Example 39

A cricket ball is thrown by a fielder. It leaves his hand at a height of 2 metres above the ground and the wicketkeeper takes the ball 60 metres away again at a height of 2 metres. It is known that after the ball has gone 25 metres it is 15 metres above the ground. The path of the cricket ball is a parabola with equation $y = ax^2 + bx + c$.



a Find the values of a, b, and c.

b Find the maximum height of the ball above the ground.

c Find the height of the ball 5 metres horizontally before it hits the wicketkeeper's gloves.

Solution

a The data can be used to obtain three equations:

$$\begin{aligned} 2 &= c && (1)\\ 15 &= (25)^2a + 25b + c && (2)\\ 2 &= (60)^2a + 60b + c && (3) \end{aligned}$$

Substitute equation (1) in equations (2) and (3).

$$\begin{aligned} \therefore 13 &= 625a + 25b && (1')\\ 0 &= 3600a + 60b && (2') \end{aligned}$$

Simplify (2′) by dividing both sides by 60.

$$0 = 60a + b \qquad (2')$$

Multiply this by 25 and subtract from equation (1′).

$$\begin{aligned} 13 &= -875a\\ \therefore a &= -\frac{13}{875} \quad \text{and} \quad b = \frac{156}{175}\\ \therefore y &= -\frac{13}{875}x^2 + \frac{156}{175}x + 2 \end{aligned}$$

b The maximum height occurs when $x = 30$ and $y = \frac{538}{35}$.

$\therefore$ maximum height is $\frac{538}{35}$ m.

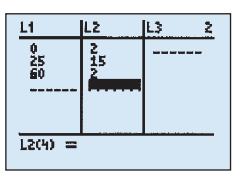
c When $x = 55$, $y = \frac{213}{35}$

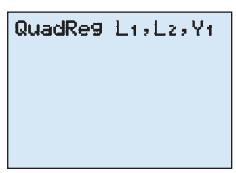
$\therefore$ height of ball is $\frac{213}{35}$ m.

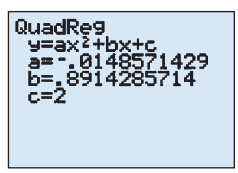
Using a graphics calculator

This equation can also be obtained on a graphics calculator. Enter the data in **L1** and **L2** as shown and then select **QuadReg** from the **CALC** submenu of **STAT**.

Complete in the Home screen as shown. Press ENTER to obtain the values of *a, b,* and *c*.





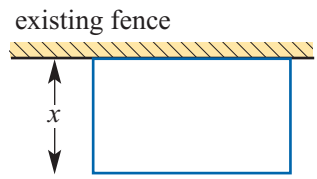


Obtain the graph of the quadratic by pressing GRAPH for a suitably chosen window.

Exercise 3M

Example 38

1 A farmer has 60 m of fencing with which to construct three sides of a rectangular yard connected to an existing fence.

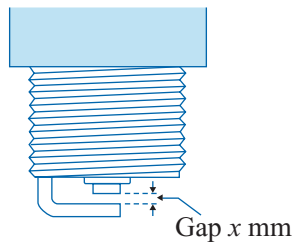


a If the width of paddock is x m and the area inside the yard A m^2, write down the rule connecting A and x.

b Sketch the graph of A against x.

c Determine the maximum area that can be formed for the yard.

2 The efficiency rating, E, of a particular spark plug when the gap is set at x mm is said to be $400(x - x^2)$.



a Sketch the graph of E against x for $0 \leq x \leq 1$.

b What values of x give a zero efficiency rating?

c What value of x gives the maximum efficiency rating?

d Use the graph, or otherwise, to determine the values of x between which the efficiency rating is 70 or more.

3 A piece of wire 68 cm in length is bent into the shape of a rectangle.

a If x cm is the length of the rectangle and A cm^2 is the area enclosed by the rectangular shape, write down a formula which connects A and x.

b Sketch the graph of A against x for suitable x-values.

c Use your graph to determine the maximum area formed.

4 A construction firm has won a contract to build cable-car pylons at various positions on the side of a mountain. Because of difficulties associated with construction in alpine areas, the construction firm will be paid an extra amount, C(\$), given by the formula $C = 240h + 100h^2$, where h is the height in km above sea level.

a Sketch the graph of C as a function of h. Comment on the possible values of h.

b Does C have a maximum value?

c What is the value of C for a pylon built at an altitude of 2500 m?

5 A tug-o-war team produces a tension in a rope described by the rule $T = 290(8t - 0.5t^2 - 1.4)$ units when t is the number of seconds after commencing the pull.

a Sketch a graph of T against t, stating the practical domain.

b What is the greatest tension produced during a 'heave'?

6 A cricketer struck a cricket ball such that its height, d metres, after it had travelled x metres horizontally was given by the rule $d = 1 + \frac{3}{5}x - \frac{1}{50}x^2, x \geq 0$.

a Use a graphics calculator to graph d against x for values of x ranging from 0 to 30.

b **i** What was the maximum height reached by the ball?

ii If a fielder caught the ball when it was 2 m above the ground, how far was the ball from where it was hit?

iii At what height was the ball when it was struck?

7 Find the equation of the quadratic which passes through the points with coordinates:

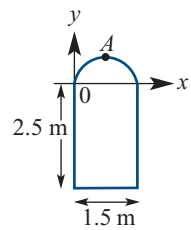
a $(-2, -1), (1, 2), (3, -16)$

b $(-1, -2), (1, -4), (3, 10)$

c $(-3, 5), (3, 20), (5, 57)$

Example 39

8 An arch on the top of a door is parabolic in shape. The point A is 3.1 m above the bottom of the door. The equation $y = ax^2 + bx + c$ can be used to describe the arch. Find the values of a, b, and c.



9 It is known that the daily spending of a government department follows a quadratic model. Let t be the number of days after 1 January and s be the spending in hundreds of thousands of dollars on a particular day, where $s = at^2 + bt + c$.

t	30	150	300
s	7.2	12.5	6

a Find the values of a, b and c.

b Sketch the graph for $0 \leq t \leq 360$. (Use a graphics calculator.)

c Find an estimate for the spending when:

i $t = 180$ **ii** $t = 350$

MC TEST D&D TEST

Chapter summary

- The general expression for a quadratic function is $y = ax^2 + bx + c$.
- **Expansions**

e.g. **i** $2x(x - 3) = 2x^2 - 6x$

ii $(2x - 3)(3x + 4) = 2x(3x + 4) - 3(3x + 4)$
$= 6x^2 + 8x - 9x - 12$
$= 6x^2 - x - 12$

iii $(x + a)^2 = x^2 + 2ax + a^2$

iv $(x - a)(x + a) = x^2 - a^2$

- **Factorising**

Type 1 Removing the highest common factor

e.g. $9x^3 + 27x^2 = 9x^2(x + 3)$

Type 2 Difference of two squares: $x^2 - a^2 = (x - a)(x + a)$

e.g. $16x^2 - 36 = (4x - 6)(4x + 6)$

Type 3 Grouping of terms

e.g. $x^3 + 4x^2 - 3x - 12 = (x^3 + 4x^2) - (3x + 12)$
$= x^2(x + 4) - 3(x + 4)$
$= (x^2 - 3)(x + 4)$

Type 4 Factorising quadratic expressions

e.g. **i** $x^2 + 2x - 8 = (x + 4)(x - 2)$

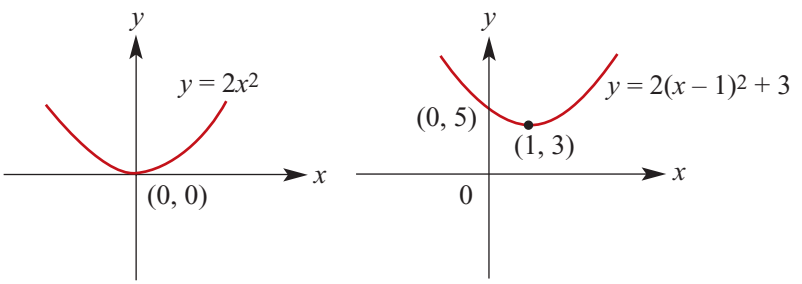
ii $6x^2 - 13x - 15$

$6x^2$	-15	-13
$6x$	$+5$	$+5x$
x	-3	-18
		$-13x$

$6x^2 - 13x - 15 = (6x + 5)(x - 3)$

Review

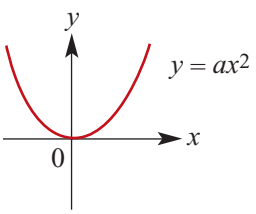
- The graphs of a quadratic may be sketched by first expressing the rule in the form $y = a(x - h)^2 + k$. The graph of a quadratic of this form is obtained by translating the graph of $y = ax^2$ h units in the positive direction of the x-axis and k units in the positive direction of the y-axis (h, k positive).
  e.g. for $y = 2(x - 1)^2 + 3$



- A quadratic equation may be solved by:
  - **i** Factorising
    e.g. $2x^2 + 5x - 12 = 0$
    $(2x - 3)(x + 4) = 0 \quad \therefore x = \frac{3}{2}$ or -4
  - **ii** Completing the square
    e.g. $x^2 + 2x - 4 = 0$
    Add and subtract $\left(\frac{b}{2}\right)^2$ to 'complete the square'
    $x^2 + 2x + 1 - 1 - 4 = 0$
    $\therefore (x + 1)^2 - 5 = 0$
    $\therefore (x + 1)^2 = 5$
    $\therefore x + 1 = \pm\sqrt{5} \ \therefore x = -1 \pm \sqrt{5}$
  - **iii** Using the general quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
    e.g. $-3x^2 - 12x - 7 = 0$
    $x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)}$
    $= \frac{6 \pm \sqrt{15}}{-3}$

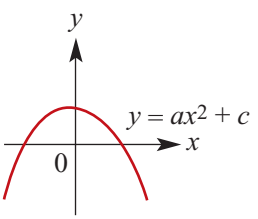
- The following can be used for sketching the graph of the quadratic relation $y = ax^2 + bx + c$:
  - **i** If $a > 0$, the function has a minimum value.
  - **ii** If $a < 0$, the function has a maximum value.
  - **iii** The value of c gives the y-axis intercept.
  - **iv** The equation of the axis of symmetry is $x = -\frac{b}{2a}$.
  - **v** The x-axis intercepts are determined by solving the equation $ax^2 + bx + c = 0$.
- From the general solution for the quadratic equation $ax^2 + bx + c = 0$, the number of x-axis intercepts can be determined:
  - **i** If $b^2 - 4ac > 0$, the equation has two distinct real roots a and b.
  - **ii** If $b^2 - 4ac = 0$, there is one root, $-\frac{b}{2a}$.
  - **iii** If $b^2 - 4ac < 0$, the equation has no real roots.
- To find a quadratic rule to fit given points:

i



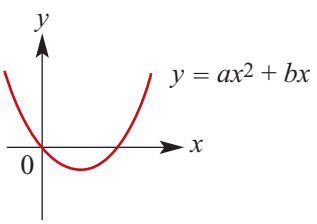
a can be calculated if one point is known.

ii



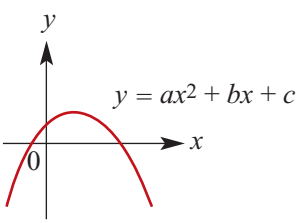
Two points are needed to determine a and c.

iii



Two points required to determine a and b.

iv



Three points required to determine a, b and c.

Multiple-choice questions

1 The linear factors of $12x^2 + 7x - 12$ are

A $4x - 3$ and $3x + 4$ **B** $3x - 4$ and $4x + 3$ **C** $3x - 2$ and $4x + 6$
D $3x + 2$ and $4x - 6$ **E** $6x + 4$ and $2x - 3$

2 The solutions of the equation $x^2 - 5x - 14 = 0$ are

A $x = -7$ only **B** $x = -7, x = 2$ **C** $x = -2, x = 7$ **D** $x = -2, x = -7$
E $x = -2$ only

Review

3 For $y = 8 + 2x - x^2$, the maximum value of y is

A $-3\frac{1}{4}$ **B** $5\frac{1}{4}$ **C** 9 **D** $9\frac{1}{2}$ **E** 10

4 If the graph of $y = 2x^2 - kx + 3$ touches the x-axis then the possible values of k are

A $k = 2$ or $k = -3$ **B** $k = 1$ **C** $k = -3$ or $k = -\frac{1}{2}$

D $k = 1$ or $k = 3$ **E** $k = 2\sqrt{6}$ or $k = -2\sqrt{6}$

5 The solutions of the equation $x^2 - 56 = x$ are

A $x = -8$ or 7 **B** $x = -7$ or 8 **C** $x = 7$ or 8 **D** $x = -9$ or 6

E $x = 9$ or -6

6 The value of the discriminant of $x^2 + 3x - 10$ is

A 5 **B** -5 **C** 49 **D** 7 **E** -2

7 The coordinates of the turning point of the graph with equation $y = 3x^2 + 6x - 1$ are

A $\left(\frac{1}{3}, -2\right)$ **B** $\left(-\frac{1}{3}, 2\right)$ **C** $\left(-\frac{1}{3}, -4\right)$ **D** $(1, -4)$ **E** $(-1, -4)$

8 The equation $5x^2 - 10x - 2$ in turning point form $a(x - h)^2 + k$, by completing the square, is

A $(5x + 1)^2 + 5$ **B** $(5x - 1)^2 - 5$ **C** $5(x - 1)^2 - 5$

D $5(x + 1)^2 - 2$ **E** $5(x - 1)^2 - 7$

9 The value(s) of m that will give the equation $mx^2 + 6x - 3 = 0$ two real roots is (are)

A $m = -3$ **B** $m = 3$ **C** $m = 0$ **D** $m > -3$ **E** $m < -3$

10 $6x^2 - 8xy - 8y^2$ is equal to

A $(3x + 2y)(2x - 4y)$ **B** $(3x - 2y)(6x + 4y)$ **C** $(6x - 4y)(x + 2y)$

D $(3x - 2y)(2x + 4y)$ **E** $(6x + y)(x - 8y)$

Short-answer questions (technology-free)

1 Express each of the following in the form $(ax + b)^2$:

a $x^2 + 9x + \frac{81}{4}$ **b** $x^2 + 18x + 81$ **c** $x^2 - \frac{4}{5}x + \frac{4}{25}$

d $x^2 + 2bx + b^2$ **e** $9x^2 - 6x + 1$ **f** $25x^2 + 20x + 4$

2 Expand each of the following products:

a $-3(x - 2)$ **b** $-a(x - a)$

c $(7a - b)(7a + b)$ **d** $(x + 3)(x - 4)$

e $(2x + 3)(x - 4)$ **f** $(x + y)(x - y)$

g $(a - b)(a^2 + ab + b^2)$ **h** $(2x + 2y)(3x + y)$

i $(3a + 1)(a - 2)$ **j** $(x + y)^2 - (x - y)^2$

k $u(v + 2) + 2v(1 - u)$ **l** $(3x + 2)(x - 4) + (4 - x)(6x - 1)$

3 Express each of the following as a product of factors:

a $4x - 8$ **b** $3x^2 + 8x$ **c** $24ax - 3x$

d $4 - x^2$ **e** $au + 2av + 3aw$ **f** $4a^2b^2 - 9a^4$

g $1 - 36x^2a^2$ **h** $x^2 + x - 12$ **i** $x^2 + x - 2$

j $2x^2 + 3x - 2$ **k** $6x^2 + 7x + 2$ **l** $3x^2 - 8x - 3$

m $3x^2 + x - 2$ **n** $6a^2 - a - 2$ **o** $6x^2 - 7x + 2$

4 Sketch the graphs of each of the following:

a $y = 2x^2 + 3$ **b** $y = -2x^2 + 3$ **c** $y = 2(x - 2)^2 + 3$

d $y = 2(x + 2)^2 + 3$ **e** $y = 2(x - 4)^2 - 3$ **f** $y = 9 - 4x^2$

g $y = 3(x - 2)^2$ **h** $y = 2(2 - x)^2 + 3$

5 Express in the form $y = a(x - h)^2 + k$ and hence sketch the graphs of the following:

a $y = x^2 - 4x - 5$ **b** $y = x^2 - 6x$ **c** $y = x^2 - 8x + 4$

d $y = 2x^2 + 8x - 4$ **e** $y = -3x^2 - 12x + 9$ **f** $y = -x^2 + 4x + 5$

6 Find:

i the x- and y-intercepts **ii** the axis of the symmetry **iii** the turning point

and hence sketch the graphs of each of the following:

a $y = x^2 - 7x + 6$ **b** $y = -x^2 - x + 12$ **c** $y = -x^2 + 5x + 14$

d $y = x^2 - 10x + 16$ **e** $y = 2x^2 + x - 15$ **f** $y = 6x^2 - 13x - 5$

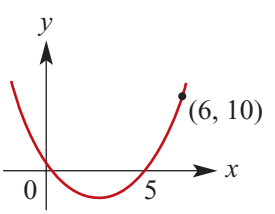
g $y = 9x^2 - 16$ **h** $y = 4x^2 - 25$

7 Use the quadratic formula to solve each of the following:

a $x^2 + 6x + 3 = 0$ **b** $x^2 + 9x + 12 = 0$ **c** $x^2 - 4x + 2 = 0$

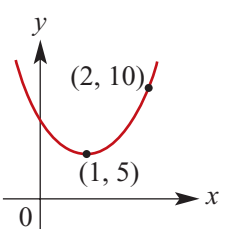
d $2x^2 + 7x + 2 = 0$ **e** $2x^2 + 7x + 4 = 0$ **f** $3x^2 + 9x - 1 = 0$

8 Find the equation of the quadratic, the graph of which is shown.



9 A parabola has the same shape as $y = 3x^2$ but its vertex is at (5, 2). Find the equation corresponding to this parabola.

10 Find the rule of the quadratic relation which describes the graph.



11 Find the coordinates of the point of intersection of the graphs with equations:

a $y = 2x + 3$ and $y = x^2$ **b** $y = 8x + 11$ and $y = 2x^2$

c $y = 3x^2 + 7x$ and $y = 2$ **d** $y = 2x^2$ and $y = 2 - 3x$

Review

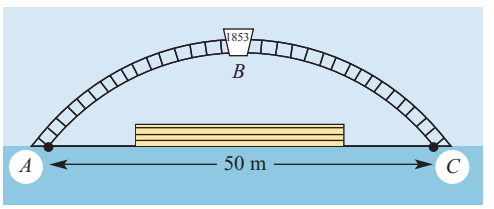
12 **a** For what value(s) of m does the equation $2x^2 + mx + 1 = 0$ have exactly one solution.

b For what values of m does the equation $x^2 - 4mx + 20 = 0$ have real solutions.

c Show that there are real solutions of the equation $4mx^2 + 4(m - 1)x + m - 2 = 0$ for all real x.

Extended-response questions

1 The diagram shows a masonry arch bridge of span 50 m. The shape of the curve, ABC, is a parabola. AC is the water level.



a Taking A as the origin and the maximum height of the arch above the water level as 4.5 m, write down a formula for the curve of the arch where y is the height of the arch above AC and x is the horizontal distance from A.

b Calculate a table of values and accurately plot the graph of the curve.

c At what horizontal distance from A is the height of the arch above the water level equal to 3 m?

d What is the height of the arch at a horizontal distance from A of 12 m?

e A floating platform 20 m wide is towed under the bridge. What is the greatest height of the deck above water level if the platform is to be towed under the bridge with at least 30 cm horizontal clearance on either side?

2 A piece of wire 12 cm long is cut into two pieces. One piece is used to form a square shape and the other a rectangular shape in which the length is twice its width.

a If x is the side length of the square, write down the dimensions of the rectangle in terms of x.

b Formulate a rule for A, the combined area of the square and rectangle in cm^2, in terms of x.

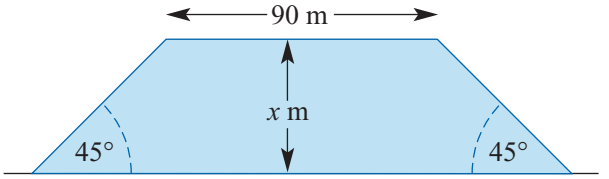
c Determine the lengths of the two parts if the sum of the areas is to be a minimum.

3 Water is pumped into an empty metal tank at a steady rate of 0.2 litres/min. After 1 hour the depth of water in the tank is 5 cm; after 5 hours the depth is 10 cm.

a If the volume of water in the tank is V litres when the depth is x cm and there is a quadratic relationship between V and x, write down a rule which connects V and x.

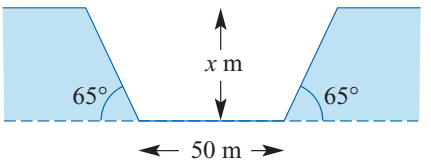
b It is known that the maximum possible depth of water in the tank is 20 cm. For how long, from the beginning, can water be pumped into the tank at the same rate without overflowing?

4 The figure shows a section view of a freeway embankment to be built across a flood-prone river flat. The height of the embankment is x m and width at the top is 90 m.



a Find a formula, in terms of x, for V, the volume of earth in m^3 required to build a 120 m length of freeway embankment.

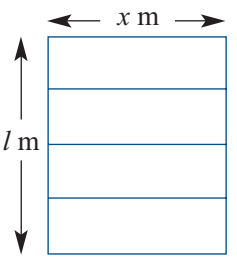
This figure shows another section of the freeway which is to be constructed by cutting through a hillside. The depth of the cutting is x m and the width of the cutting at the base is 50 m.



b Find a formula for the volume of earth, in m^3, which would have to be excavated to form a straight 100 m section of the cutting.

c If $x = 4$ m, what length of embankment could be constructed from earth taken from the cutting?

5 100 m of angle steel is used to make a rectangular frame with three crossbars as shown in the figure.



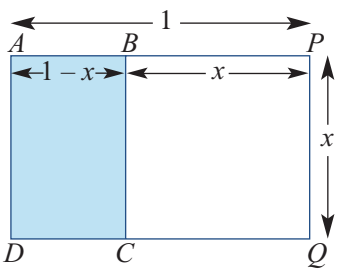
a If the width of the frame is x m, determine an expression for l, the length of the frame in metres, in terms of x.

b The frame is to be covered by light aluminium sheeting. If the area of this sheeting is A m^2, formulate a rule connecting A and x.

c Sketch a graph of A against x, stating the axes intercepts and the turning point.

d What is the maximum area and the value of x which gives this area?

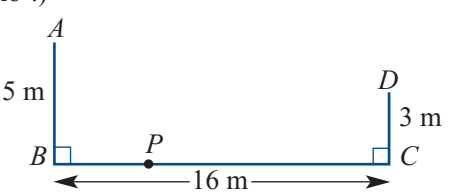
6 A shape which has been of interest to architects and artists over the centuries is the 'golden rectangle'. Many have thought that it gave the perfect proportions for buildings. The rectangle is such that if a square is drawn on one of the longer sides then the new rectangle is similar to the original.



Let the length of $AP = 1$ unit, then $AB = 1 - x$ units and $\frac{AP}{AD} = \frac{AD}{AB}$.

Find the value of x. (x is known as the 'golden ratio'.)

7 P is a point on line BC x m from B.



a Find distance PA in terms of x.

b **i** Find distance PC in terms of x.

ii Find distance PD in terms of x.

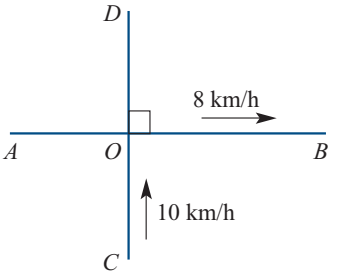
c Find x if $PA = PD$.

d Find x if $PA = 2PD$. (Answer correct to 3 decimal places.)

e Find x if $PA = 3PD$. (Answer correct to 3 decimal places.)

Review

8 AB and CD are crossroads. A jogger runs along road AB at a speed of 8 km/hour and passes O at 1.00 pm. Another runner is moving along road CD. The second runner is moving at 10 km/hour and passes O at 1.30 pm.



a Let y km be their distance apart t hours after 1.00 pm.

i Find an expression for y in terms of t.

ii Plot the graph of y against t on a graphics calculator.

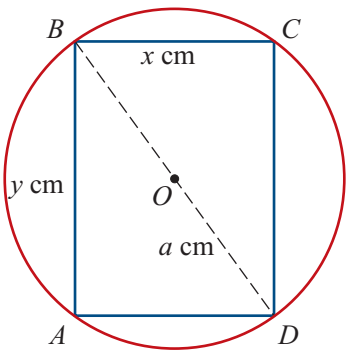
iii Find the time(s) when the runners are 4 km apart. (Use a graphics calculator.)

iv Find the time at which the runners are closest and their distance apart at this time.

b Find the exact value(s) of t for which:

i $y = 5$ **ii** $y = 6$

9 A rectangle of perimeter b cm is inscribed in a circle of radius a cm. The rectangle has width x cm and length y cm.



a Apply Pythagoras' theorem in triangle BCD to show that $x^2 + y^2 = 4a^2$.

b Form a second equation involving x, y and b.

c Eliminate y from these equations to form a quadratic equation in terms of x, a and b.

d As x, y and $2a$ are the sides of a triangle, $x + y > 2a$. Use this result and apply the discriminant to the quadratic equation formed in part **c** to show that the rectangle can be inscribed in the circle only if $4a < b \leq 4\sqrt{2}a$.

e **i** If $a = 5$ and $b = 24$, find the value(s) of x and y.

ii If $b = 4\sqrt{2}a$, find the value of x and y in terms of a.

f If $\frac{b}{a} = 5$, find the values of x and y in terms of a.

g Write a program to solve the quadratic equation found in part c for suitable choices of a and b and state the values of x and y. (Answers correct to 2 decimal places.)

10 The equation of curve B is $y = -6 + 4x - x^2$.

a $(h, 3)$ is the vertex of a parabola A, with equation $y = x^2 + bx + c$. Find the values of b, c and h for $h > 0$.

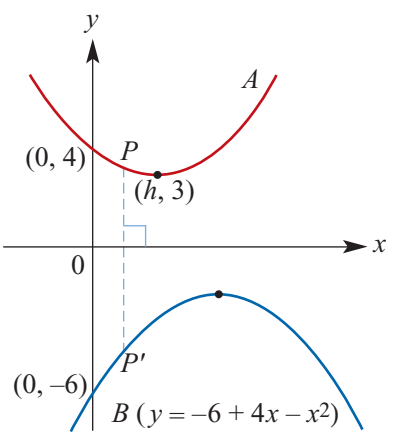
b Let P be a point on curve A, and P' be a point on curve B such that PP' is perpendicular to the x-axis.

i The coordinates of P are $(x, x^2 + bx + c)$. State the coordinates of P' in terms of x.

ii Find the coordinates of M, the midpoint of PP' in terms of x.

iii Find the coordinates of M for $x = 0, 1, 2, 3, 4$.

iv Give the equation of the straight line on which all of these points lie. (This is called the locus of the midpoints.)



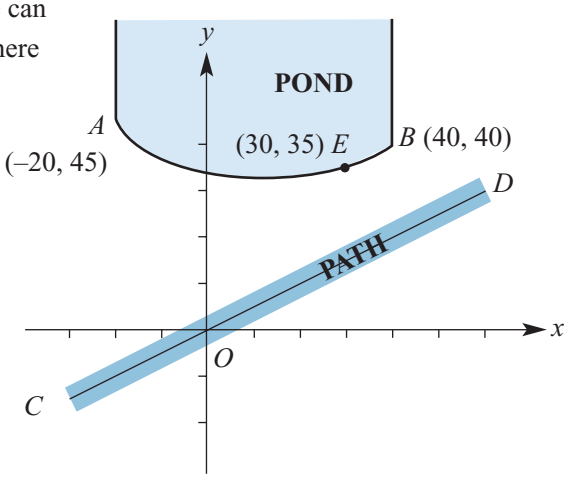
c Let d be the distance PP'.

i Express d in terms of x.

ii Sketch the graph of d against x.

iii Find the minimum value of d and the value of x for which this occurs.

11 A path cuts across a park. Its centreline can be described by the equation $y = \frac{x}{2}$, where the origin is at a point O in the park.



The path starts at a point $C(-30, -15)$ and finishes at a point $D(60, 30)$.

a How long is the path?

One boundary of the pond in the park is parabolic in shape. The boundary passes through the points $A(-20, 45)$, $B(40, 40)$ and $E(30, 35)$. The equation of the parabola is of the form $y = ax^2 + bx + c$.

b **i** Find the equation of the parabola.

ii Find the coordinates of the vertex of the parabola.

c On the one set of axes sketch the graphs of $y = \frac{x}{2}$ and the parabola. (Use a graphics calculator to help.)

d Consider the rule $y = (ax^2 + bx + c) - \frac{1}{2}x$, where a, b and c have been determined in part b i.

i What does this expression determine?

ii Find the minimum value of this expression and the value of x for which this occurs.

CHAPTER

4

A Gallery of Graphs

Objectives

- To recognise the rules of a number of common algebraic relationships: $y = x^{-1}$, $y = x^{-2}$, $y = x^{\frac{1}{2}}$ and $x^2 + y^2 = 1$.
- To be able to sketch the graphs and simple transformations of these relations.

In the previous chapter quadratics written in 'turning point form' were considered. The technique of sketching their graphs through employing transformations of the graph of the basic quadratic $y = x^2$ was introduced.

In this chapter some other common algebraic relations will be considered. Methods similar to those developed in Chapter 3 will be used to sketch the graphs of these relations.

First, it is necessary to be able to recognise the basic rule of each type of relation and its corresponding graph.

In this chapter use of a graphics calculator is recommended to obtain the initial graphs.

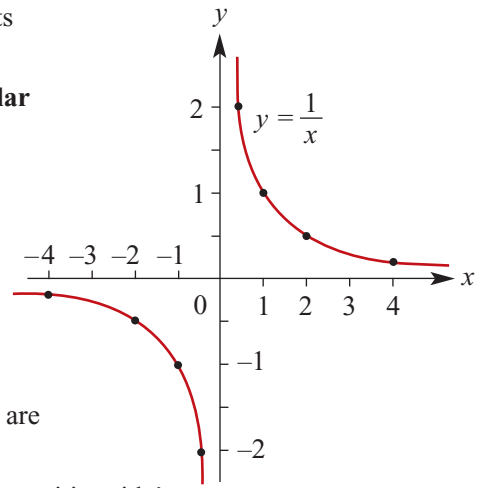
4.1 Rectangular hyperbolas

Consider the rule $y = \frac{1}{x} = x^{-1}$, for $x \neq 0$.

We can construct a table of values for $y = \frac{1}{x}$ for values of x between -4 and 4 as follows.

x	-4	-3	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

We can plot these points and then connect the dots to produce a continuous curve.



A graph of this type is an example of a **rectangular hyperbola**.

It should be noted that when $x = 0$, y is undefined and there is no x-value that will produce the value $y = 0$.

As x approaches infinity in either direction, the value of y approaches zero. The following notation will be used to state this.

As $x \to \infty$, $y \to 0^+$. As $x \to -\infty$, $y \to 0^-$. These are read:

'As x approaches infinity, y approaches 0 from the positive side' and 'as x approaches negative infinity, y approaches 0 from the negative side'.

As x approaches zero from either direction, the magnitude of y becomes very large. The following notation will be used to state this.

As $x \to 0^+$, $y \to \infty$ and as $x \to 0^-$, $y \to -\infty$. These are read:

'As x approaches zero from the positive side, y approaches infinity' and 'as x approaches zero from the negative side, y approaches negative infinity.'

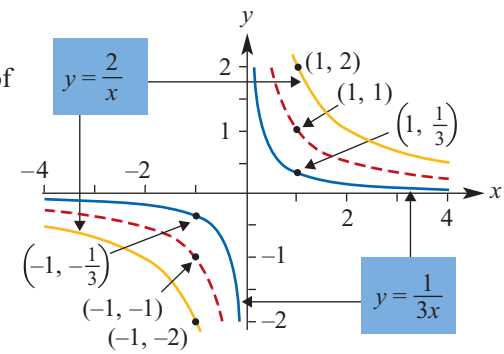
The graph approaches both the x-axis (the line $y = 0$) and the y-axis (the line $x = 0$) but does not cross either of these lines.

We refer to these lines as **asymptotes**.

Hence, for the graph of $y = \frac{1}{x}$, the equations of the asymptotes are $y = 0$ and $x = 0$.

In the diagram on the right, the graphs of $y = \frac{1}{x}$, $y = \frac{2}{x}$ and $y = \frac{1}{3x}$ are shown.

The asymptotes are the x-axis and the y-axis, and they have equations $y = 0$ and $x = 0$ respectively.



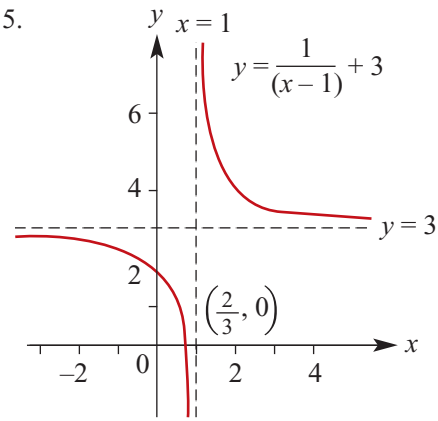
As can be seen from the diagram, the graphs of $y = \frac{2}{x}$ and $y = \frac{1}{3x}$ have the same 'shape' and asymptotes as $y = \frac{1}{x}$ but have been 'stretched' (or **dilated** from the x-axis). It is said that the **dilation** from the x-axis, which takes the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{2}{x}$, is by factor 2 and the point with coordinates (1, 1) on the graph of $y = \frac{1}{x}$ is taken to the point with coordinates (1, 2) on the graph of $y = \frac{2}{x}$.

Dilations will be considered formally in the Chapter 5.

Now let us consider the graph of $y = \dfrac{1}{(x-1)} + 3$.

The basic graph of $y = \dfrac{1}{x}$ has been translated 1 unit to the right and 3 units up. The equation of the vertical asymptote is now $x = 1$ and the equation of the horizontal asymptote is now $y = 3$.

The graph now has x-axis and y-axis intercepts. These can be calculated in the usual way to give further detail to the graph.



x-axis intercept: let $y = 0$

$$0 = \frac{1}{x-1} + 3$$
$$-3 = \frac{1}{x-1}$$
$$-3(x-1) = 1$$
$$x = \frac{2}{3}$$

$\therefore$ intercept is $\left(\dfrac{2}{3}, 0\right)$

y-axis intercept: let $x = 0$

$$y = \frac{1}{(0-1)} + 3$$
$$y = 2$$

$\therefore$ intercept is $(0, 2)$

Using the above technique we are therefore able to sketch graphs of all rectangular hyperbolas of the form $y = \dfrac{a}{x-h} + k$.

Example 1

Sketch the graph of $y = \dfrac{2}{x+1} - 3$.

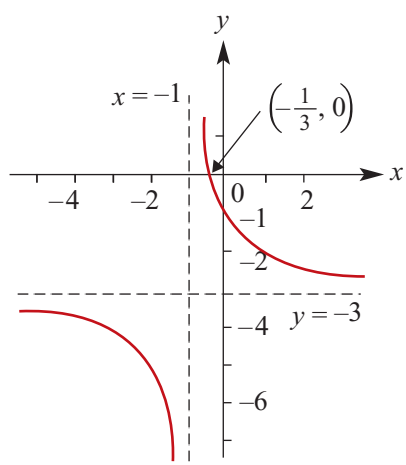
Solution

The graph of $\dfrac{2}{x}$ has been translated 1 unit to the left and 3 units down.

Asymptotes have equations $x = -1$ and $y = -3$.

y-axis intercept: $(0, -1)$

x-axis intercept: $\left(-\dfrac{1}{3}, 0\right)$



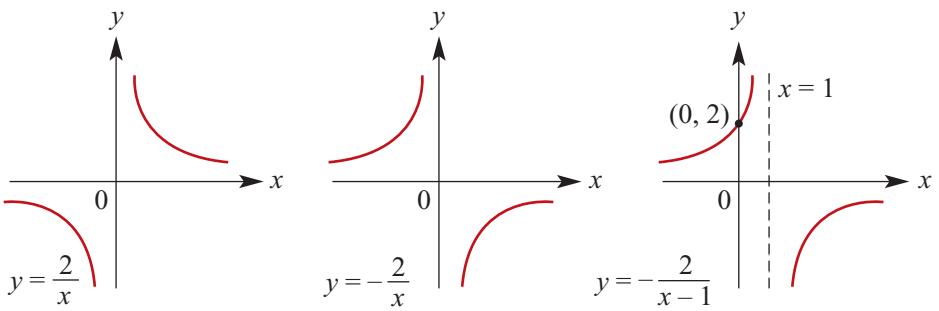
Example 2

Sketch the graph of $y = \dfrac{-2}{x-1}$.

Solution

The graph of $y = -\dfrac{2}{x}$ is obtained from the graph of $y = \dfrac{2}{x}$ by reflection in the x-axis.

This graph is then translated 1 unit to the right to obtain the graph of $y = \dfrac{-2}{x-1}$.



Exercise 4A

1 Sketch the graphs of the following, showing all important features of the graphs:

a $y = \dfrac{1}{x}$ **b** $y = \dfrac{2}{x}$ **c** $y = \dfrac{1}{2x}$ **d** $y = \dfrac{-3}{x}$

e $y = \dfrac{1}{x} + 2$ **f** $y = \dfrac{1}{x} - 3$ **g** $y = \dfrac{2}{x} - 4$ **h** $y = \dfrac{-1}{2x} + 5$

i $y = \dfrac{1}{x-1}$ **j** $y = \dfrac{-1}{x+2}$ **k** $y = \dfrac{1}{x+1} + 3$ **l** $y = \dfrac{-2}{x-3} - 4$

2 Write down the equations of the asymptotes for each of the graphs in Question 1.

4.2 The truncus

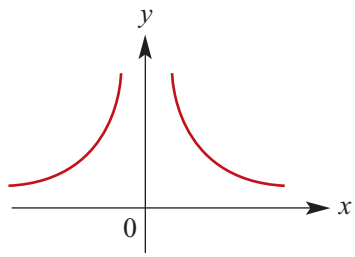
Consider the rule $y = \dfrac{1}{x^2} = x^{-2}$. Constructing a table of values for $y = \dfrac{1}{x^2}$ for values of x between -4 and 4 we have as follows:

x	-4	-3	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{9}$	$\frac{1}{4}$	1	4	4	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$

We can plot these points and then connect the dots to produce a continuous curve.

A graph of this shape is called a **truncus**.

Note that when $x = 0$, y is undefined, and that there is no x-value that will produce a negative value of y.



Note: As $x \to \infty$, $y \to 0^+$. As $x \to -\infty$, $y \to 0^+$.

As $x \to 0^+$, $y \to \infty$ and, as $x \to 0^-$, $y \to \infty$.

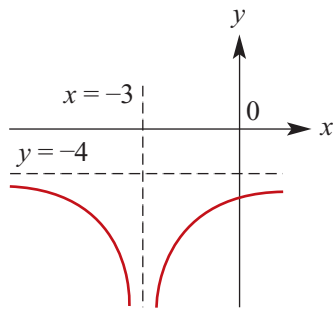
The graph of $y = \dfrac{1}{x^2}$, has asymptotes $y = 0$ and $x = 0$.

All graphs of the form $y = \dfrac{a}{(x-h)^2} + k$ will have the same basic 'truncus' shape.

Asymptotes will be the lines with equations $y = k$ and $x = h$.

Example 3

Sketch the graph of $y = \dfrac{-1}{2(x+3)^2} - 4$.



Solution

The graph of $y = \dfrac{-1}{2x^2}$ is translated 3 units to the left and 4 units down.

Exercise 4B

1 Sketch the graphs of the following, showing all important features:

a $y = \dfrac{1}{(x+3)^2}$ **b** $y = \dfrac{1}{x^2} - 4$ **c** $y = \dfrac{-1}{(x-2)^2}$

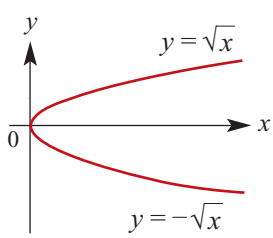
d $y = \dfrac{1}{(x-1)^2} + 3$ **e** $y = \dfrac{1}{2(x+3)^2} - 4$ **f** $y = \dfrac{-2}{(x-2)^2} + 1$

g $y = \dfrac{3}{(x+3)^2} - 6$ **h** $y = \dfrac{-1}{(x-4)^2} + 2$

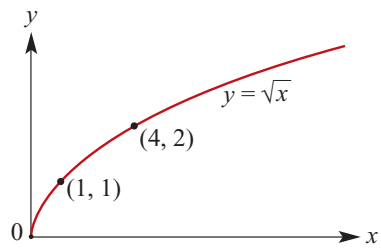
2 Write down the equations of the asymptotes of each of the graphs in Question 1.

4.3 $y = \sqrt{x} = x^{1/2}$

The rule $y = \sqrt{x} = x^{1/2}$ for $x \geq 0$ corresponds to the graph shown opposite. It is one arm of the parabola $x = y^2$.



Coordinates of points on the graph of $y = \sqrt{x}$ include (0, 0), (1, 1), (4, 2) and (9, 3).



All graphs of the form $y = a\sqrt{x - h} + k$ will have the same basic shape as the graph of $y = \sqrt{x}$.

The rule $y = \sqrt{-x}$ for $x \leq 0$ yields a graph which is the reflection of the graph of $y = \sqrt{x}$ in the y-axis. All graphs of the form $y = a\sqrt{-(x - h)} + k$ will have the same basic shape as the graph of $y = \sqrt{-x}$.

Example 4

Sketch the graph of $y = -2\sqrt{x - 1} + 3$.

Solution

The graph is formed by dilating the graph of $y = \sqrt{x}$ by factor 2 from the x-axis, reflecting this in the x-axis and translating it 1 unit to the right and 3 units up.

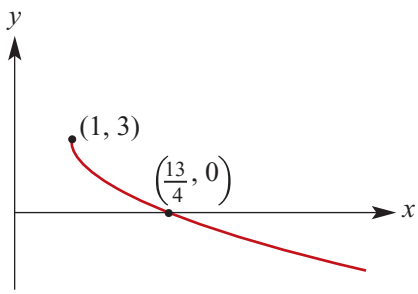
When $y = 0$, $-2\sqrt{x - 1} + 3 = 0$.

Therefore $2\sqrt{x - 1} = 3$.

Square both sides to give $4(x - 1) = 9$

Therefore $x = \frac{9}{4} + 1 = \frac{13}{4}$.

The rule is defined for $x \geq 1$ and the values the rule can take (the range) is all numbers less than or equal to 3, i.e. $y \leq 3$.



Example 5

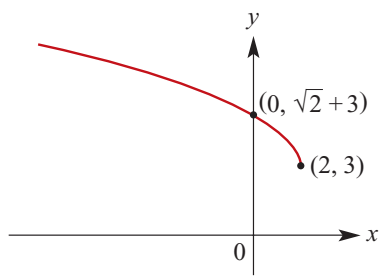
Sketch the graph of $y = \sqrt{-(x - 2)} + 3$.

Solution

When $x = 0$, $y = \sqrt{-(0 - 2)} + 3$

$= \sqrt{2} + 3$

The rule is defined for $x \leq 2$ and the values the rule can take (the range) is all numbers greater than or equal to 3, i.e. $y \geq 3$.



Exercise 4C

For each of the following rules, sketch the corresponding graph, giving the axes intercepts when they exist, the set of x-values for which the rule is defined and the set of y-values which the rule takes:

a $y = 2\sqrt{x} + 3$ **b** $y = \sqrt{x-2} + 3$ **c** $y = \sqrt{x-2} - 3$

d $y = \sqrt{x+2} + 1$ **e** $y = -\sqrt{x+2} + 3$ **f** $y = 2\sqrt{x+2} - 3$

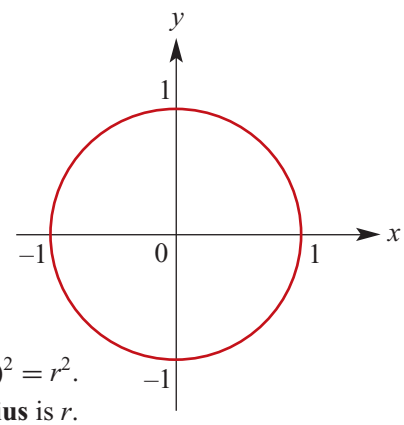
g $y = -\sqrt{x-2} + 3$ **h** $y = \sqrt{-(x-4)} - 2$ **i** $y = -2\sqrt{-(x+4)} - 1$

4.4 Circles

Excel

The equation $x^2 + y^2 = r^2$ defines a circle whose centre is the origin (0, 0), with radius r.

To the right is the graph of the circle with equation $x^2 + y^2 = 1$.



All circles can be considered as being transformations of this basic graph.

Changing the value of r obviously enlarges or reduces the circle.

As has been seen with other graphs, the basic graph may be translated horizontally and/or vertically.

The general equation for a circle is $(x-h)^2 + (y-k)^2 = r^2$.

The **centre** of the circle is the point (h, k) and the **radius** is r.

If the radius and coordinates of the centre of the circle are given, the equation of the circle can be determined.

For example, if the radius is 2 and the centre is the point $(-3, 5)$ then the equation will be

$$(x-(-3))^2 + (y-5)^2 = 4$$
$$(x+3)^2 + (y-5)^2 = 4$$

If the equation of the circle is given, the radius and the centre of the circle can be determined and the graph sketched.

Example 6

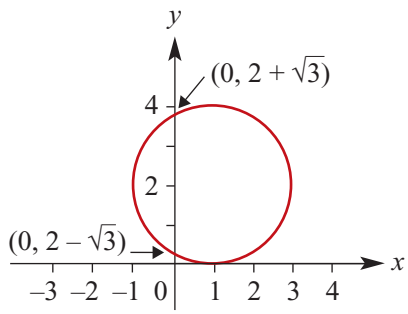
Sketch the graph of $(x-1)^2 + (y-2)^2 = 4$.

Solution

The equation $(x-1)^2+(y-2)^2=4$ defines a circle of radius 2, with centre at the point (1, 2).

When $x=0$, $1+(y-2)^2=4$.

Therefore $(y-2)^2=3$ and $y=2\pm\sqrt{3}$.

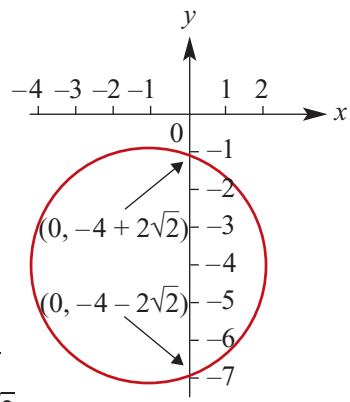


Example 7

Sketch the graph of the circle $(x+1)^2+(y+4)^2=9$.

Solution

The circle has a radius of 3, and its centre at $(-1,-4)$. y-axis intercepts can be found in the usual way.



When $x=0$, $\quad 1+(y+4)^2=9$

i.e. $(y+4)^2=8$

$y=-4\pm\sqrt{8}$

$=-4\pm2\sqrt{2}$

The equation of a circle may not always be written in the form $(x-h)^2+(y-k)^2=r^2$.

Expanding the general equation of a circle gives

$$(x-h)^2+(y-k)^2=r^2$$
$$x^2-2hx+h^2+y^2-2ky+k^2=r^2$$
$$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$$

Let $h^2+k^2-r^2=c$, then an alternative form of the circle equation is achieved:

$$x^2+y^2-2hx-2ky+c=0$$

Notice that in the above form of the circle equation the coefficients of x^2 and y^2 are both 1 and there is no xy term.

In order to sketch a circle with equation expressed in this form, the equation can be converted to the 'centre–radius' form by completing the square for both x and y.

Example 8

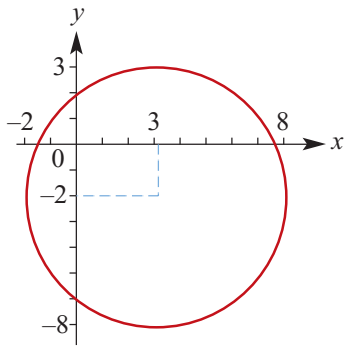
Find the radius and the coordinates of the centre of the circle with equation $x^2+y^2-6x+4y-12=0$ and hence sketch the graph.

Solution

By completing the square for both x and y we have

$$x^2 + y^2 - 6x + 4y - 12 = 0$$
$$(x^2 - 6x + 9) - 9 + (y^2 + 4y + 4) - 4 - 12 = 0$$
$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 25$$
$$(x - 3)^2 + (y + 2)^2 = 5^2$$

$\therefore$ radius $= 5$, centre is at $(3, -2)$.



Semicircles

Transposing the general equation of the circle $x^2 + y^2 = r^2$ to make y the subject, we have

$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

We can now consider two separate rules, $y = +\sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$, which represent the top half and bottom half of the circle respectively.

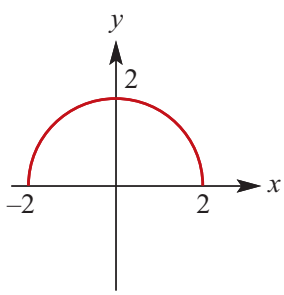
Example 9

Sketch the graphs of:

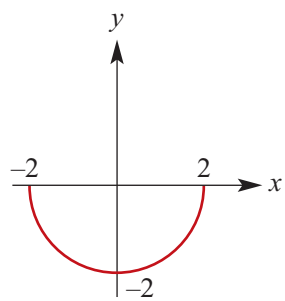
a $y = +\sqrt{4 - x^2}$ **b** $y = -\sqrt{4 - x^2}$

Solution

a

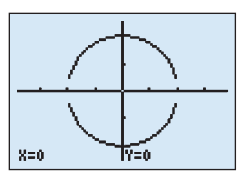


b

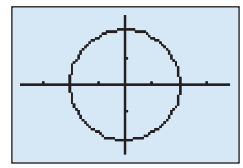


Using a graphics calculator

A graphics calculator can be used to draw circles in two ways. For the circle $x^2 + y^2 = 4$, two semicircles can be used. Hence enter Y1 $= \sqrt{4 - x^2}$ and Y2 $= -\sqrt{4 - x^2}$.



Circles can also be drawn through the DRAW menu. Select **9:Circle** and complete in the Home screen as Circle(0, 0, 2). The result is as shown. The first two entries are the coordinates of the centre and the third is the radius.



Exercise 4D

Example 6

1 Write down the equations of the following circles with centres at $C(h, k)$ and radius r.

a $C(0, 0)$, $r = 3$ **b** $C(0, 0)$, $r = 4$ **c** $C(1, 3)$, $r = 5$

d $C(2, -4)$, $r = 3$ **e** $C(-3, 4)$, $r = \frac{5}{2}$ **f** $C(-5, -6)$, $r = 4.6$

2 Find the centre, C, and the radius, r, of the following circles:

Example 7

a $(x - 1)^2 + (y - 3)^2 = 4$ **b** $(x - 2)^2 + (y + 4)^2 = 5$

c $(x + 3)^2 + (y - 2)^2 = 9$ **d** $x^2 + y^2 - 6y - 16 = 0$

Example 8

e $(x + 3)^2 + (y + 2)^2 = 36$ **f** $x^2 + y^2 - 6x + 4y + 9 = 0$

g $x^2 + y^2 + 4x - 6y - 12 = 0$ **h** $x^2 + y^2 - 8x + 4y + 1 = 0$

3 Sketch the graphs of each of the following:

a $x^2 + y^2 = 64$ **b** $x^2 + (y - 4)^2 = 9$

c $(x + 2)^2 + y^2 = 25$ **d** $(x + 1)^2 + (y - 4)^2 - 169 = 0$

e $(2x - 3)^2 + (2y - 5)^2 = 36$ **f** $x^2 + y^2 - 6y - 16 = 0$

g $x^2 + y^2 + 4x - 6y - 3 = 0$ **h** $x^2 + y^2 - 8x + 22y + 27 = 0$

Example 9

i $y = +\sqrt{9 - x^2}$ **j** $y = -\sqrt{25 - x^2}$

k $y = \sqrt{36 - (x - 2)^2}$ **l** $y - 2 = \sqrt{4 - (x + 2)^2}$

4 Find the equation of the circle with centre $(2, -3)$ which touches the x-axis.

5 Find the equation of the circle whose centre is at the point $(2, 1)$ and which passes through the point $(4, -3)$.

6 Find the equation of the circle whose centre lies on the line $y = 4$ and which passes through the points $(2, 0)$ and $(6, 0)$.

7 The equation to a circle is $x^2 + y^2 + 4x - 6y = 23$. Find the centre and radius.

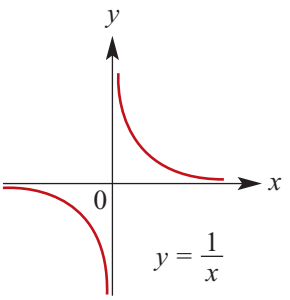
8 Find the length cut off on the x-axis and y-axis by the circle $x^2 + y^2 - 2x - 4y = 20$.

D&D TEST MC TEST

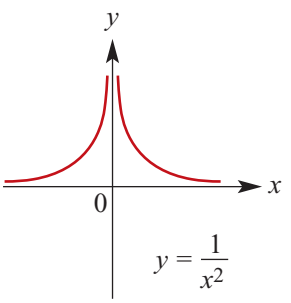
Review

Chapter summary

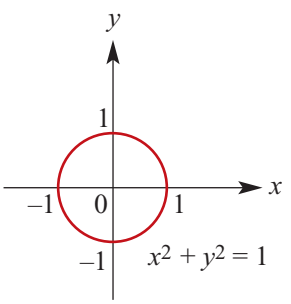
- The standard graphs:



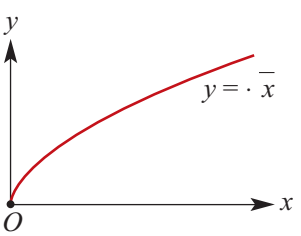
Rectangular hyperbola



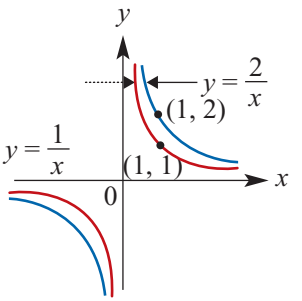
Truncus

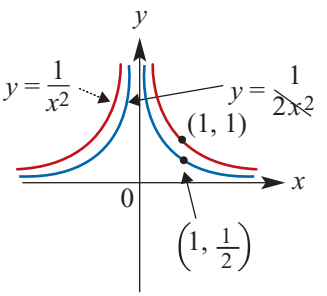


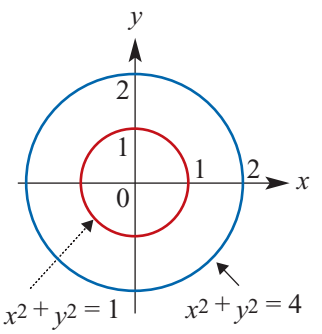
Circle

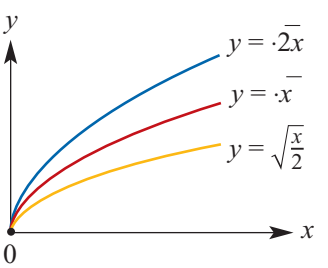


- Dilations of these graphs:

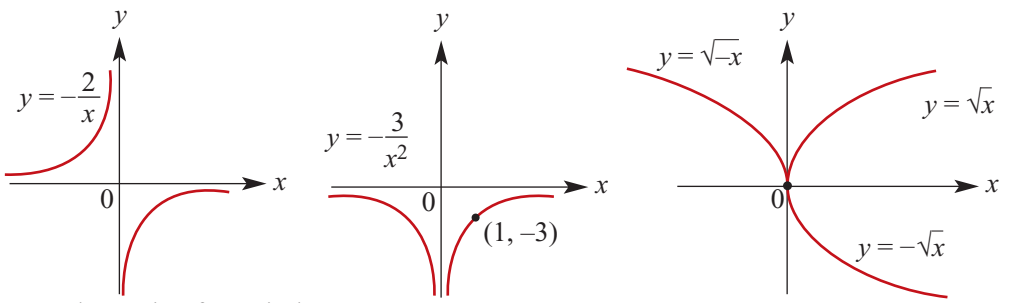








- Reflections in the axes:



- General equation for a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

The centre is at (h, k) and the radius is r.

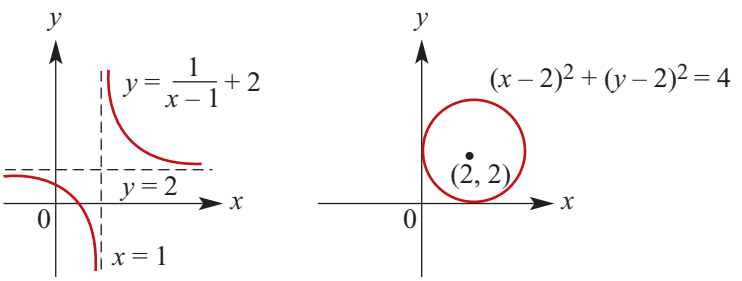
Alternative form:

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

where $h^2 + k^2 - r^2 = c$.

- Translations of graphs:

Examples



Multiple-choice questions

1 The circle with equation $(x - a)^2 + (y - b)^2 = 36$ has its centre on the x-axis and passes through the point with coordinates (6, 6). The values of a and b are

A $a = 0$ and $b = 6$ **B** $a = 0$ and $b = 0$ **C** $a = 2$ and $b = 0$

D $a = -6$ and $b = 0$ **E** $a = 6$ and $b = 0$

2 The equations of the asymptotes of the graph of $y = 5 - \frac{1}{3x - 5}$ are

A $x = 5, y = \frac{3}{5}$ **B** $y = 5, x = \frac{5}{3}$ **C** $x = 5, y = \frac{5}{3}$

D $y = 5, x = \frac{3}{5}$ **E** $x = 5, y = -\frac{5}{3}$

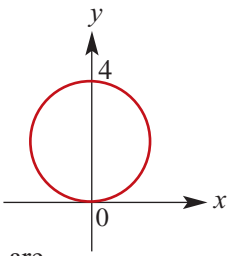
Review

3 For the rule $y = \frac{5}{x^2} + 3$, when $x = \frac{a}{2}$, $y =$

A $\frac{5}{4a^2} + 3$ **B** $\frac{12a^2 + 5}{4a^2}$ **C** $\frac{20a^2 + 3}{4a^2}$

D $\frac{12a^2 + 5}{a^2}$ **E** $\frac{20}{a^2} + 3$

4 If the y-axis is an axis of symmetry and the circle passes through the origin and (0, 4), the equation of the circle shown is



A $x^2 + (y - 2)^2 = 4$ **B** $(x - 2)^2 + y^2 = 2$

C $(x + 2)^2 + y^2 = 4$ **D** $x^2 + (y - 2)^2 = 4$

E $x^2 + y^2 - 2 = 4$

5 The equations of the asymptotes of the graph of $y = 5 + \frac{1}{(x - 2)^2}$ are

A $x = 2, y = 5$ **B** $x = -2, y = 5$ **C** $x = 5, y = 4$

D $x = 5, y = 2$ **E** $x = 4, y = 5$

6 The coordinates of the centre and the radius of the circle with the equation $(x - 5)^2 + (y + 2)^2 = 9$ are

A $(-5, 2)$ and 3 **B** $(-5, 2)$ and 9 **C** $(5, -2)$ and 9

D $(5, -2)$ and 3 **E** $(-2, 5)$ and 3

7 For $y = -2\sqrt{x} + 3$ and $x \geq 0$, y will take the following values

A $[3, \infty)$ **B** $(-3, \infty)$ **C** $[-3, \infty)$ **D** $(-\infty, 3]$ **E** $(3, \infty)$

8 The equation of the circle which has a diameter with end points at $(-2, 8)$ and $(6, 8)$ is

A $(x + 2)^2 + (y - 8)^2 = 16$ **B** $(x + 2)^2 + (y + 8)^2 = 64$

C $(x - 2)^2 + (y - 8)^2 = 16$ **D** $(x - 2)^2 + (y + 8)^2 = 4$

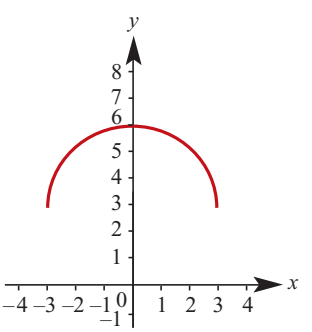
E $(x - 2)^2 + (y - 8)^2 = 16$

9 Which of the following is the equation for the graph of a circle?

A $y = 16 - x^2$ **B** $y^2 = 16 + x^2$ **C** $x + y^2 = 16$

D $x^2 - y^2 = 16$ **E** $y^2 = 16 - x^2$

10 The equation of the semicircle shown is



A $x^2 + (y - 3)^2 = 9$

B $y = \sqrt{9 - x^2} + 3$

C $y = \sqrt{9 + x^2} - 3$

D $y = -\sqrt{9 - x^2} - 3$

E $y = -\sqrt{9 - x^2} + 3$

Short-answer questions (technology-free)

1 Sketch the graphs of each of the following:

a $y = \dfrac{-3}{x}$ **b** $y = \dfrac{2}{x^2}$ **c** $y = \dfrac{1}{x-1}$ **d** $y = \dfrac{2}{x+1} + 1$

e $y = \dfrac{-2}{x^2}$ **f** $y = \dfrac{-1}{x-1}$ **g** $y = \dfrac{4}{2-x} + 3$ **h** $y = \dfrac{-3}{x^2} + 1$

i $y = 2\sqrt{x} + 2$ **j** $y = 2\sqrt{x-3} + 2$ **k** $y = -2\sqrt{x+2} + 2$

2 By completing the square, write each of the following in the form $(x-a)^2 + (y-b)^2 = r^2$:

a $x^2 + y^2 - 6x + 4y - 12 = 0$ **b** $x^2 + y^2 - 3x + 5y - 4 = 0$

c $2x^2 + 2y^2 - x + y - 4 = 0$ **d** $x^2 + y^2 + 4x - 6y = 0$

e $x^2 + y^2 = 6(x + y)$ **f** $x^2 + y^2 = 4x - 6y$

3 Find the equation of the diameter of the circle $x^2 + y^2 - 4x + 6y = 14$ which passes through the origin.

4 Find the equation of the diameter of the circle $x^2 + y^2 - 3x + 2y = 26$ which cuts the x-axis at an angle of $45°$.

5 Find the equation to the circle with centre C and radius r for each of the following and sketch the graph:

a $C(3, 4), r = 5$ **b** $C(-1, 0), r = 1$

c $C(4, 4), r = 2$ **d** $C\left(\dfrac{1}{2}, \dfrac{-1}{3}\right), r = \dfrac{1}{6}$

6 Sketch the graphs of the following semicircles:

a $y = \sqrt{9 - x^2}$ **b** $y = -\sqrt{16 - (x+1)^2}$

c $y - 2 = -\sqrt{1 - x^2}$ **d** $y + 3 = \sqrt{4 - (x+2)^2}$

Extended-response questions

The following questions also involve techniques developed in chapters 2 and 3.

1 The line with equation $y = mx$ is tangent to the circle with centre $(10, 0)$ and radius 5 at the point $P(x, y)$.

a Find the equation of the circle.

b Show that the x-coordinate of the point P satisfies the equation $(1 + m^2)x^2 - 20x + 75 = 0$.

c Use the discriminant for this equation to find the exact value of m.

d Find the coordinates of P. (There are two such points.)

e Find the distance of P from the origin.

Review

2 A circle has its centre at the origin and radius 4.

a Find the equation of this circle.

b Two lines which pass through the point (8, 0) are tangents to this circle.

i Show that the equations of these tangents are of the form $y = mx - 8m$.

ii Use techniques similar to those used in Question 1 to find the value of m and, hence, the equations of the tangents.

3 A circle has centre at the origin and radius 5. The point $P(3, 4)$ lies on the circle.

a Find the gradient of the line segment OP.

b Find the gradient of the tangent to the circle at P.

c Find the equation of the tangent at P.

d If the tangent crosses the x-axis at A and the y-axis at B, find the length of line segment AB.

4 Let $P(x_1, y_1)$ be a point on the circle with equation $x^2 + y^2 = a^2$.

a **i** Give the gradient of the line segment OP, where O is the origin.

ii Give the gradient of the tangent to the circle at P.

b Show that the equation of the tangent at $P(x_1, y_1)$ is $x_1x + y_1y = a^2$.

c If $x_1 = y_1$ and $a = 4$, find the equations of the possible tangents.

5 The line with equation $x = -a$ is the equation of the side BC of an equilateral triangle ABC circumscribing the circle with equation $x^2 + y^2 = a^2$.

a Find the equations of AB and AC.

b Find the equation of the circle circumscribing triangle ABC.

6 Consider the curve with equation $y = \sqrt{x - b} + c$.

a Show that if the curve meets the line with equation $y = x$ at the point (a, a) then a satisfies the equation $a^2 - (2c + 1)a + c^2 + b = 0$.

b **i** If the line is a tangent to the curve, show that $c = \dfrac{4b - 1}{4}$.

ii Sketch the graph of $y = \sqrt{x} - \frac{1}{4}$ and find the coordinates of the point on the graph at which the line with equation $y = x$ is a tangent.

c Find the values of k for which $y = x + k$:

i meets the curve with equation $y = \sqrt{x} - \frac{1}{4}$ twice

ii meets the curve with equation $y = \sqrt{x} - \frac{1}{4}$ once

iii does not meet the curve with equation $y = \sqrt{x} - \frac{1}{4}$.

7 For the curve with equation $y = \sqrt{x} - 1$ and the straight line with equation $y = kx$, find the values of k such that:

a the line meets the curve twice

b the line meets the curve once.

CHAPTER

5

Functions, Relations and Transformations

Objectives

- To understand and use the **notation of sets**, including the symbols $\in$, $\subseteq$, $\cap$, $\cup$, $\varnothing$ and $\setminus$.
- To use the notation for **sets of numbers**.
- To understand the concept of **relation**.
- To understand the terms **domain** and **range**.
- To understand the concept of **function**.
- To understand the term **one-to-one**.
- To understand the terms **implied (maximal) domain, restriction of a function** and **hybrid function**.
- To be able to find the inverse of a one-to-one function.
- To define dilations from the axes, reflections in the axes and translations.
- To be able to apply transformations to graphs of relations.
- To apply a knowledge of functions to **solving problems**.

Sections 5.1 and 5.2 of this chapter introduce the notation that will be used throughout the rest of the book. You will have met much of it before and this will serve as revision. The language introduced in this chapter helps to express important mathematical ideas precisely. Initially they may seem unnecessarily abstract, but later in the book you will find them used more and more in practical situations.

5.1 Set notation and sets of numbers

Set notation

Set notation is used widely in mathematics and in this book where appropriate. This section summarises all of the set notation you will need.

By a **set** we mean a collection of objects. The objects that are in the set are known as **elements** or members of the set. If x is an element of a set A we write $x \in A$. This can also be read as 'x is a member of the set of A' or 'x belongs to A' or 'x is in A'.

The notation $x \notin A$ means x is *not* an element of A. For example:

$$2 \notin \text{set of odd numbers}$$

Set B is called a subset of set A, *if* and *only if*

$$x \in B \text{ implies } x \in A$$

To indicate that B is a subset of A, we write $B \subseteq A$.

This expression can also be read as 'B is contained in A' or 'A contains B'.

The set of elements common to two sets A and B is called the **intersection** of A and B and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

If the sets A and B have no elements in common, we say A and B are **disjoint**, and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.

The **union** of sets A and B, written $A \cup B$, is the set of elements that are either in A or in B. This does not exclude objects that are elements of both A and B.

Example 1

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \cap B$ **b** $A \cup B$

Solution

a $A \cap B = \{3, 7\}$ **b** $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

In Example 1, $3 \in A$ and $5 \notin A$ and $\{2, 3\} \subseteq A$.

Finally we introduce the **set difference** of two sets A and B:

$$A \setminus B = \{x : x \in A, x \notin B\}$$

$A \setminus B$ is the set of elements of A that are not elements of B. For sets A and B in Example 1, $A \setminus B = \{1, 2\}$ and $B \setminus A = \{4, 5, 6\}$.

Sets of numbers

We begin by recalling that the elements of $\{1, 2, 3, 4, \ldots\}$ are called the **natural numbers** and the elements of $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are called **integers**.

The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rationals are called **irrational** (e.g. π and $\sqrt{2}$).

The rationals may be characterised by the property that each rational number may be written as a terminating or recurring decimal.

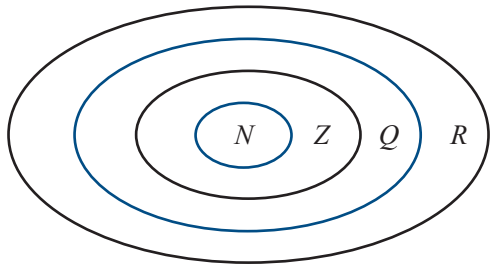
The set of real numbers will be denoted by R.

The set of rational numbers will be denoted by Q.

The set of integers will be denoted by Z.

The set of natural numbers will be denoted by N.

It is clear that $N \subseteq Z \subseteq Q \subseteq R$, and this may be represented by this diagram.



The set of all x such that $(\ldots)$ is denoted by $\{x: (\ldots)\}$.

Thus $\{x: 0 < x < 1\}$ is the set of all real numbers between 0 and 1.

$\{x: x > 0, x \text{ rational}\}$ is the set of all positive rational numbers.

$\{2n: n = 0, 1, 2, \ldots\}$ is the set of all even numbers.

Among the most important subsets of R are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers and that $a < b$:

$$(a, b) = \{x: a < x < b\} \qquad [a, b] = \{x: a \leq x \leq b\}$$
$$(a, b] = \{x: a < x \leq b\} \qquad [a, b) = \{x: a \leq x < b\}$$
$$(a, \infty) = \{x: a < x\} \qquad [a, \infty) = \{x: a \leq x\}$$
$$(-\infty, b) = \{x: x < b\} \qquad (-\infty, b] = \{x: x \leq b\}$$

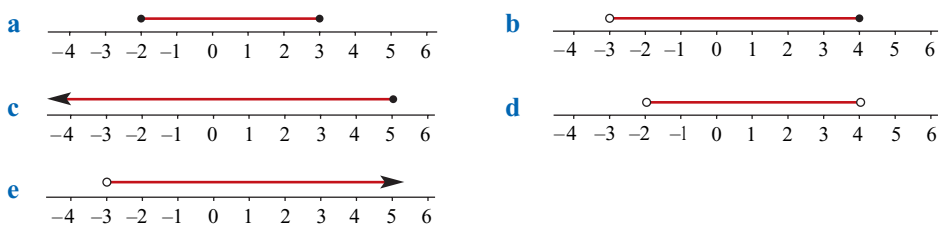
Intervals may be represented by diagrams as shown in Example 2.

Example 2

Illustrate each of the following intervals of real numbers:

a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 5]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

Solution



The 'closed' circle (●) indicates that the number is included.

The 'open' circle (○) indicates that the number is not included.

The following are subsets of the real numbers for which we have special notations:

$R^+ = \{x: x > 0\}$

$R^- = \{x: x < 0\}$

$R \setminus \{0\}$ is the set of real numbers excluding 0.

$Z^+ = \{x: x \in Z, x > 0\}$

Skillsheet TEACHER

Exercise 5A

Example 1

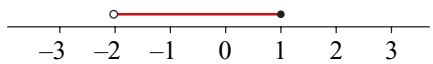
1 For $A = \{1, 2, 3, 5, 7, 11, 15\}$, $B = \{7, 11, 25, 30, 32\}$, $C = \{1, 7, 11, 25, 30\}$, find:

a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup B$

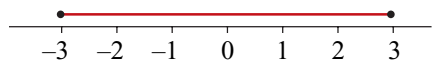
d $A \setminus B$ **e** $C \setminus B$ **f** $A \cap C$

2 Describe each of the following subsets of the real number line using the interval notation $[a, b)$, (a, b), etc:

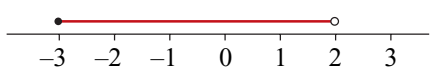
a



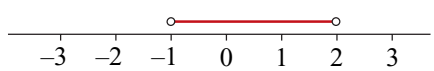
b



c



d



Example 2

3 Illustrate each of the following intervals on a number line:

a $[-3, 4)$ **b** $(-\infty, 3]$ **c** $[-2, -1]$

d $(-2, \infty)$ **e** $(-2, 3)$ **f** $(-2, 4]$

4 Use the appropriate interval notation, i.e. $[a, b]$, (a, b) etc., to describe each of the following sets:

a $\{x: -1 \leq x \leq 2\}$ **b** $\{x: -4 < x \leq 2\}$ **c** $\{y: 0 < y < \sqrt{2}\}$

d $\left\{y: -\frac{\sqrt{3}}{2} < y \leq \frac{1}{\sqrt{2}}\right\}$ **e** $\{x: x > -1\}$ **f** $\{x: x \leq -2\}$

g R **h** $R^+ \cup \{0\}$ **i** $R^- \cup \{0\}$

5 For $A = \{1, 2, 3, 5, 7, 11, 15\}$, $B = \{7, 11, 25, 30, 32\}$, $C = \{1, 7, 11, 25, 30\}$, find:

a $[-3, 8] \cap C$ **b** $(-2, 10] \cap B$ **c** $(3, \infty) \cap B$ **d** $(2, \infty) \cup B$

6 For each of the following use one number line on which to represent the sets:

a $[-2, 5]$, $[3, 4]$, $[-2, 5] \cap [3, 4]$ **b** $[-2, 5]$, $R \setminus [-2, 5]$

c $[3, \infty)$, $(-\infty, 7]$, $[3, \infty) \cap (-\infty, 7]$ **d** $[-2, 3]$, $R \setminus [-2, 3]$

5.2 Relations, domain and range

In previous chapters we have looked at how to sketch the graphs of various mathematical relations. We will now look at this aspect of defining relations in a more formal way.

An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first element and y the second.

A **relation** is a set of ordered pairs. The following are examples of relations:

a $S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}$ **b** $T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}$

Every relation determines two sets. The set of all the first elements of the ordered pairs is called the **domain**. The set of all the second elements of ordered pairs is called the **range**.

In the above examples:

a domain of $S = \{1, 3, 5\}$, range of $S = \{1, 2, 4, 6\}$

b domain of $T = \{-3, 4, 5, 7\}$, range of $T = \{5, 12, -6\}$

Some relations may be defined by a **rule** relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example, the set

$$\{(x, y): y = x + 1, x \in \{1, 2, 3, 4\}\}$$

is the relation

$$\{(1, 2)(2, 3)(3, 4)(4, 5)\}$$

The **domain** is the set $X = \{1, 2, 3, 4\}$ and the **range** is the set $Y = \{2, 3, 4, 5\}$.

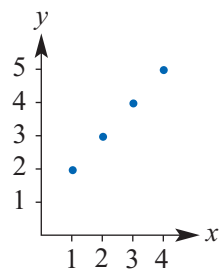
Graphing relations

We can represent a relation as a graph on a set of cartesian axes.

On the right is the graph of the relation

$$\{(x, y): y = x + 1, x \in \{1, 2, 3, 4\}\}$$

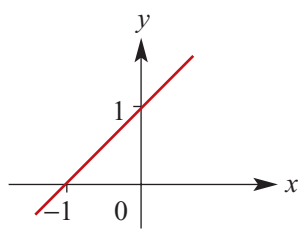
Note that we only graph the individual points of this relation.



If the domain of the relation is the set of real numbers, R, then there is an infinite number of points and the graph of

$$y = x + 1, x \in R$$

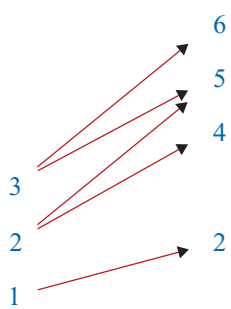
is a continuous straight line.



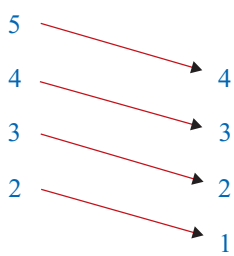
Mapping diagrams

A relation may be represented by a mapping diagram.

Mapping diagram 1



Mapping diagram 2



Mapping diagram 1 represents the relation $\{(3, 6), (3, 5), (2, 5), (2, 4), (1, 2)\}$.
Mapping diagram 2 represents the relation $\{(5, 4), (4, 3), (3, 2), (2, 1)\}$.

In summary, a relation may be written as:

i a listed set of ordered pairs (not always convenient or possible)

ii a rule with a specified or implied domain.

A relation may be represented by a graph or a mapping diagram.

Example 3

Sketch a graph of each of the following and state its domain and range:

a $\{(-2, -1), (-1, -1), (-1, 1), (0, 1), (1, -1)\}$

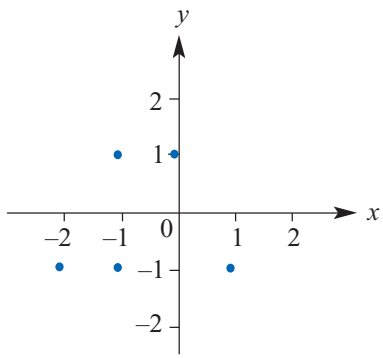
b $\{(x, y): x^2 + y^2 = 1, x \in [-1, 1]\}$

c $\{(x, y): 2x + 3y = 6, x \geq 0\}$

d $\{(x, y): y = 2x - 1, x \in [-1, 2]\}$

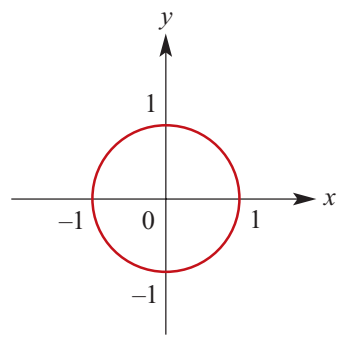
Solution

a



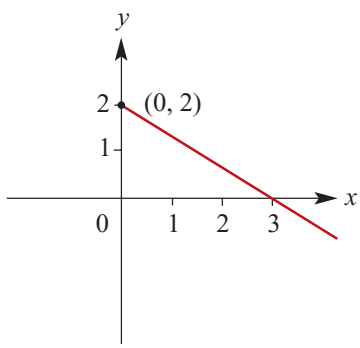
Domain $= \{-2, -1, 0, 1\}$
Range $= \{-1, 1\}$

b



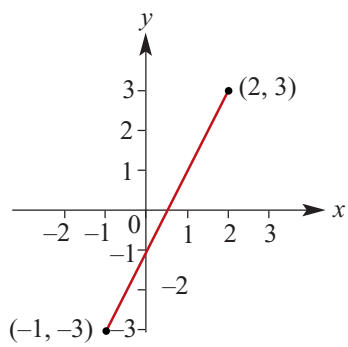
Domain $= \{x: -1 \leq x \leq 1\}$
Range $= \{y: -1 \leq y \leq 1\}$

c



Domain $= R^+ \cup \{0\}$
Range $= (-\infty, 2]$

d



Domain $= [-1, 2]$
Range $= [-3, 3]$

Often set notation is not used in the specification of a relation.

For example:

- $\{(x, y): y = x^2\}$ is written as $y = x^2$
- $\{(x, y): y = x + 1\}$ is written as $y = x + 1$.

This has been the case in your previous considerations of relations.

Note: In order to determine the range of a relation it is necessary to consider the graph. This strategy is used in the following examples.

Example 4

Find the range of the relation with rules:

a $y = x^2 - 4x + 5$ **b** $y = -x^2 + 4x - 5$

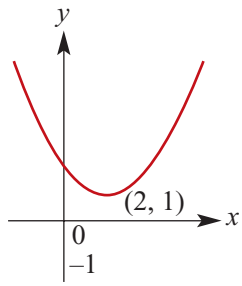
Solution

a Completing the square gives

$$y = x^2 - 4x + 5 = (x - 2)^2 + 1$$

The vertex of the corresponding parabola is at the point with coordinates (2, 1).

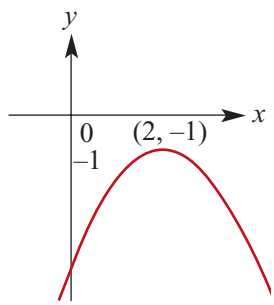
Therefore the minimum value of the relation is 1 and the range is $[1, \infty)$.



b Completing the square gives

$$y = -x^2 + 4x - 5$$
$$= -(x^2 - 4x + 5) = -(x - 2)^2 - 1$$

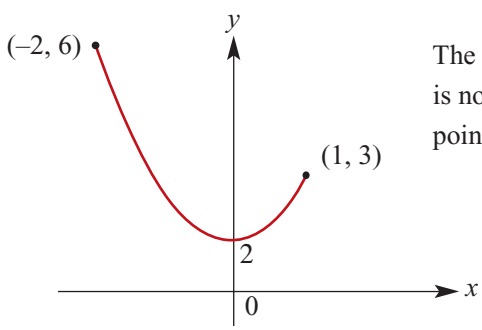
The vertex of the corresponding parabola is at the point with coordinates $(2, -1)$. Therefore the maximum value of the relation is -1 and the range is $(-\infty, -1]$.



Example 5

Sketch the graph of the relation $y = x^2 + 2$ for $x \in [-2, 1]$ and state the range.

Solution



The range is [2, 6]. Note that the range is not determined by considering the end points alone.

Implied (maximal) domain

When the rule for a relation is written and no domain is stipulated then it is understood that the domain taken is the largest for which the rule has meaning. This domain is called the **maximal or implied domain**.

For example, the maximal domain of $y = x^2$ is R and for $x^2 + y^2 = 1$ the maximal domain is $[-1, 1]$. This concept is considered again in Section 5.3.

Example 6

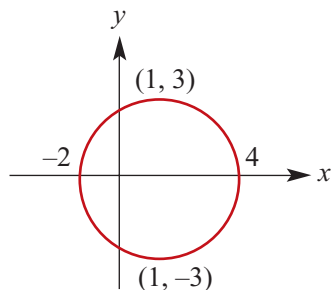
For each of the following relations state the maximal domain and the range:

a $(x-1)^2 + y^2 = 9$ **b** $y = \sqrt{9 - x^2} + 1$

Solution

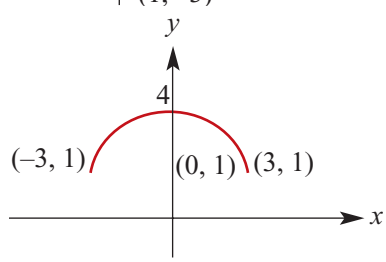
a This relation is a circle with centre (1, 0) and radius 3 units.

The maximal domain is $[-2, 4]$ and the range is $[-3, 3]$.



b This relation is a semicircle with centre (0, 1) and radius 3 units.

The maximal domain is $[-3, 3]$ and the range is $[1, 4]$.

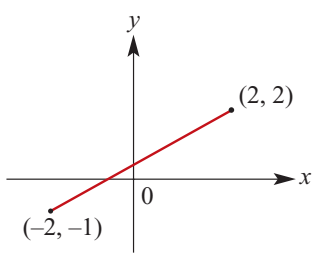


Skillsheet TEACHER Example 3

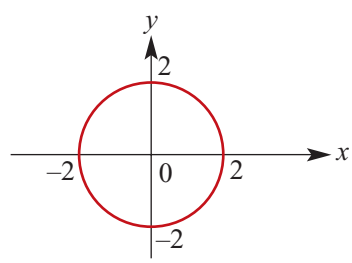
Exercise 5B

1 State the domain and range for the relations represented by each of the following graphs:

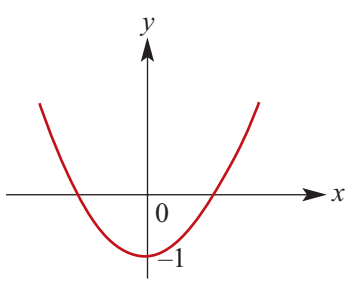
a



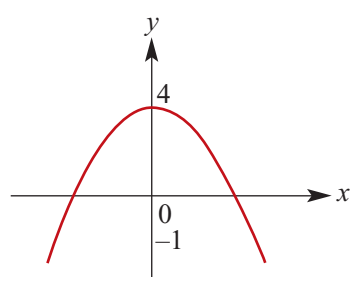
b



c



d



Example 3 **2** Sketch the graphs of each of the following and state the range of each:

a $y = x + 1$ $x \in [2, \infty)$ **b** $y = -x + 1$ $x \in [2, \infty)$
c $y = 2x + 1$ $x \in [-4, \infty)$ **d** $y = 3x + 2$ $x \in (-\infty, 3)$
e $y = x + 1$ $x \in (-\infty, 3]$ **f** $y = -3x - 1$ $x \in [-2, 6]$
g $y = -3x - 1$ $x \in [-5, -1]$ **h** $y = 5x - 1$ $x \in (-2, 4)$

Examples 4, 5 **3** Sketch the graphs of each of the following relations, stating the range of each:

a $\{(x, y): y = x^2 + 1\}$ **b** $\{(x, y): y = x^2 + 2x + 1\}$
c $\{(x, y): y = 4 - x^2, x \in [-2, 2]\}$ **d** $\{(x, y): y = x^2 + 2x + 3\}$
e $\{(x, y): y = -x^2 + 2x + 3\}$ **f** $\{(x, y): y = x^2 - 2, x \in [-1, 2]\}$
g $\{(x, y): y = 2x^2 - 3x + 6\}$ **h** $\{(x, y): y = 6 - 3x + x^2\}$

Example 6 **4** Sketch the graphs of each of the following relations, stating the maximal domain and range of each:

a $\{(x, y): x^2 + y^2 = 9\}$ **b** $(x - 2)^2 + (y - 3)^2 = 16$
c $(2x - 1)^2 + (2y - 4)^2 = 1$ **d** $y = \sqrt{25 - x^2}$
e $y = -\sqrt{25 - x^2}$ **f** $\left\{(x, y): y = -\sqrt{25 - (x - 2)^2}\right\}$

5.3 Functions

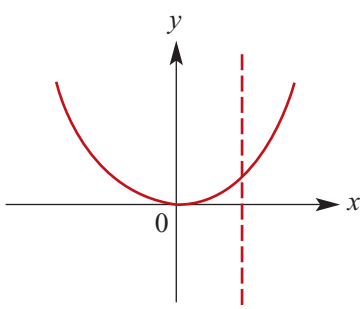
A **function** is a relation for which for each x-value of an ordered pair there is a unique y-value of the ordered pair. This means that if (a, b) and (a, c) are ordered pairs of a function then $b = c$.

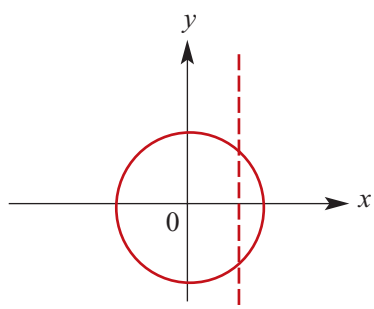
One way to identify if a relation is a function is to draw a graph of the relation and apply the **vertical line test**.

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, the relation is a **function**.

Examples:

$y = x^2$ is a function $x^2 + y^2 = 4$ is not a function





Function notation

Functions are usually denoted with lower case letters such as f, g, h.

The definition of a function tells us that for each x in the domain there is a unique element y in the range such that $(x, y) \in f$.

The element y is called the **image** of x under f or the **value** of f at x and x is called the **pre-image** of y.

Since the y-value obtained is a **function** of the x-value that was substituted into the rule, we use the notation $f(x)$ (read 'f of x') in place of y.

i.e. Instead of $y = 2x + 1$ we write $f(x) = 2x + 1$.

$f(2)$ means the y-value obtained when $x = 2$.

e.g. $f(2) = 2(2) + 1 = 5$

$f(-4) = 2(-4) + 1 = -7$

$f(a) = 2a + 1$

By incorporating the mapping notation we have an alternative way of writing functions.

i For the function $\{(x, y): y = x^2\}$, with domain $= R$ we write $f: R \to R, f(x) = x^2$.

ii For the function $\{(x, y): y = 2x - 1, x \in [0, 4]\}$ we write $f: [0, 4] \to R, f(x) = 2x - 1$.

iii For the function $\{(x, y): y = \frac{1}{x}\}$ with domain $= R \setminus \{0\}$ we write $f: R \setminus \{0\} \to R, f(x) = \frac{1}{x}$.

If the domain is R we often just write the rule. For example in **i** $f(x) = x^2$.

Note that in using the notation $f: X \to Y$, X is the domain but Y is not necessarily the range. It is a set that contains the range and is called the **co-domain**. With this notation for function we write domain of f as dom f and range of f as ran f.

A function $f: R \to R, f(x) = a$ is called a **constant function**.

For such a f, dom $f = R$ and ran $f = \{a\}$, e.g. let $f(x) = 7$. The dom $f = R$ and ran $f = \{7\}$.

A function $f: R \to R, f(x) = mx + c$ is called a **linear function**, e.g. let $f(x) = 3x + 1$.

Then dom $f = R$ and ran $f = R$.

Note if the domain of a linear function is R and $m \neq 0$, the range of $f = R$.

Example 7

If $f(x) = 2x^2 + x$, find $f(3)$, $f(-2)$ and $f(x - 1)$.

Solution

$$\begin{aligned} f(3) &= 2(3)^2 + 3 = 21 \\ f(-2) &= 2(-2)^2 - 2 = 6 \\ f(x - 1) &= 2(x - 1)^2 + x - 1 \\ &= 2(x^2 - 2x + 1) + x - 1 \\ &= 2x^2 - 3x + 1 \end{aligned}$$

Example 8

If $f(x) = 2x + 1$, find $f(-2)$ and $f\left(\frac{1}{a}\right)$, $a \neq 0$.

Solution

$$f(-2) = 2(-2) + 1 = -3$$

$$f\left(\frac{1}{a}\right) = 2\left(\frac{1}{a}\right) + 1 = \frac{2}{a} + 1$$

CAS APPENDIX

Example 9

Consider the function defined by $f(x) = 2x - 4$ for all $x \in R$.

a Find the value of $f(2)$, $f(-1)$ and $f(t)$.

b For what values of t is $f(t) = t$?

c For what values of x is $f(x) \geq x$?

Solution

a
$$f(2) = 2(2) - 4 = 0$$
$$f(-1) = 2(-1) - 4 = -6$$
$$f(t) = 2t - 4$$

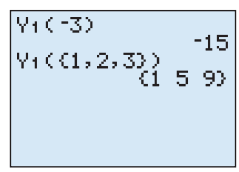
b
$$f(t) = t$$
$$\therefore 2t - 4 = t$$
$$\therefore t - 4 = 0$$
$$\therefore t = 4$$

c
$$f(x) \geq x$$
$$\therefore 2x - 4 \geq x$$
$$\therefore x - 4 \geq 0$$
$$\therefore x \geq 4$$

Using a graphics calculator

The graphics calculator uses function notation with the evaluation of functions.

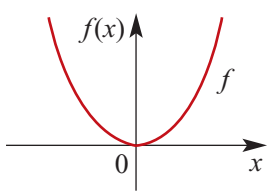
For example for $f(x) = 4x - 3$ enter Y1 $= 4x - 3$ in the **Y=** screen. In the Home screen enter Y1 (-3) to evaluate $f(-3)$ and enter Y1$(\{1, 2, 3\})$ to evaluate a list of values.



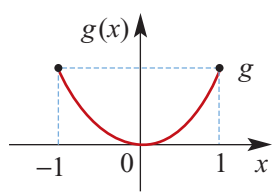
Note: The symbol Y1 is found by pressing VARS, and selecting **Y-VARS** and then **1:Function** and finally **1:Y_1**.

Restriction of a function

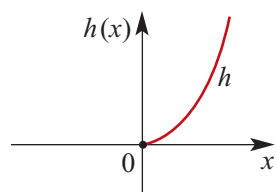
Consider the following functions:



$f(x) = x^2, x \in R$



$g(x) = x^2, -1 \leq x \leq 1$



$h(x) = x^2, x \in R^+ \cup \{0\}$

The different letters, f, g, and h, used to name the functions emphasise the fact that there are three different functions, even though they each have the same rule. They are different because they are defined for different domains. We call g and h restrictions of f, since their domains are subsets of the domain of f.

CAS APPENDIX

Example 10

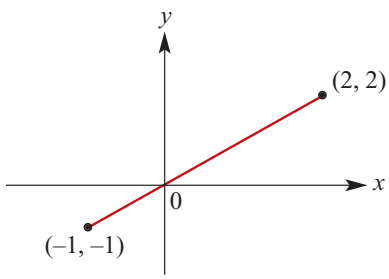
Sketch the graph of each of the following functions and state its range.

a $f: [-1, 2] \to R, f(x) = x$ **b** $f: [-1, 1] \to R, f(x) = x^2 + x$

c $f: (0, 2] \to R, f(x) = \frac{1}{x}$ **d** $f: R \to R, f(x) = x^2 - 2x + 8$

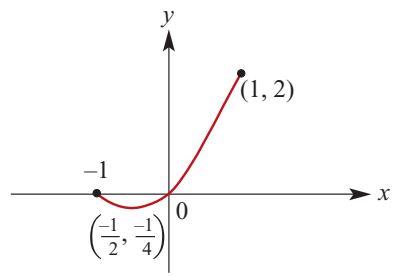
Solution

a



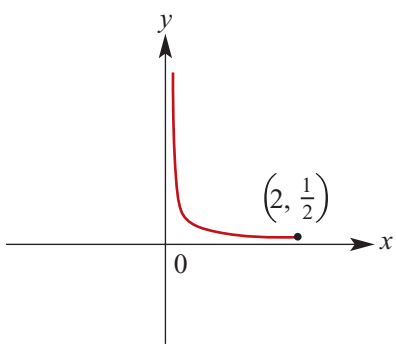
Range is $[-1, 2]$

b



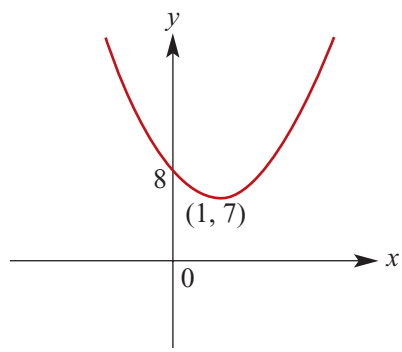
Range is $\left[\frac{-1}{4}, 2\right]$

c



Range is $\left[\frac{1}{2}, \infty\right)$

d



$f(x) = x^2 - 2x + 8 = (x - 1)^2 + 7$

Range is $[7, \infty)$

Exercise 5C

1 Sketch the graph of each of the following relations, then state the range of each and specify whether the relation is a function or not:

a $y = x^2, x \in [0, 4]$ **b** $\{(x, y): x^2 + y^2 = 4\}, x \in [0, 2]$

c $\{(x, y): 2x + 8y = 16, x \in [0, \infty)\}$ **d** $y = \sqrt{x}, x \in R^+$

e $\{(x, y): y = \frac{1}{x^2}, x \in R \setminus \{0\}\}$ **f** $\{(x, y): y = \frac{1}{x}, x \in R^+\}$

g $y = x^2, x \in [-1, 4]$ **h** $\{(x, y): x = y^2, x \in R^+\}$

2 Which of the following relation are functions? State the domain and range for each:

a $\{(0, 1), (0, 2), (1, 2), (2, 3), (3, 4)\}$

b $\{(-2, -1), (-1, -2), (0, 2), (1, 4), (2, -5)\}$

c $\{(0, 1), (0, 2), (-1, 2), (3, 4), (5, 6)\}$ **d** $\{(1, 3), (2, 3), (4, 3), (5, 3), (6, 3)\}$
e $\{(x, -2): x \in R\}$ **f** $\{(3, y): y \in Z\}$
g $y = -x + 3$ **h** $y = x^2 + 5$
i $\{(x, y): x^2 + y^2 = 9\}$

Example 7

3 **a** Given that $f(x) = 2x - 3$, find:
i $f(0)$ **ii** $f(4)$ **iii** $f(-1)$ **iv** $f(6)$

b Given that $g(x) = \dfrac{4}{x}$, find:
i $g(1)$ **ii** $g(-1)$ **iii** $g(3)$ **iv** $g(2)$

c Given that $g(x) = (x - 2)^2$, find:
i $g(4)$ **ii** $g(-4)$ **iii** $g(8)$ **iv** $g(a)$

Example 8

d Given that $f(x) = 1 - \dfrac{1}{x}$, find:
i $f(1)$ **ii** $f(1 + a)$ **iii** $f(1 - a)$ **iv** $f\left(\dfrac{1}{a}\right)$

4 Find the value(s) of x for which the function has the given value:
a $f(x) = 5x - 2, f(x) = 3$ **b** $f(x) = \dfrac{1}{x}, f(x) = 6$
c $f(x) = x^2, f(x) = 9$ **d** $f(x) = (x + 1)(x - 4), f(x) = 0$
e $f(x) = x^2 - 2x, f(x) = 3$ **f** $f(x) = x^2 - x - 6, f(x) = 0$

5 Let $g(x) = x^2 + 2x$ and $h(x) = 2x^3 - x^2 + 6$.
a Evaluate $g(-1)$, $g(2)$ and $g(-2)$. **b** Evaluate $h(-1)$, $h(2)$ and $h(-2)$.
c Express the following in terms of x:
i $g(-3x)$ **ii** $g(x - 5)$ **iii** $h(-2x)$
iv $g(x + 2)$ **v** $h(x^2)$

6 Consider the function $f(x) = 2x^2 - 3$. Find:
a $f(2), f(-4)$ **b** the range of f

Example 9

7 Consider the function $f(x) = 3x + 1$. Find:
a the image of 2 **b** the pre-image of 7 **c** $\{x: f(x) = 2x\}$

8 Consider the function $f(x) = 3x^2 + 2$. Find:
a the image of 0 **b** the pre-image(s) of 5 **c** $\{x: f(x) = 11\}$

9 Consider the functions $f(x) = 7x + 6$ and $g(x) = 2x + 1$. Find:
a $\{x: f(x) = g(x)\}$ **b** $\{x: f(x) > g(x)\}$ **c** $\{x: f(x) = 0\}$

10 Rewrite each of the following using the $f: X \to Y$ notation:
a $\{(x, y): y = 3x + 2\}$ **b** $\{(x, y): 2y + 3x = 12\}$
c $\{(x, y): y = 2x + 3, x \geq 0\}$ **d** $y = 5x + 6, -1 \leq x \leq 2$
e $y + x^2 = 25, -5 \leq x \leq 5$ **f** $y = 5x - 7, 0 \leq x \leq 1$

Example 10

11 Sketch the graphs of each of the following functions and state the range of each:

a $f: [-1, 2] \to R, f(x) = x^2$ **b** $f: [-2, 2] \to R, f(x) = x^2 + 2x$

c $f: (0, 3] \to R, f(x) = \dfrac{1}{x}$ **d** $f: R \to R, f(x) = x^2 - 2x + 3$

5.4 Special types of functions and implied domains

Types of relations

There are four types of relations:

1	**One-to-one**	Each x-value maps onto a unique y-value.	e.g. $y = 2x$
2	**Many-to-one**	More than one x-value maps onto the same y-value.	e.g. $y = x^2$
3	**One-to-many**	An x-value maps onto more than one y-value.	e.g. $y = \pm\sqrt{x}$
4	**Many-to-many**	More than one x-value maps onto more than one y-value.	e.g. $x^2 + y^2 = 4$

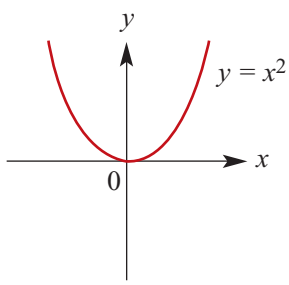
As stated earlier, a function is a relation in which no two ordered pairs have the same x-value. Functions are relations that are either **many-to-one** or **one-to-one**.

One-to-one functions

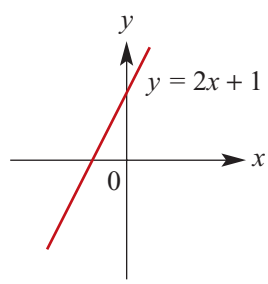
The **vertical line test** can be used to determine whether a relation is a function or not.

Similarly there is a geometric test that determines whether a function is **one-to-one** or not.

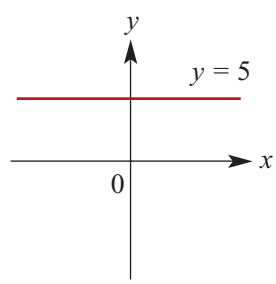
If a horizontal line can be drawn anywhere on the graph of a function and only ever intersects the graph a maximum of once, the function is **one-to-one**.



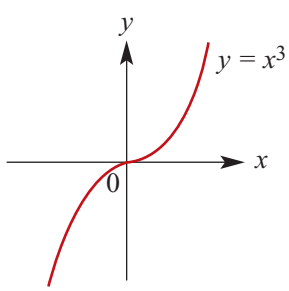
not one-to-one



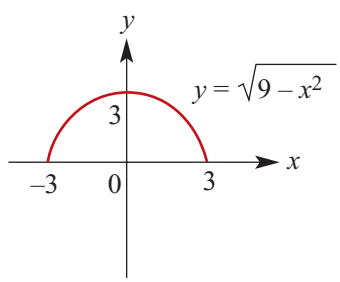
one-to-one



not one-to-one



one-to-one



not one-to-one

When the domain of a relation is not explicitly stated, it is assumed to consist of all real numbers for which the rule has a meaning. We refer to the **implied (maximal)** domain of a relation, because the domain is implied by the rule.

For example:

$S = \{(x, y): y = x^2\}$ has the **implied (maximal) domain** R.

and $T = \{(x, y): y = \sqrt{x}\}$ has the **implied (maximal) domain** $[0, \infty)$.

Example 11

State the maximal domain, sketch the graph and find the corresponding range of each of the following:

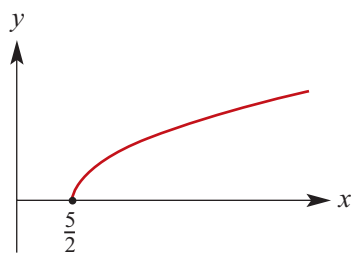
a $y = \sqrt{2x - 5}$ **b** $y = \dfrac{1}{2x - 5}$

Solution

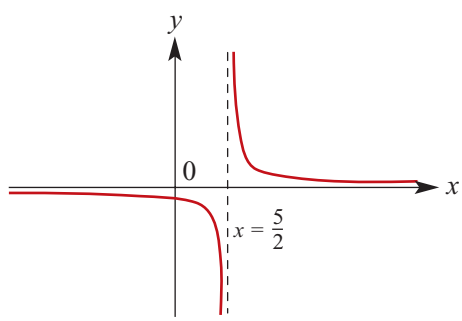
a To be defined $2x - 5 \geq 0$. That is, $x \geq \dfrac{5}{2}$. Hence the maximal domain is $[\dfrac{5}{2}, \infty)$.

The range of the function is $R^+ \cup \{0\} = [0, \infty)$.

The range of the function is $R \setminus \{0\}$.



b To be defined $2x - 5 \neq 0$, i.e. $x \neq \dfrac{5}{2}$. Hence maximal domain is $R \setminus \{\dfrac{5}{2}\}$ and the range is $R \setminus \{0\}$.



Exercise 5D

1 State which of the following functions are one-to-one:

a $\{(1, 3), (2, 4), (4, 4), (3, 6)\}$ **b** $\{(1, 3), (2, 4), (3, 6), (7, 9)\}$

c $\{(x, y): y = x^2\}$ **d** $\{(x, y): y = 3x + 1\}$

e $f(x) = x^3 + 1$ **f** $f(x) = 1 - x^2$

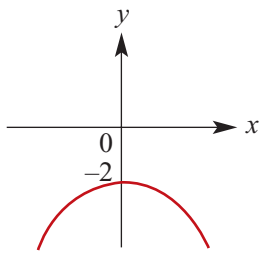
g $y = x^2, x \geq 0$

2 The following are the graphs of a relation:

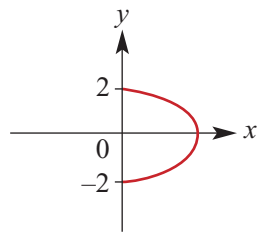
i State which are the graphs of a function.

ii State which are the graphs of a one-to-one function.

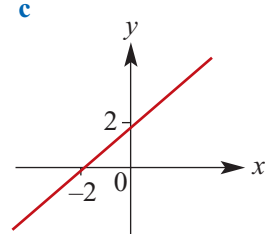
a



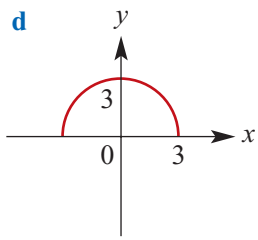
b



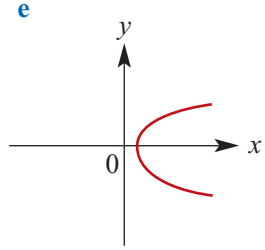
c



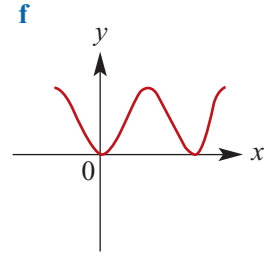
d



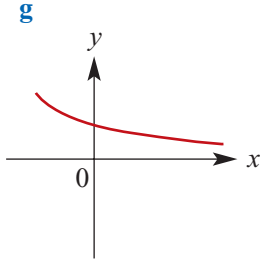
e



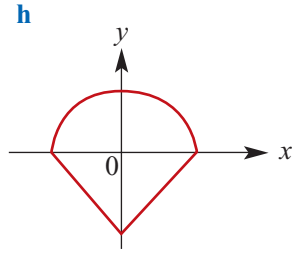
f



g



h



Example 11

3 For each of the following, find the maximal domain and the corresponding range for the function defined by the rule:

a $y = 7 - x$ **b** $y = 2\sqrt{x}$ **c** $y = x^2 + 1$

d $y = -\sqrt{9 - x^2}$ **e** $y = \dfrac{1}{\sqrt{x}}$ **f** $y = 3 - 2x^2$

g $y = \sqrt{x - 2}$ **h** $y = \sqrt{2x - 1}$ **i** $y = \sqrt{3 - 2x}$

j $y = \dfrac{1}{2x - 1}$ **k** $y = \dfrac{1}{(2x - 1)^2} - 3$ **l** $y = \dfrac{1}{2x - 1} + 2$

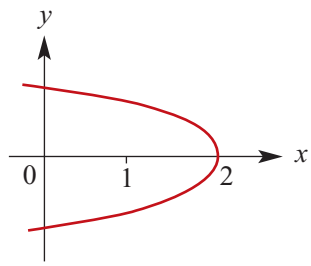
4 Each of the following is the rule of a function. In each case write down the maximal domain and the range:

a $f(x) = 3x + 4$ **b** $g(x) = x^2 + 2$

c $y = -\sqrt{16 - x^2}$ **d** $y = \dfrac{1}{x + 2}$

5 The graph shown is of the relation $\{(x, y): y^2 = -x + 2, x \leq 2\}$.

From this relation, form two functions and specify the range of each.



6 **a** Draw the graph of $f: R \to R, f(x) = x^2 - 2$.

b By restricting the domain of f, form two one-to-one functions that have the same rule as f.

5.5 Hybrid functions

Functions which have different rules for different subsets of the domain, are called **hybrid functions**.

Example 12

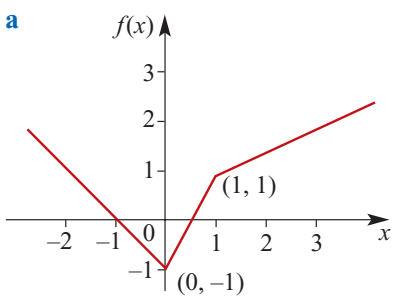
a Sketch the graph of the function f given by:

$$f(x) = \begin{cases} -x - 1 & \text{for } x < 0 \\ 2x - 1 & \text{for } 0 \leq x \leq 1 \\ \frac{1}{2}x + \frac{1}{2} & \text{for } x \geq 1 \end{cases}$$

b State the range of f.

Solution

a



b The range is $[-1, \infty)$.

Exercise 5E

Example 12

1 Sketch the graph of each of the following functions and state its range:

a $h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

b $h(x) = \begin{cases} x - 1, & x \geq 1 \\ 1 - x, & x < 1 \end{cases}$

c $h(x) = \begin{cases} -x, & x \geq 0 \\ x, & x < 0 \end{cases}$

d $h(x) = \begin{cases} 1+x, & x \geq 0 \\ 1-x, & x < 0 \end{cases}$

e $h(x) = \begin{cases} x, & x \geq 1 \\ 2-x, & x < 1 \end{cases}$

2 **a** Sketch the graph of the function: **b** What is the range of f?

$$f(x) = \begin{cases} \frac{2}{3}x + 3, & x < 0 \\ x + 3, & 0 \leq x \leq 1 \\ -2x + 6, & x > 1 \end{cases}$$

3 Sketch the graph of the function:

$$g(x) = \begin{cases} -x - 3, & x < 1 \\ x - 5, & 1 \leq x \leq 5 \\ 3x - 15, & x > 5 \end{cases}$$

4 **a** Sketch the graph of the function: **b** State the range of h.

$$h(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ 1 - x, & x < 0 \end{cases}$$

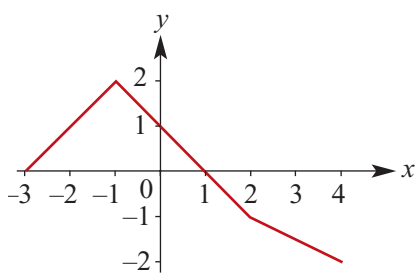
5 **a** Sketch the graph of the function: **b** State the range of f.

$$f(x) = \begin{cases} x + 3, & x < -3 \\ x^2 - 9, & -3 \leq x \leq 3 \\ x - 3, & x > 3 \end{cases}$$

6 **a** Sketch the graph of the function: **b** State the range of f.

$$f(x) = \begin{cases} \frac{1}{x}, & x > 1 \\ x, & x \leq 1 \end{cases}$$

7 Specify the function represented by this graph:



5.6 Miscellaneous exercises

Example 13

The volume of a sphere of radius r is determined by the function with rule $V(r) = \frac{4}{3}\pi r^3$. State the practical domain of the function V and find $V(10)$.

Solution

The practical domain is $(0, \infty)$.

$$V(10) = \frac{4}{3} \times \pi \times 10^3$$
$$= \frac{4000\pi}{3}$$

The volume is $\frac{4000\pi}{3}$ cubic units.

Example 14

If $f: R \to R, f(x) = ax + b, f(1) = 7$ and $f(5) = 19$, find a and b and sketch the graph of $y = f(x)$.

Solution

Since $f(1) = 7$ and $f(5) = 19$:

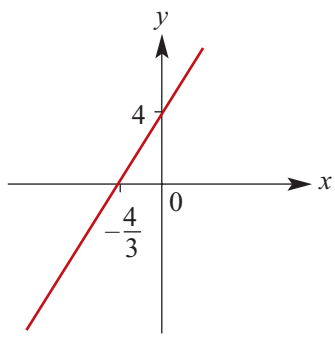
$$7 = a + b \qquad (1)$$

and $$19 = 5a + b \qquad (2)$$

Subtract (1) from (2):

$$12 = 4a$$
$$a = 3, \text{ and substituting in (1) gives } b = 4$$
$$a = 3, b = 4$$

and $f(x) = 3x + 4$



Example 15

Find the quadratic function f such that $f(4) = f(-2) = 0$ and $f(0) = 16$.

Solution

4 and -2 are solutions to the quadratic equations $f(x) = 0$.

Thus $f(x) = k(x - 4)(x + 2)$

As $f(0) = 16$

$16 = k(-4)(2)$

$\therefore$ $k = -2$

and $f(x) = -2(x - 4)(x + 2)$

$= -2(x^2 - 2x - 8)$

$= -2x^2 + 4x + 16$

Exercise 5F

Example 14

1 If $f(x) = a + bx$ and $f(4) = -1$ and $f(8) = 1$:

a Find a and b. **b** Solve the equation $f(x) = 0$.

2 Find a linear function f such that $f(0) = 7$, whose graph is parallel to that of the function with rule $g(x) = 2 - 5x$.

3 f is a linear function such that $f(-5) = -12$ and $f(7) = 6$.

a Find:

i $f(0)$ **ii** $f(1)$

b Solve the equation $f(x) = 0$.

4 If $f(x) = 2x + 5$, find:

a $f(p)$ **b** $f(p + h)$

c $f(p + h) - f(p)$ **d** $f(p + 1) - f(p)$

5 If $f(x) = 3 - 2x$, find $f(p + 1) - f(p)$.

6 A metal bar is L cm long when its temperature is C degrees centigrade. L and C are approximately related by the formula $L = 0.002C + 25$.

a L is a function of C and the rule can be written $L(C) = 0.002C + 25$. State a possible practical domain for the function.

b Find:

i $L(30)$ **ii** $L(16)$ **iii** $L(100)$ **iv** $L(500)$

Example 15

7 Find the quadratic function f such that $f(2) = f(4) = 0$ and 7 is the greatest value of $f(x)$.

8 Write $f(x) = x^2 - 6x + 16$ in the form $f(x) = (x - 3)^2 + p$ and hence state the range of f.

9 State the range of each of the following:

a $f(x) = -2x^2 + x - 2$ **b** $f(x) = 2x^2 - x + 4$

c $f(x) = -x^2 + 6x + 11$ **d** $g(x) = -2x^2 + 8x - 5$

10 $f: [-1, 6] \to R, f(x) = 5 - 3x$

a Sketch the graph of f. **b** State the range of f.

11 $f: [-1, 8] \to R, f(x) = (x - 2)^2$

a Sketch the graph of f. **b** State the range of f.

12 State the implied domain and range of each of the following relations:

a $x^2 + y^2 = 9$ **b** $(x - 2)^2 + y^2 = 1$

c $(2x - 1)^2 + (2y - 1)^2 = 1$ **d** $(x - 4)^2 + y^2 = 25$

e $(y - 2)^2 + x^2 = 16$

13 The domain of the function f is $\{1, 2, 3, 4\}$. Find the range of f if:

a $f(x) = 2x$ **b** $f(x) = 5 - x$ **c** $f(x) = x^2 - 4$ **d** $f(x) = \sqrt{x}$

14 $f: R \to R, f(x) = ax^2 + bx + c$. Find a, b and c if $f(0) = 2, f(4) = 0$ and $f(5) = 0$.

15 Find two quadratic functions f and g such that $f(1) = 0$, $g(1) = 0$ and $f(0) = 10$, $g(0) = 10$ and both have a maximum value of 18.

16 **a** Find the set of values of k for which $f(x) = 3x^2 - 5x - k$ is greater than 1 for all real x.

b Show that, for all k, the minimum value of $f(x)$ occurs when $x = \frac{5}{6}$. Find k if this minimum value is zero.

5.7 Inverse functions

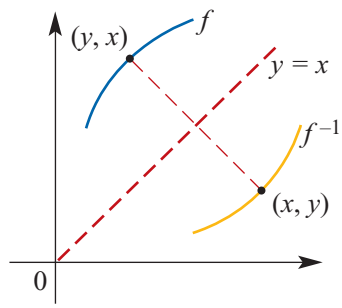
Excel

If f is a one-to-one function then for each number y in the range of f there is exactly one number x in the domain of f such that $f(x) = y$.

Thus, if f is a one-to-one function, a new function f^{-1}, called the inverse of f, may be defined by:

$$f^{-1}(x) = y \text{ if } f(y) = x, \text{ for } x \in \text{ran } f, y \in \text{dom } f$$

It is not difficult to see what the relation between f and f^{-1} means geometrically. The point (x, y) is on the graph of f^{-1} if the point (y, x) is on the graph of f. Therefore to get the graph of f^{-1} from the graph of f, the graph of f is to be reflected in the line $y = x$.



From this the following is evident:

$$\text{dom } f^{-1} = \text{ran } f$$
$$\text{ran } f^{-1} = \text{dom } f$$

A function has an inverse function if and only if it is one-to-one.

CAS APPENDIX

Example 16

Find the inverse function f^{-1} of the function $f(x) = 2x - 3$ and sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the one set of axes.

Solution

The graph of f has equation $y = 2x - 3$ and the graph of f^{-1} has equation $x = 2y - 3$, i.e. x and y are interchanged.

Solve for y:

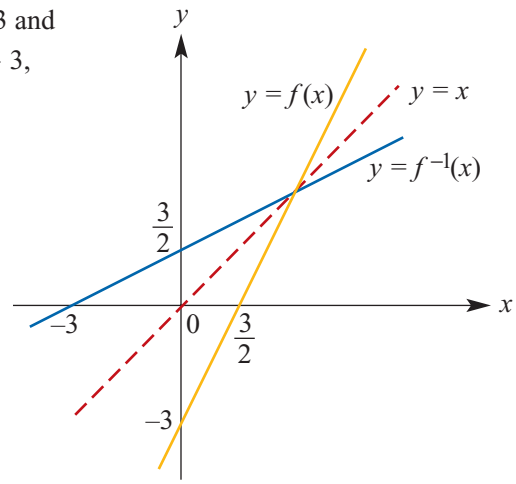
$$x + 3 = 2y$$

and $$y = \frac{1}{2}(x + 3)$$

$$\therefore \quad f^{-1}(x) = \frac{1}{2}(x + 3)$$

$$\text{dom } f = \text{ran } f^{-1} = R$$

and $$\text{ran } f = \text{dom } f^{-1} = R$$



Example 17

Let $f: [3, 6] \to R, f(x) = \left(\frac{x}{3}\right)^2$. Find f^{-1} and state its domain and range.

Solution

Let $y = \left(\frac{x}{3}\right)^2$. Then the inverse function has rule

$$x = \left(\frac{y}{3}\right)^2$$

$$\therefore \quad \pm\sqrt{x} = \frac{y}{3}$$

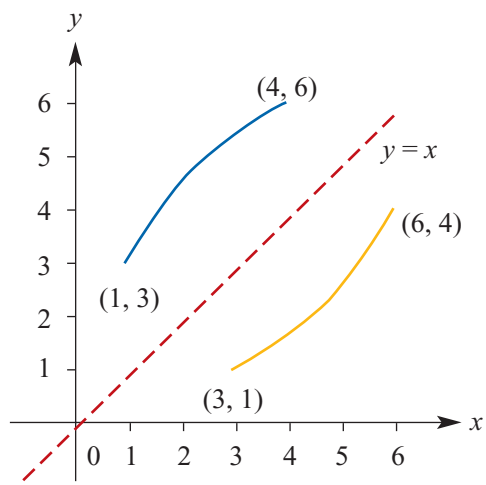
and $$y = \pm 3\sqrt{x}$$

But ran $f^{-1} = \text{dom } f = [3, 6]$

$$\therefore \quad f^{-1}(x) = 3\sqrt{x}$$

and dom $f^{-1} = \text{ran } f = [1, 4]$

i.e. $f^{-1}: [1, 4] \to R, f^{-1}(x) = 3\sqrt{x}$



Example 18

Find the inverse of each of the following functions:

a $\{(1, 2), (3, 4), (5, 6), (7, 8)\}$

b $f: [1, \infty) \to R, f(x) = (x - 1)^2 + 4$

Solution

a $\{(2, 1), (4, 3), (6, 5), (8, 7)\}$

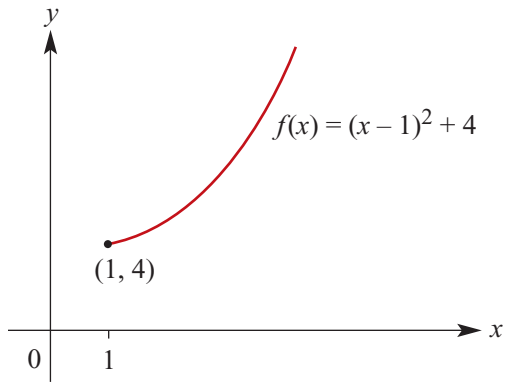
b The inverse has rule

$$x = (y-1)^2 + 4$$
$$\therefore \quad (y-1)^2 = x - 4$$
$$\therefore \quad y - 1 = \pm\sqrt{x-4}$$
$$\therefore \quad y = 1 \pm \sqrt{x-4}$$

But ran f^{-1} = dom $f = [1, \infty)$

$$\therefore \quad f^{-1}(x) = 1 + \sqrt{x-4}$$

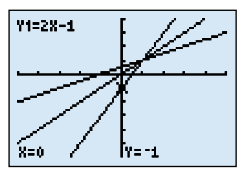
Also dom f^{-1} = ran $f = [4, \infty)$



Using a graphics calculator

Drawing the graph of the inverse of a function

Enter Y1 $= 2x - 1$ and Y2 $= x$ in the **Y=** window and choose a suitable window. Go to the Home screen and enter **8:DrawInv** Y1 and press ENTER. The result is as shown.



Note: **8:DrawInv** is obtained from the **DRAW** menu (2ND PRGM) and the symbol Y1 is found by pressing VARS and selecting **Y-VARS**, then **1:Function** and finally **1:Y1**.

The graphs of $y = 2x - 1$, $y = x$ and the inverse of $y = 2x - 1$, which has rule $y = \dfrac{x+1}{2}$, are displayed.

Exercise 5G

Example 16

1 Find the inverse function of each of the following, clearly stating the domain and range of f^{-1}:

a $\{(1, 3), (-2, 6), (4, 5), (7, 1)\}$

b $f: R \to R,\ f(x) = 6 - 2x$

c $f: [1, 5] \to R,\ f(x) = 3 - x$

d $f: R^+ \to R,\ f(x) = x + 4$

Example 17

e $f: (-\infty, 4] \to R,\ f(x) = x + 4$

f $f: [0, \infty) \to R,\ f(x) = x^2$

Example 18

g $f: [2, \infty) \to R,\ f(x) = (x-2)^2 + 3$

h $f: (-\infty, 4] \to R,\ f(x) = (x-4)^2 + 6$

i $f: [0, 1] \to R,\ f(x) = \sqrt{1-x}$

j $f: [0, 4] \to R,\ f(x) = \sqrt{16 - x^2}$

k $f: [-1, 7] \to R,\ f(x) = 16 - 2x$

l $f: [0, \infty) \to R,\ f(x) = (x+4)^2 + 6$

2 **a** On the one set of axes sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, where $f(x) = 2x - 6$.

b Find the coordinates of the point for which $f(x) = f^{-1}(x)$.

3 **a** On the one set of axes sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, where $f: [0, \infty) \to R$, $f(x) = x^2$.

b Find the coordinates of the point(s) for which $f(x) = f^{-1}(x)$.

4 $f: R \to R$, $f(x) = ax + b$, where a and b are non-zero constants, and $f(1) = 2$ and $f^{-1}(1) = 3$. Find the values of a and b.

5 $f: (-\infty, a] \to R, f(x) = \sqrt{a - x}$.

a Find $f^{-1}(x)$.

b If the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at $x = 1$, find the possible values for a.

5.8 Translations of functions[1]

In this and the following two sections a formal study of transformations is undertaken using function notation. These sections also provide systematic methods for applying and determining transformations.

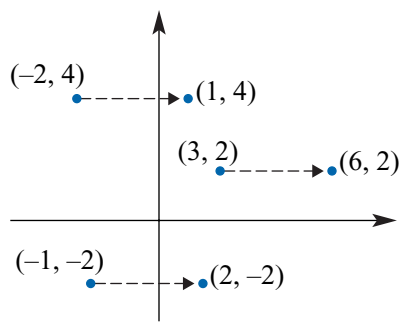
Let $R^2 = \{(x, y): x, y \in R\}$. That is R^2 is the set of all ordered pairs of real numbers. Transformations associate each ordered pair of R^2 with a unique ordered pair.

For example, the translation 3 units in the positive direction of the x-axis (to the right) associates with each ordered pair (x, y) a new ordered pair $(x + 3, y)$.

The notation used is $(x, y) \to (x + 3, y)$. For example, $(3, 2) \to (3 + 3, 2)$, i.e. $(6, 2)$. This second ordered pair is uniquely determined by the first.

In the diagram it is seen that the point with coordinates $(3, 2)$ is mapped to the point with coordinates $(6, 2)$.

Also $(-2, 4) \to (1, 4)$ and $(-1, -2) \to (2, -2)$.



In applying a transformation it is useful to think of every point (x, y) on the plane as being mapped to a new point (x', y'). This point (x, y) is the only point which maps to (x', y'). The following can be written with the translation considered above:

$$x' = x + 3 \text{ and } y' = y$$

The translation 2 units in the positive direction of the x-axis (to the right) and 4 units in the positive direction of the y-axis can be described by the rule $(x, y) \to (x + 2, y + 4)$.

[1] These sections could be omitted but they form a sound foundation for further study.

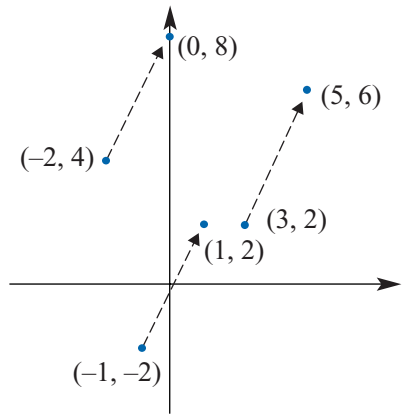
For example, $(3, 2) \rightarrow (3 + 2, 2 + 4)$.

The following can be written for this second translation considered above:

$$x' = x + 2 \text{ and } y' = y + 4$$

In the diagram it is seen that the point with coordinates (3, 2) is mapped to the point with coordinates (5, 6).

Also $(-2, 4) \rightarrow (0, 8)$ and $(-1, -2) \rightarrow (1, 2)$.



In general

- A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule

  $$(x, y) \rightarrow (x + h,\ y + k)$$

  or $\quad x' = x + h \quad$ and $\quad y' = y + k$

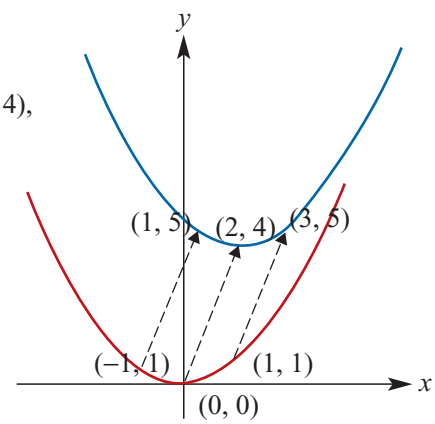
  where h and k are positive numbers.
- A translation in the negative direction of the x-axis and the negative direction of the y-axis is described by

  $$(x, y) \rightarrow (x - h,\ y - k)$$

  or $\quad x' = x - h \quad$ and $\quad y' = y - k$

  where h and k are positive numbers.

Consider translating the set of points defined by a function such as $\{(x, y): y = x^2\}$. Now with the transformation defined by the rule $(x, y) \rightarrow (x + 2, y + 4)$, it is known that $x' = x + 2$ and $y' = y + 4$ and hence $x = x' - 2$ and $y = y' - 4$. Thus $\{(x, y): y = x^2\}$ maps to $\{(x', y'): y' - 4 = (x' - 2)^2\}$. This means the points on the curve with equation $y = x^2$ are mapped to the curve with equation $y' - 4 = (x' - 2)^2$.



In general, the curve with equation $y = f(x)$ is mapped to the curve with equation $y - k = f(x - h)$ by the translation with rule $(x, y) \rightarrow (x + h, x + k)$.

This generalisation can now be applied to other functions.

Example 19

Find the image of the curve with equation $y = f(x)$, where $f(x) = \frac{1}{x}$ under a translation 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

Solution

The rule is $(x, y) \to (x + 3, y - 2)$. Let (x', y') be the image of the point (x, y), where (x, y) is a point on the graph of $y = f(x)$. Then $x' = x + 3$ and $y' = y - 2$ and hence $x = x' - 3$ and $y = y' + 2$. The graph of $y = f(x)$ is mapped to the graph of $y' + 2 = f(x' - 3)$, i.e. $y = \frac{1}{x}$ is mapped to $y' + 2 = \frac{1}{x' - 3}$.

Using a graphics calculator

Example 20

Use a graphics calculator to sketch the graph of the function obtained from the graph of the function with equation $y = \sqrt{x}$ by a translation of:

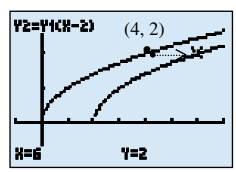
a 2 units in the positive direction of the x-axis

b 3 units in the negative direction of the y-axis

Solution

a A translation of h units ($h > 0$) in the positive direction of the x-axis transforms the function with rule $y = f(x)$ into the function with rule $y = f(x - h)$.

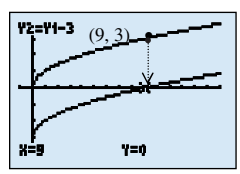
Therefore $Y1 = \sqrt{X}$ $(= f(x))$
and $Y2 = Y1(X - 2)$ $(= f(x - h))$



The transformed graph of Y2 is shown to the right, together with the graph of the original function.

b A translation of k units ($k > 0$) in the negative direction of the y-axis transforms the function with rule $y = f(x)$ into the function with rule $y = f(x) - k$.

Therefore $Y1 = \sqrt{X}$ $(= f(x))$
and $Y2 = Y1 - 3$ $(= f(x) - k)$



The transformed graph of Y2 is shown to the right, together with the graph of the original function.

Applying translations to sketch graphs

Recognising that a transformation has been applied makes it easy to sketch many graphs.

For example, in order to sketch the graph of $y = \frac{1}{x-2}$ note that it is of the form $y = f(x-2)$ where $f(x) = \frac{1}{x}$. That is, the graph of $y = \frac{1}{x}$ is translated 2 units in the positive direction of the x-axis. Examples of the graphs of two other functions to which this translation is applied are:

- $f(x) = x^2$ $\quad f(x-2) = (x-2)^2$
- $f(x) = \sqrt{x}$ $\quad f(x-2) = \sqrt{x-2}$

Exercise 5H

1 Sketch the graphs of each of the following, labelling asymptotes and axes intercepts:

a $y = \frac{1}{x} + 3$ **b** $y = \frac{1}{x^2} - 3$ **c** $y = \frac{1}{(x+2)^2}$

d $y = \sqrt{x-2}$ **e** $y = \frac{1}{x-1}$ **f** $y = \frac{1}{x} - 4$

g $y = \frac{1}{x+2}$ **h** $y = \frac{1}{x-3}$ **i** $f(x) = \frac{1}{(x-3)^2}$

j $f(x) = \frac{1}{(x+4)^2}$ **k** $f(x) = \frac{1}{x-1} + 1$ **l** $f(x) = \frac{1}{x-2} + 2$

Example 19

2 For $y = f(x) = \frac{1}{x}$, sketch the graph of each of the following, labelling asymptotes and axes intercepts:

a $y = f(x-1)$ **b** $y = f(x) + 1$ **c** $y = f(x+3)$

d $y = f(x) - 3$ **e** $y = f(x+1)$ **f** $y = f(x) - 1$

3 For $y = f(x) = x^2$, sketch the graph of each of the following, labelling axes intercepts:

a $y = f(x-1)$ **b** $y = f(x) + 1$ **c** $y = f(x+3)$

d $y = f(x) - 3$ **e** $y = f(x+1)$ **f** $y = f(x) - 1$

4 For $y = f(x) = x^2$, sketch the graph of each of the following, labelling axes intercepts:

a $y = f(x-1) + 2$ **b** $y = f(x-3) + 1$ **c** $y = f(x+3) - 5$

d $y = f(x+1) - 3$ **e** $y + 2 = f(x+1)$ **f** $y = f(x-5) - 1$

5 Sketch the graphs of each of the following, stating the equations of asymptotes, the axes intercepts and the range of each function:

a $y = \frac{1}{x^2} + 1$ **b** $y = \frac{3}{x^2}$ **c** $y = \frac{1}{(x-1)^2}$ **d** $y = \frac{1}{x^2} - 4$

5.9 Dilations and reflections

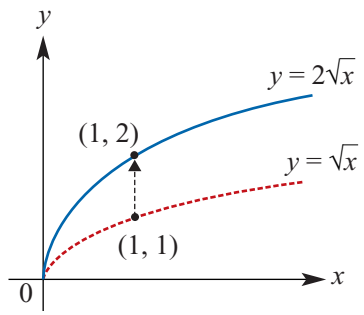
Dilations

A dilation of factor 2 from the x-axis can be defined by the rule $(x, y) \rightarrow (x, 2y)$. Hence the point with coordinates $(1, 1) \rightarrow (1, 2)$.

A dilation of factor 2 from the y-axis can be defined by the rule $(x, y) \rightarrow (2x, y)$. Hence the point with coordinates $(1, 1) \rightarrow (2, 1)$.

Dilation from the x-axis

The curve with equation $y = \sqrt{x}$ is considered. Let (x', y') be the image of the point with coordinates (x, y) on the curve with equation $y = \sqrt{x}$ under dilation of factor 2 from the x-axis. Hence $x' = x$ and $y' = 2y$. Then $x = x'$ and $y = \frac{y'}{2}$ and the curve with equation $y = \sqrt{x}$ maps to the curve with equation $\frac{y'}{2} = \sqrt{x'}$, i.e. the curve with equation $y = 2\sqrt{x}$.



In general

- A dilation of a units from the x-axis is described by the rule

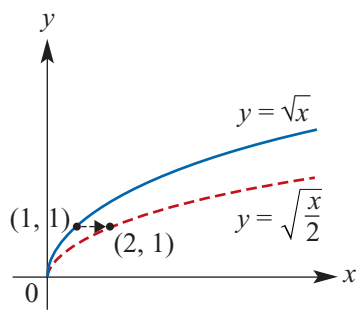
  $(x, y) \rightarrow (x, ay)$

  or $x' = x$ and $y' = ay$

  where a is a positive number.
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = af(x)$ by the transformation with rule $(x, y) \rightarrow (x, ay)$.

Dilation from the y-axis

The curve with equation $y = \sqrt{x}$ is again considered. Let (x', y') be the image of the point with coordinates (x, y) on the curve with equation $y = \sqrt{x}$ under dilation of factor 2 from the y-axis. Hence $x' = 2x$ and $y' = y$. Then $x = \frac{x'}{2}$ and $y = y'$ and the curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{\frac{x'}{2}}$.



In general

- A dilation of a units from the y-axis is described by the rule

  $(x, y) \rightarrow (ax, y)$

  or $x' = ax$ and $y' = y$

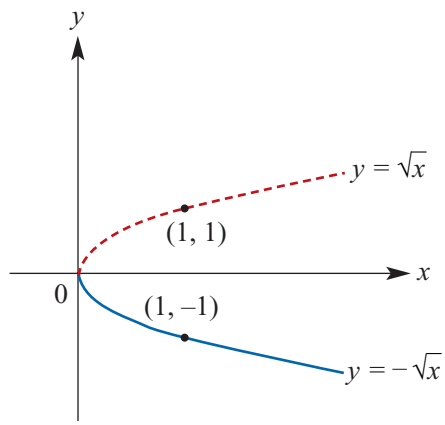
  where a is a positive number.
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f(\frac{x}{a})$ by the transformation with rule $(x, y) \rightarrow (ax, y)$.

Reflections in the axes

Reflection in the *x*-axis

A reflection in the x-axis can be defined by the rule $(x, y) \rightarrow (x, -y)$. Hence the point with coordinates $(1, 1) \rightarrow (1, -1)$.

The curve with equation $y = \sqrt{x}$ is again considered. Let (x', y') be the image of the point with coordinates (x, y) on the curve with equation $y = \sqrt{x}$ under a reflection in the x-axis. Hence $x' = x$ and $y' = -y$. Then $x = x'$ and $y = -y'$ and the curve with equation $y = \sqrt{x}$ maps to the curve with equation $-y' = \sqrt{x}$, i.e. the curve with equation $y = -\sqrt{x}$.



In general

- A reflection in the x-axis is described by the rule

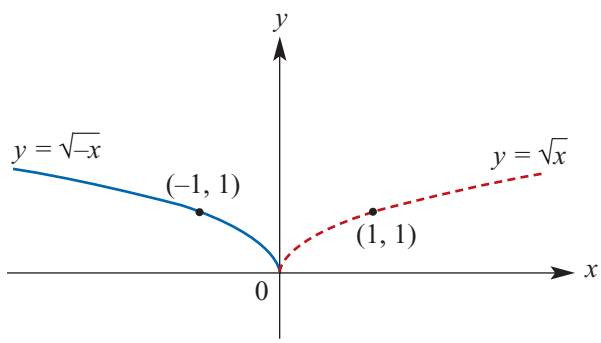
  $(x, y) \rightarrow (x, -y)$

  or $x' = x$ and $y' = -y$
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = -f(x)$ by the transformation with rule $(x, y) \rightarrow (x, -y)$.

Reflection in the *y*-axis

A reflection in the y-axis can be defined by the rule $(x, y) \rightarrow (-x, y)$. Hence the point with coordinates $(1, 1) \rightarrow (-1, 1)$.

The curve with equation $y = \sqrt{x}$ is again considered. Let (x', y') be the image of the point with coordinates (x, y) on the curve with equation $y = \sqrt{x}$ under a reflection in the y-axis. Hence $x' = -x$ and $y' = y$. Then $x = -x'$ and $y = y'$ and the curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{-x'}$, i.e. the curve with equation $y = \sqrt{-x}$.



In general

- A reflection in the y-axis is described by the rule

  $(x, y) \rightarrow (-x, y)$

  or $x' = -x$ and $y' = y$
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f(-x)$ by the transformation with rule $(x, y) \rightarrow (-x, y)$.

Example 21

Determine the rule of the image when the graph of $y = \frac{1}{x^2}$ is dilated by a factor of 4:

a from the y-axis **b** from the x-axis

Solution

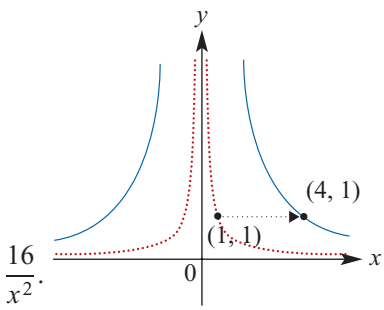
a $(x, y) \to (4x, y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = 4x$, $y' = y$.

Rearranging gives $x = \frac{x'}{4}$, $y = y'$.

Therefore $y = \frac{1}{x^2}$ becomes $y' = \frac{1}{\left(\frac{x'}{4}\right)^2}$.

So the rule of the transformed function is $y = \frac{16}{x^2}$.



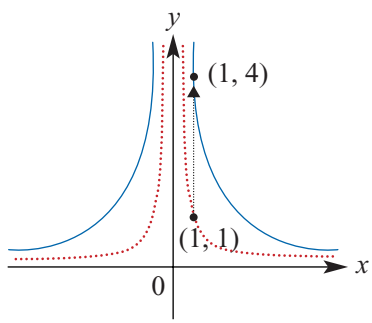
b $(x, y) \to (x, 4y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = x$, $y' = 4y$.

Rearranging gives $x = x'$, $y = \frac{y'}{4}$.

Therefore $y = \frac{1}{x^2}$ becomes $\frac{y'}{4} = \frac{1}{(x')^2}$.

So the rule of the transformed function is $y = \frac{4}{x^2}$.



Using a graphics calculator

Example 22

Use the graphics calculator to sketch the graph of the function obtained when the graph of the function with equation $y = \sqrt{x}$ is dilated by factor:

a $\sqrt{2}$ from the x-axis **b** $\frac{1}{2}$ from the y-axis

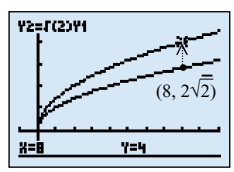
Solution

a A dilation of factor k from the x-axis transforms the function with rule $y = f(x)$ into the function with rule $y = kf(x)$.

Therefore $\quad Y1 = \sqrt{X} \qquad (= f(x))$

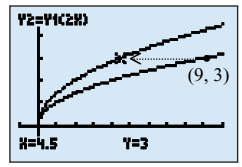
and $\quad Y2 = \sqrt{(2)}Y1 \quad (= kf(x))$

The transformed graph of Y2 is shown to the right, together with the original function.



Note: Y1 is obtained by pressing VARS and selecting the **Y-VARS** submenu, then **1:Function** and finally Y1.

b A dilation of factor k from the y-axis transforms the function with rule $y = f(x)$ into the function with rule $y = f(\frac{x}{k})$.



Therefore $Y1 = \sqrt{X}$ $(= f(x))$

and $Y2 = Y1(2X)$ $\left(= f\left(\frac{x}{k}\right)\right)$

The transformed graph of Y2 is shown to the right, together with the original function.

Note: In this example the dilation of factor $\sqrt{2}$ from the x-axis has produced the same graph as a dilation of factor $\frac{1}{2}$ from the y-axis.

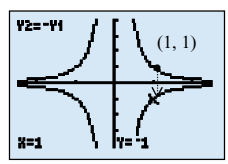
Example 23

Use the graphics calculator to sketch the graph of the function obtained when the graph of the function with equation $y = \frac{1}{x^2}$ is reflected in:

a the x-axis **b** the y-axis

Solution

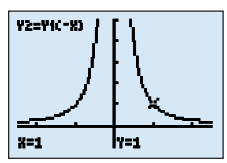
a A reflection in the x-axis transforms the function with rule $y = f(x)$ into the function with rule $y = -f(x)$.



Therefore $Y1 = 1/X^2$ $(= f(x))$

and $Y2 = -Y1$ $(= -f(x))$

b A reflection in the y-axis transforms the function with rule $y = f(x)$ into the function with rule $y = f(-x)$.



Therefore $Y1 = 1/X^2$ $(= f(x))$

and $Y2 = Y1(-X)$ $(= f(-x))$

The transformed graph of Y2 is shown to the right, together with the original function.

Note: The reflection in the y-axis has produced the graph of the original function.

Applying dilations and reflections to sketch graphs

In order to sketch the graph of $y = \sqrt{\frac{x}{2}}$ note that it is of the form $y = f\left(\frac{x}{2}\right)$, where $f(x) = \sqrt{x}$. This is the graph of $y = \sqrt{x}$ dilated by a factor 2 from the y-axis. Examples of other functions under this dilation are:

- $f(x) = x^2, \quad f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$
- $f(x) = \frac{1}{x}, \quad f\left(\frac{x}{2}\right) = \frac{1}{x/2} = \frac{2}{x}$

It should be noted that each of these functions formed by a dilation of factor 2 from the y-axis can also be formed by a dilation from the x-axis. This result is not true in general, as will be seen when new functions are introduced in chapters 12 and 13.

- $y = \sqrt{\frac{x}{2}} = \frac{1}{\sqrt{2}}\sqrt{x} = \frac{1}{\sqrt{2}} f(x)$, where $f(x) = \sqrt{x}$. That is, it is formed by a dilation of factor $\frac{1}{\sqrt{2}}$ from the x-axis.
- $y = \frac{x^2}{4} = \frac{1}{4}x^2 = \frac{1}{4} f(x)$, where $f(x) = x^2$. That is, it is formed by a dilation of factor $\frac{1}{4}$ from the x-axis.
- $y = \frac{2}{x} = 2\frac{1}{x} = 2 f(x)$, where $f(x) = \frac{1}{x}$. That is, it is formed by a dilation of factor 2 from the x-axis.

Exercise 5I

Example 21

1 Write down the equation of the rule obtained when the graph of each of the functions below is transformed by:

i a dilation of factor $\frac{1}{2}$ from the y-axis **ii** a dilation of factor 5 from the y-axis

iii a dilation of factor $\frac{2}{3}$ from the x-axis **iv** a dilation of factor 4 from the x-axis

v a reflection in the x-axis **vi** a reflection in the y-axis

a $y = x^2$ **b** $y = \frac{1}{x^2}$ **c** $y = \frac{1}{x}$ **d** $y = \sqrt{x}$

2 Sketch the graphs of each of the following:

a $y = 3\sqrt{x}$ **b** $y = -\frac{1}{x}$ **c** $y = \frac{3}{x}$ **d** $y = \frac{1}{2x}$

e $y = \sqrt{3x}$ **f** $y = \frac{3}{2x}$

5.10 Combinations of transformations

In this section sequences of transformations are applied.

For example, first consider:

- a dilation of factor 2 from the x-axis followed by
- a reflection in the x-axis.

The rule becomes $(x, y) \rightarrow (x, 2y) \rightarrow (x, -2y)$. First the dilation and then the reflection is applied.

For example, $(1, 1) \rightarrow (1, 2) \rightarrow (1, -2)$.

Another example is:

- a dilation of factor 2 from the x-axis followed by
- a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

The rule becomes $(x, y) \rightarrow (x, 2y) \rightarrow (x + 2, 2y - 3)$. First the dilation and then the translation is applied.

For example, $(1, 1) \rightarrow (1, 2) \rightarrow (3, -1)$.

Example 24

Find the equation of the image of $y = \sqrt{x}$ under:

a a dilation of factor 2 from the x-axis followed by a reflection in the x-axis

b a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

Solution

a From the discussion above, $(x, y) \rightarrow (x, 2y) \rightarrow (x, -2y)$. Hence if (x', y') is the image of (x, y) under this map, $x' = x$ and $y' = -2y$. Hence $x = x'$ and $y = \frac{y'}{-2}$. The graph of the image will have equation $\frac{y'}{-2} = \sqrt{x'}$ and hence $y' = -2\sqrt{x'}$.

b From the discussion above, $(x, y) \rightarrow (x, 2y) \rightarrow (x + 2, 2y - 3)$. Hence if (x', y') is the image of (x, y) under this map, $x' = x + 2$ and $y' = 2y - 3$. Hence $x = x' - 2$ and $y = \frac{y' + 3}{2}$. Thus the graph of the image will have equation $\frac{y' + 3}{2} = \sqrt{x' - 2}$ or $y' = 2\sqrt{x' - 2} - 3$.

Determining transformations

The method that has been used to find the effect of transformations can be reversed to determine the sequence of transformations used to take a graph to its image. For the example above, in order to find the sequence of transformations which map $y = \sqrt{x}$ to $y' = -2\sqrt{x'}$, work backwards through the steps in the solution.

$y = \sqrt{x}$ maps to $\frac{y'}{-2} = \sqrt{x'}$

Hence $x = x'$ and $y = \frac{y'}{-2}$.

Therefore $x' = x$ and $y' = -2y$.

The transformation is a dilation of factor 2 from the x-axis followed by a reflection in the x-axis.

This can also be done by inspection, of course, as you recognise the form of the image. For the combination of transformations in this course it is often simpler to do this.

Example 25

Find a sequence of transformations which take the graph of $y = x^2$ to the graph of $y = 2(x - 2)^2 + 3$.

Solution

By inspection

By inspection it is a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

By the method

$$y = x^2 \text{ maps to } y' = 2(x' - 2)^2 + 3$$

Rearranging the expression on the right

$$\frac{y' - 3}{2} = (x' - 2)^2$$

It can be seen that $y = \frac{y' - 3}{2}$ and $x = x' - 2$. Solving for x' and y' gives $y' = 2y + 3$ and $x' = x + 2$.

The transformation is a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

Exercise 5J

Example 24

1 Find the equation of the image of the graph $y = \sqrt{x}$ when each of the following sequences of transformations have been applied:

- **a** a translation of 2 units in the positive direction of the x-axis followed by a dilation of factor 3 from the x-axis
- **b** a translation of 3 units in the negative direction of the x-axis followed by a reflection in the x-axis
- **c** a reflection in the x-axis followed by a dilation of factor 3 from the x-axis
- **d** a reflection in the x-axis followed by a dilation of factor 2 from the y-axis

e a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

f a dilation of factor 2 from the y-axis followed by a translation of 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis

2 Repeat Question 1 for $y = \frac{1}{x}$.

Example 25

3 For each of the following find a sequence of transformations that take:

a the graph of $y = x^2$ to the graph of

i $y = 2(x-1)^2 + 3$ **ii** $y = -(x+1)^2 + 2$ **iii** $y = (2x+1)^2 - 2$

b the graph of $y = \frac{1}{x}$ to the graph of

i $y = \frac{2}{x+3}$ **ii** $y = \frac{1}{x+3} + 2$ **iii** $y = \frac{1}{x-3} - 2$

c the graph of $y = \sqrt{x}$ to the graph of

i $y = \sqrt{x+3} + 2$ **ii** $y = 2\sqrt{3x}$ **iii** $y = -\sqrt{x} + 2$

5.11 Functions and modelling exercises

Example 26

A householder has six laying hens and wishes to construct a rectangular enclosure to provide a maximum area for the hens, using a 12 m length of fencing wire. Construct a function that will give the area of the enclosure, A, in terms of the length, l. By sketching a graph find the maximum area that can be fenced.

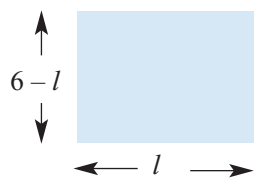
Solution

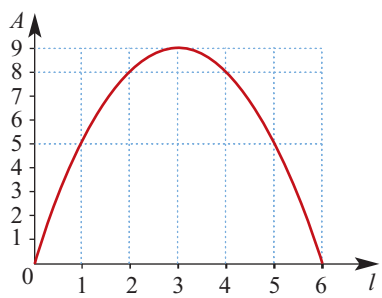
If l = length of the enclosure

$$\text{then width} = \frac{12 - 2l}{2}$$
$$= 6 - l$$
$$\therefore \text{ area } A(l) = l(6 - l)$$
$$= 6l - l^2$$

The domain of A is the interval $(0, 6)$.

The maximum area is 9 m^2 and occurs when $l = 3$ m, i.e. the enclosure is a square.





Example 27

The following list shows Australia Post airmail rates for articles:

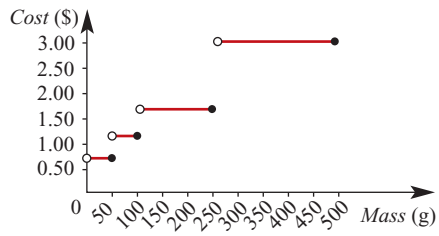
Mass (m, g)	Cost (C, \$)
Up to 50 g	\$0.70
Over 50 g up to 100 g	\$1.15
Over 100 g up to 250 g	\$1.70
Over 250 g up to 500 g	\$3.00

Sketch a graph of the cost function, C, giving its domain and range and the rules that define it.

Solution

The rules are $C = 0.70$ for $0 < m \leq 50$
$= 1.15$ for $50 < m \leq 100$
$= 1.70$ for $100 < m \leq 250$
$= 3.00$ for $250 < m \leq 500$

The graph is as follows:



Range $= \{0.70, 1.15, 1.70, 3.00\}$

Example 28

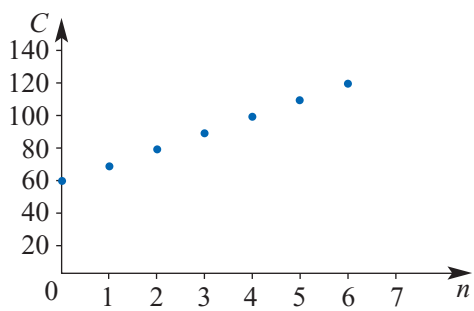
A book club has a membership fee of \$60.00 and each book purchased is \$10.00. Construct a cost function that can be used to determine the cost of different numbers of books, then sketch its graph.

Solution

Let C denote the cost (in dollars) and n denote the number of books purchased, then

$$C = 60 + 10n$$

The domain of this function is $Z^+ \cup \{0\}$, the set of positive integers, and its graph will be as shown.



The range of this function is $\{x: x \in Z \text{ and } x \geq 60\}$. Sometimes to simplify the situation we represent such functions by a continuous line. Strictly, this is not mathematically correct but may aid our understanding of the situation.

Exercise 5K

Example 26

1 **a** **i** Find an expression for the area A in terms of x and y.

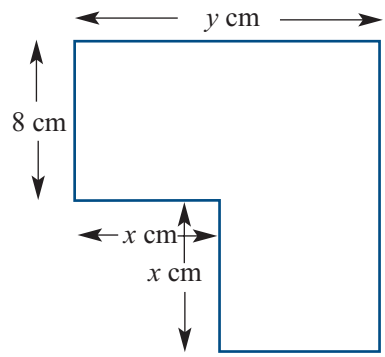
ii Find an expression for the perimeter P in terms of x and y.

b **i** If $P = 64$ cm, find A in terms of x.

ii Find the allowable values for x.

iii Sketch the graph of A against x for these values.

iv What is the maximum area?



Example 27

2 Suppose Australia Post charged the following rates for airmail letters to Africa: \$1.20 up to 20 g; \$2.00 over 20 g and up to 50 g; \$3.00 over 50 g and up to 150 g.

a Write a cost function, C (\$), in terms of the mass, m (g) for letters up to 150 g.

b Sketch the graph of the function, stating the domain and range.

3 Telenet listed the following scale of charges for a 3 minute STD call between the hours of 6 pm and 10 pm on Monday to Friday.

Distance, d (km)	Up to 25 km (not including 25)	25 up to 50 km (not including 50)	50 up to 85 km (not including 85)	85 up to 165 km (not including 165)	165 up to 745 km (not including 745)	745 km and over
Cost, C (\$)	0.30	0.40	0.70	1.05	1.22	1.77

a Write a cost function, C (\$), in terms of distance, d (km).

b Sketch the graph of the function.

4 Self-Travel, a car rental firm, has two methods of charging for car rental:

i Method 1: \$64 per day + 25 cents per kilometre

ii Method 2: \$89 per day with unlimited travel

a Write a rule for each method if C_1 is the cost, in \$, using method 1 for x kilometres travelled, and C_2 is the cost using method 2.

b Draw a graph of each rule the same axes.

c Determine, from the graph, the distance which must be travelled per day if method 2 is cheaper than method 1.

Review

Chapter summary

- **Set notation**

  $x \in A$ x is an element of A

  $x \notin A$ x is not an element of A

  $A \subseteq B$ A is a subset of B

  $A \cap B$ $x \in A \cap B$ if and only if $x \in A$ and $x \in B$

  $A \cup B$ $x \in A \cup B$ if and only if $x \in A$ or $x \in B$

  $A \setminus B$ $\{x: x \in A, x \notin B\}$

- **Sets of numbers**

  N Natural numbers

  Z Integers

  Q Rational numbers

  R Real numbers

- **Interval notation**

  $(a, b) = \{x: a < x < b\}$ $[a, b] = \{x: a \leq x \leq b\}$

  $(a, b] = \{x: a < x \leq b\}$ $[a, b) = \{x: a \leq x < b\}$

  $(a, \infty) = \{x: a < x\}$ $[a, \infty) = \{x: a \leq x\}$

  $(-\infty, b) = \{x: x < b\}$ $(-\infty, b] = \{x: x \leq b\}$

- A **relation** is a set of ordered pairs.
  The **domain** of a relation is the set of all first elements of the ordered pairs of the relation.
  The **range** of a relation is the set of all second elements of the ordered pairs of the relation.
- For a function f and an element x of the domain of f there is unique element y in the range such that $(x, y) \in f$. The element y is called the **image** of x under f.
  If $(x, y) \in f$, then x is called a **pre-image** of y.
- **One-to-one function**
  A function is said to be one-to-one if, for $a, b \in \text{dom } f$, $a \neq b$, then $f(a) \neq f(b)$.
  In other words f is called one-to-one if every image under f has a unique pre-image.
- **Implied domain**
  The implied domain of a function is the largest subset of R for which the rule is defined.
- **Restrictions of a function**
  For a function f with domain D a new function g may be defined with domain $A \subseteq D$ and rule defined by $g(x) = f(x)$ for all $x \in A$. The function g is called a **restriction** of f.

Inverses of functions

If f is a one-to-one function then for each number y in the range of f there is exactly one number x in the domain of f such that $f(x) = y$.

Thus if f is a one-to-one function, a new function f^{-1}, called the inverse of f, may be defined by:

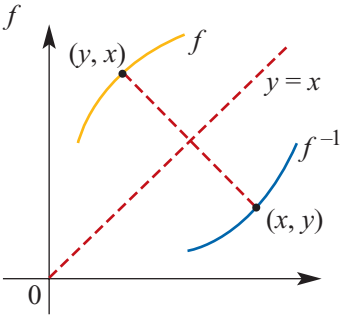
$$f^{-1}(x) = y, \text{ if } f(y) = x, \text{ for } x \in \text{ran } f, y \in \text{dom } f$$

The point (x, y) is on the graph of f^{-1} if the point (y, x) is on the graph of f.

Therefore to get the graph of f^{-1} from the graph of f, the graph of f is to be reflected in the line $y = x$.

From this the following is evident:

$$\text{dom } f^{-1} = \text{ran } f$$
$$\text{ran } f^{-1} = \text{dom } f$$



Transformations of the graphs of functions

In general, a **translation** of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule $(x, y) \to (x + h, y + k)$ or $x' = x + h$ and $y' = y + k$, where h and k are positive numbers.

For **translations** in the negative direction of the x-axis, $(x, y) \to (x - h, y - k)$ or $x' = x - h$ and $y' = y - k$, where h and k are positive numbers.

In general:

- The curve with equation $y = f(x)$ is mapped to the curve with equation $y - k = f(x - h)$ by the **translation** with rule $(x, y) \to (x + h, y + k)$.
- A **dilation of a units from the x-axis** is described by the rule $(x, y) \to (x, ay)$ or $x' = x$ and $y' = ay$, where a is a positive number.
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = af(x)$ by the **dilation** with rule $(x, y) \to (x, ay)$.
- A **dilation of a units from the y-axis** is described by the rule $(x, y) \to (ax, y)$ or $x' = ax$ and $y' = y$, where a is a positive number.
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f\left(\frac{x}{a}\right)$ by the **dilation** with rule $(x, y) \to (ax, y)$.
- A **reflection in the x-axis** is described by the rule $(x, y) \to (x, -y)$ or $x' = x$ and $y' = -y$.
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = -f(x)$ by the **reflection** with rule $(x, y) \to (x, -y)$.
- A **reflection in the y-axis** is described by the rule $(x, y) \to (-x, y)$ or $x' = -x$ and $y' = y$.
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f(-x)$ by the **reflection** with rule $(x, y) \to (-x, y)$.

Review

Multiple-choice questions

1 For $f(x) = 10x^2 + 2$, $f(2a)$ equals

A $20a^2 + 2$ **B** $40a^2 + 2$ **C** $2a^2 + 2a$ **D** $100a^2 + 2$ **E** $10a^2 + 2a$

2 The maximal domain of the function f with rule $f(x) = \sqrt{3x + 5}$ is

A $(0, \infty)$ **B** $\left(-\frac{5}{3}, \infty\right)$ **C** $(5, \infty)$ **D** $[-5, \infty)$ **E** $\left[-\frac{5}{3}, \infty\right)$

3 The range of the relation $x^2 + y^2 > 9$ when $x, y \in R$ is

A $[0, \infty)$ **B** R **C** $(-\infty, 0]$ **D** $(3, \infty) \cup (-\infty, -3)$ **E** $(-\infty, 0)$

4 For $f(x) = 7x - 6$, $f^{-1}(x)$ equals

A $7x + 4$ **B** $\frac{1}{7}x + 6$ **C** $\frac{1}{7}x + \frac{6}{7}$ **D** $\frac{1}{7x - 6}$ **E** $\frac{1}{7}x - 6$

5 For $f: (a, b] \to R$, $f(x) = 3 - x$

A $(3 - a, 3 - b)$ **B** $(3 - a, 3 - b]$ **C** $(3 - b, 3 - a)$

D $(3 - b, 3 - a]$ **E** $[3 - b, 3 - a)$

6 Which of the following functions is not one-to-one?

A $f(x) = 9 - x^2, x \geq 0$ **B** $f(x) = \sqrt{9 - x^2}$ **C** $f(x) = 1 - 9x$

D $f(x) = \sqrt{x}$ **E** $f(x) = \frac{9}{x}$

7 The graph of $y = \frac{2}{x} + 3$ is reflected in the x-axis and then in the y-axis. The equation of the final image is

A $y = -\frac{2}{x} + 3$ **B** $y = -\frac{2}{x} - 3$ **C** $y = \frac{2}{x} + 3$

D $y = \frac{2}{x} - 3$ **E** $y = 2x - 3$

8 The sequence of transformations which takes the graph of $y = x^2$ to the graph of $y = -(2x - 6)^2 + 4$ is

A a reflection in the y-axis followed by a dilation of $\frac{1}{2}$ from the y-axis and then a translation of 3 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis

B a reflection in the y-axis followed by a dilation of $\frac{1}{2}$ from the y-axis and then a translation of 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

C a reflection in the y-axis followed by a dilation of 2 from the y-axis and then a translation of 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

D a reflection in the x-axis followed by a dilation of $\frac{1}{2}$ from the y-axis and then a translation of 3 units in the negative direction of the x-axis and 4 units in the positive direction of the y-axis

E a reflection in the x-axis followed by a dilation of $\frac{1}{2}$ from the y-axis and then a translation of 3 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis

9 For $f: [-1, 5) \to R,\ f(x) = x^2$, the range is

A R **B** $[0, \infty)$ **C** $[0, 25)$ **D** $[1, 25]$ **E** $[0, 5]$

10 Which of the following rules does **not** describe a function?

A $y = x^2 - x$ **B** $y = \sqrt{4 - x^2}$ **C** $y = 3, x > 0$ **D** $x = 3$ **E** $y = 3x$

Short-answer questions (technology-free)

1 If f is the function with rule $f(x) = 2 - 6x$, find:

a $f(3)$ **b** $f(-4)$ **c** the value of x for which f maps x to 6

2 For $[-1, 6] \to R,\ f(x) = 6 - x$:

a sketch the graph of f **b** state the range of f

3 Sketch the graphs of each of the following, stating the range of each:

a $\{(x, y): 3x + y = 6\}$ **b** $\{(x, y): y = 3x - 2, x \in [-1, 2]\}$

c $\{(x, y): y = x^2, x \in [-2, 2]\}$ **d** $\{(x, y): y = 9 - x^2\}$

e $\{(x, y): y = x^2 + 4x + 6\}$ **f** $\{(1, 2)\ (3, 4)\ (2, -6)\}$

g $f: R \to R,\ f(x) = (x - 2)^2$ **h** $f: R \setminus \{0\} \to R,\ f(x) = \dfrac{1}{x} + 2$

i $\left(x - \dfrac{1}{2}\right)^2 + (y + 2)^2 = 9$ **j** $f: [-1, 3] \to R,\ f(x) = x$

4 The function f has rule $f(x) = \dfrac{a}{x} + b$ such that $f(1) = \dfrac{3}{2}$ and $f(2) = 9$.

a Find the values of a and b. **b** State the implied domain of f.

5 Given that $f: [0, 2] \to R,\ f(x) = 2x - x^2$:

a sketch the graph **b** state the range

6 Given that $f(x) = ax + b$, $f(5) = 10$ and $f(1) = -2$, find the values of a and b.

7 Given that $f(x) = ax^2 + bx + c$, $f(0) = 0$, $f(4) = 0$ and $f(-2) = -6$, find the values of a, b and c.

8 State the implied (maximal) domain for each of the following:

a $y = \dfrac{1}{x - 2}$ **b** $f(x) = \sqrt{x - 2}$ **c** $y = \sqrt{25 - x^2}$

d $f(x) = \dfrac{1}{2x - 1}$ **e** $g(x) = \sqrt{100 - x^2}$ **f** $h(x) = \sqrt{4 - x}$

9 State which of the following functions are one-to-one:

a $y = x^2 + 2x + 3$ **b** $f: [2, \infty) \to R, f(x) = (x - 2)^2$

c $f(x) = 3x + 2$ **d** $f(x) = \sqrt{x - 2}$

e $f(x) = \dfrac{1}{x - 2}$ **f** $f: [-1, \infty) \to R, f(x) = (x + 2)^2$

g $f: [-3, 5] \to R, f(x) = 3x - 2$ **h** $f(x) = 7 - x^2$

i $f(x) = \dfrac{1}{(x - 2)^2}$ **j** $h(x) = \dfrac{1}{x - 2} + 4$

Review

10 Sketch the graphs of each of the following:

a $f(x) = \begin{cases} 3x - 1, & x \in [0, \infty) \\ x^2, & x \in [-3, 0) \\ 9, & x \in (-\infty, -3) \end{cases}$ **b** $h(x) = \begin{cases} 1 - 2x, & x \in [0, \infty) \\ x^2, & x \in [-3, 0) \\ -x^2, & x \in (-\infty, -3) \end{cases}$

11 For each of the following find the inverse function, stating the rule and the domain:

a $f: [-1, 5] \to R, f(x) = 3x - 2$ **b** $f: [-2, \infty) \to R, f(x) = \sqrt{x + 2} + 2$

c $f: [-1, \infty) \to R, f(x) = 3(x + 1)^2$ **d** $f: (-\infty, 1) \to R, f(x) = (x - 1)^2$

12 For the function f with rule $f(x) = \sqrt{x}$, find the equation for the graph of the image under each of the following transformations:

a a translation of 2 in the positive direction of the x-axis and 3 in the positive direction of the y-axis

b a dilation of factor 2 from the x-axis **c** a reflection in the x-axis

d a reflection in the y-axis **e** a dilation of factor 3 from the y-axis

Extended-response questions

1 An Easyride coach leaves town X and maintains a constant speed of 80 km/h for 4 hours, stops at town Y for $\frac{3}{4}$ hour before travelling for a further $2\frac{1}{2}$ hours at 80 km/h to its destination at town Z. A second coach leaves town Z at the same time and runs express to town X, completing its journey in $5\frac{1}{2}$ hours.

a Construct functions that describe the distance, d km, from X of each coach at time t, stating the domain, range and rule for each.

b Calculate the distance, from X, at which the two coaches pass each other.

2 A parking meter is designed to accept 200 twenty-cent coins.

a Write a rule which gives the number of hours parking, P hours, in terms of n, the number of twenty-cent coins inserted, when the cost of parking is 20c for each half hour.

b Sketch the function, stating the domain and range.

3 The following table shows the amount of taxation payable for a given range of annual income from 1 July 1986.

Taxable income, $	Tax	
\$1 to \$ $(C - 1)$	Nil	
C to 12 499	Nil	plus 24.42 cents for each \$1 over C
\$12 500 to 12 599	D	plus 26.50 cents for each \$1 over 12 500
\$12 600 to 19 499	D + \$26.50	plus 29.42 cents for each \$1 over 12 600
\$19 500 to 27 999	D + \$2056.48	plus 44.25 cents for each \$1 over \$ 19 500
\$28 000 to 34 999	D + \$5817.73	plus 46.83 cents for each \$1 over \$28 000
\$35 000 and over	D + \$9095.83	plus 57.08 cents for each \$1 over \$35 000

C is an amount called the reduced taxfree threshold
and $D = 24.42\%$ of $(12\,500 - C)$
$= 0.2442\,(12\,500 - C)$

a If for a particular taxpayer C is \$4223, write a rule for the tax payable, T (\$), for a taxable income, x (\$), where $29\,000 \leq x \leq 33\,000$.

b Sketch a graph of this function, stating the range.

c Calculate the amount of tax payable on an income of \$30 000.

4 The Exhibition Centre hires a graphics company to produce a poster for an exhibit. The graphics company charges \$1000 and an additional \$5 for each poster produced.

a **i** Write the rule for a function, $C(n)$, which describes the cost to the Exhibition Centre of obtaining n posters.

ii Sketch the graph of C against n (use a continuous model).

b The Exhibition Centre is going to sell the posters for \$15.00 each.

i Write down the rule for the function $P(n)$ which gives the profit when the Exhibition Centre sells n posters.

ii Sketch the graph of this function (use a continuous model).

5 An article depreciates by 5% of its original cost each year. If the original cost was \$8000, find an expression for the value, V, of the item n years after purchase.

6 The organisers of a sporting event know that, on average, 50 000 people will visit the venue each day. They are presently charging \$15.00 for an admission ticket. Each time in the past when they have raised the admission price an average of 2500 fewer people have come to the venue for each \$1.00 increase in ticket price. Let x represent the number of \$1.00 increases.

a Write the rule for a function which gives the revenue, R, in terms of x.

b Sketch the graph of R against x.

c Find the price which will maximise the revenue.

7 A thin wire of length a cm is bent to form the perimeter of a pentagon $ABCDE$ in which $BCDE$ is a rectangle, and ABE is an equilateral triangle. Let x cm be the length of CD and $A(x)$ be the area of the pentagon.

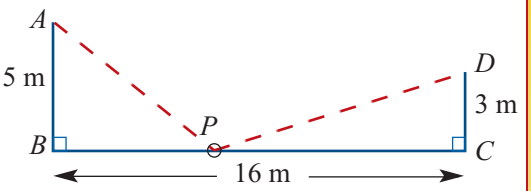
a Find $A(x)$ in terms of x.

b State the allowable values for x.

c Show that the maximum area $= \dfrac{a^2}{4(6 - \sqrt{3})}$ cm^2.

Review

8 Let P be a point between B and C on line BC.



Let $d(x)$ be the distance $(PA + PD)$ m, where x is the distance of P from B.

a **i** Find an expression for $d(x)$.

ii Find the allowable values of x.

b **i** Use a graphics calculator to plot the graph of $y = d(x)$ for a suitable window setting.

ii Find the value of x if $d(x) = 20$ (correct to 2 decimal places).

iii Find the values of x for which $d(x) = 19$ (correct to 2 decimal places).

c **i** Find the minimum value of $d(x)$ and the value of x for which this occurs.

ii State the range of the function.

9 **a** Find the coordinates of $A(x_1, y_1)$, and $B\ (x_1, y_1)$.

b Let $d(x)$ be the 'vertical' distance between the graphs for $x \in [x_2, x_1]$.

Graphics Calculator

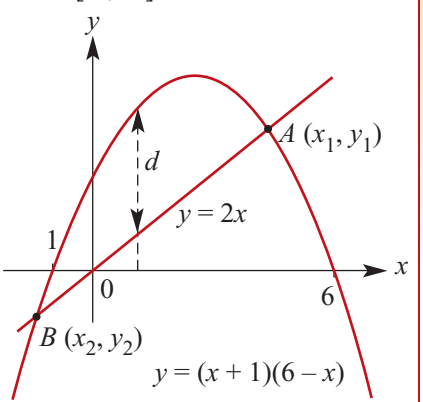
i Find $d(x)$ in terms of x.

ii Plot the graph of $d(x)$ against x for $x \in [x_1, x_2]$ and on the same screen the graphs of $y = 2x$ and $y = (x + 1)(6 - x)$.

c **i** State the maximum value of the function defined by $d(x)$ for $x \in [x_2, x_1]$.

ii State the range of this function.

d Repeat with the graphs $y = 5x$ and $y = (x + 1)(6 - x)$.



CHAPTER

6

Cubic and Quartic Functions

Objectives

- To recognise and sketch the graphs of **cubic and quartic functions**.
- To **divide** polynomials.
- To use the **remainder theorem** and the **factor theorem** to solve cubic equations.
- To find equations for given cubic graphs.
- To apply cubic and quartic functions to solving problems.
- To use **finite difference tables** to find rules of sequences generated by polynomial functions.

In Chapter 3 we looked at second degree polynomials or quadratics.

A third degree polynomial is called a cubic and is a function, f, with rule

$$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$$

A fourth degree polynomial is called a quartic and is a function, f, with rule

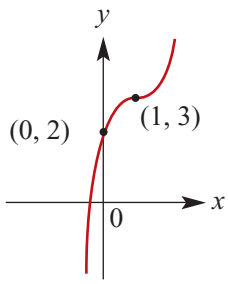
$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$$

In Chapter 3 it was shown that all quadratic functions could be written in 'perfect square' form and that the graph of a quadratic has one basic form, the parabola.

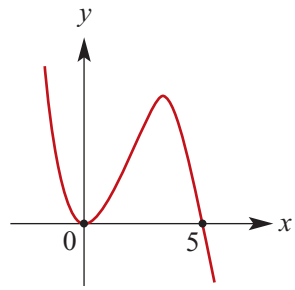
This is not true of cubic or quartic functions.

Two examples of graphs of cubic functions and two examples of quartic functions are shown.

Cubic functions

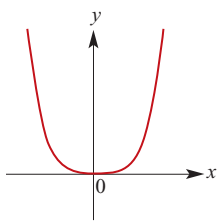


$f(x) = (x - 1)^3 + 3$
$= x^3 - 3x^2 + 3x + 2$

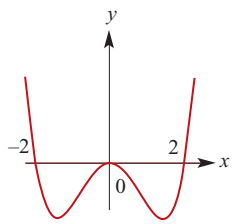


$f(x) = x^2(5 - x)$
$= -x^3 + 5x^2$

Quartic functions



$y = 2x^4$



$y = x^4 - 4x^2 = x^2(x^2 - 4)$

6.1 Functions of the form $f: R \to R$, $f(x) = a(x - h)^n + k$

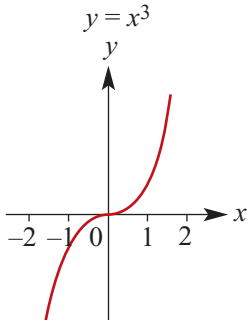
Cubic functions of this form

The graph of $f(x) = (x - 1)^3 + 3$ is obtained from the graph of $y = x^3$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

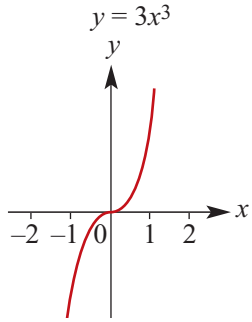
As with other graphs it has been seen that changing a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$ the graph is inverted.

For example:

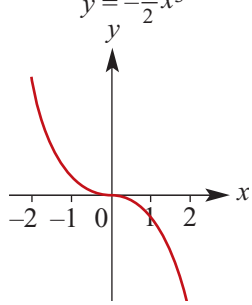
$y = x^3$



$y = 3x^3$



$y = -\frac{1}{2}x^3$



The significant feature of the graph of cubics of this form is the **point of inflexion** (a point of zero gradient).

The point of inflexion of $y = x^3$ is at the origin (0, 0).

It should be noted that the implied **domain** of all cubics is $\boldsymbol{R}$ and the **range** is also $\boldsymbol{R}$.

Translations of basic graphs will be considered in the same way as other types of graphs seen in chapters 3, 4 and 5.

Vertical translations

By adding or subtracting a constant term to $y = x^3$, the graph moves either 'up' or 'down'.

$y = x^3 + k$ is the basic graph moved k units up ($k > 0$). The point of inflexion becomes $(0, k)$. In this case the graph of $y = x^3$ is translated k units in the positive direction of the y-axis.

Horizontal translations

The graph of $y = (x - h)^3$ is simply the basic graph moved h units to the 'right' for $h > 0$.

The point of inflexion is at $(h, 0)$. In this case the graph of $y = x^3$ is translated h units in the positive direction of the x-axis.

The general form of cubics of this form is

$$y = a(x - h)^3 + k$$

The point of inflexion is at (h, k).

When sketching cubic graphs which are of the form $y = a(x - h)^3 + k$, first identify the point of inflexion. To add further detail to the graph, the x-axis and y-axis intercepts are found.

Example 1

Sketch the graph of the function $y = (x - 2)^3 + 4$.

Solution

The graph of $y = x^3$ is translated 2 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.

Point of inflexion is (2, 4).

x-axis intercept:

let $y = 0$

$0 = (x - 2)^3 + 4$

$-4 = (x - 2)^3$

$\sqrt[3]{-4} = x - 2$

$x = 2 + \sqrt[3]{-4}$

≈ 0.413

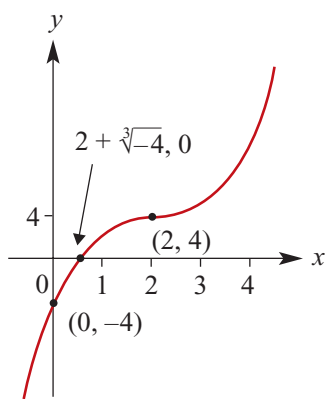
y-axis intercept:

let $x = 0$

$y = (0 - 2)^3 + 4$

$y = -8 + 4$

$y = -4$

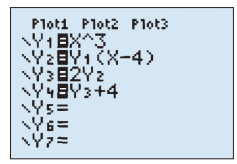


Using a graphics calculator

Plot the graphs of $y = x^3$, $y = (x - 4)^3$, $y = 2(x - 4)^3$ and $y = 2(x - 4)^3 + 4$. This procedure was introduced in sections 5.8 and 5.9.

This can be done by building up.

Y4 = Y3 + 4 yields the graph of $y = 2(x - 4)^3 + 4$.



The process is:

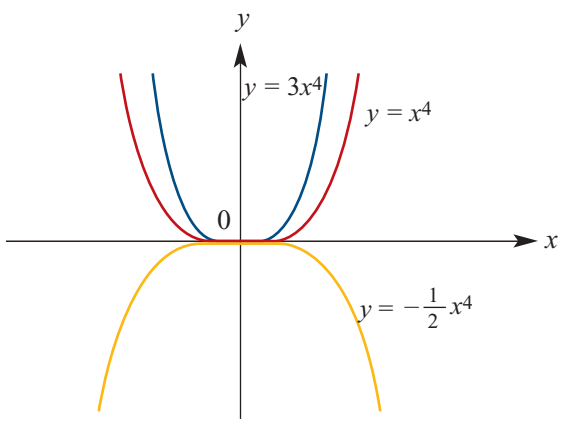
x → [-4] → $x - 4$ → [cube] → $(x - 4)^3$ → [$\times 2$] → $2(x - 4)^3$ → [$+4$] → $2(x - 4)^3 + 4$

All functions of the form $a(x - h)^3 + k$ can be built up in this way.

Quartic graphs

The graph of $f(x) = (x - 1)^4 + 3$ is obtained from the graph of $y = x^4$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

As with other graphs it has been seen that changing a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$ the graph is inverted.



The significant feature of the graph of quartics of this form is the **turning point** (a point of zero gradient).

The turning point of $y = x^4$ is at the origin (0, 0).

It should be noted that the implied **domain** of all quartics is R, but unlike cubics the range is not R.

Vertical translations

By adding or subtracting a constant term to $y = x^4$, the graph moves either up or down.

$y = x^4 + k$ is the basic graph moved k units up ($k > 0$). The turning point becomes $(0, k)$. The graph of $y = x^4$ is translated k units in the positive direction of the y-axis.

Horizontal translations

The graph of $y = (x - h)^4$ is simply the basic graph moved h units to the right for $h > 0$.

The turning point is at $(h, 0)$. The graph of $y = x^4$ is translated h units in the positive direction of the x-axis.

The general form of quartics of this form is

$$y = a(x - h)^4 + k$$

The turning point is at (h, k).

When sketching quartic graphs of the form $y = a(x - h)^4 + k$, first identify the turning point. To add further detail to the graph, the x-axis and y-axis intercepts are found.

Example 2

Sketch the graph of the function $y = (x - 2)^4 - 1$.

Solution

Turning point is $(2, -1)$.

x-axis intercept:

let $y = 0$

$0 = (x - 2)^4 - 1$

$1 = (x - 2)^4$

$\pm\sqrt[4]{1} = x - 2$

$x = 2 + 1$ or $x = 2 - 1$

$x = 3$ or $x = 1$

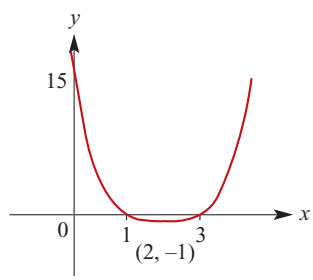
y-axis intercept:

let $x = 0$

$y = (0 - 2)^4 - 1$

$y = 16 - 1$

$y = 15$



Power functions of higher degree

Even degree

Functions with rules $f(x) = x^2$ and $f(x) = x^4$ are examples of even degree functions.

The following are properties of all even degree power functions:

- $f(-x) = f(x)$ for all x
- $f(0) = 0$
- As $x \to \pm\infty$, $y \to \infty$

Note that m and n are positive even integers and if $m > n$ then:

- $x^m > x^n$ for $x > 1$ or $x < -1$
- $x^m < x^n$ for $-1 < x < 1$ but x not 0
- $x^m = x^n$ for $x = 1$ or $x = -1$ or $x = 0$

Odd degree

The functions with rules $f(x) = x^3$ and $f(x) = x$ are examples of odd degree functions.

The following are properties of all odd degree power functions:

- $f(-x) = -f(x)$ for all x
- $f(0) = 0$
- As $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$

Note that m and n are positive odd integers and if $m > n$ then:

- $x^m > x^n$ for $x > 1$ and $-1 < x < 0$
- $x^m < x^n$ for $x < -1$ and $0 < x < 1$
- $x^m = x^n$ for $x = 1$ or $x = -1$ or $x = 0$

Exercise 6A

Example 1

1 Using the method of horizontal and vertical translations, sketch the graphs of each of the following:

a $y = (x + 2)^3 - 1$ **b** $y = (x - 1)^3 - 1$ **c** $y = (x + 3)^3 + 2$
d $y = (x - 2)^3 + 5$ **e** $y = (x + 2)^3 - 5$

2 Sketch the graphs of the following functions:

a $y = 2x^3 + 3$ **b** $y = 2(x - 3)^3 + 2$ **c** $3y = x^3 - 5$
d $y = 3 - x^3$ **e** $y = (3 - x)^3$ **f** $y = -2(x + 1)^3 + 1$
g $y = \frac{1}{2}(x - 3)^3 + 2$

Example 2

3 Using the method of horizontal and vertical translations, sketch the graphs of each of the following:

a $y = (x + 2)^4 - 1$ **b** $y = (x - 1)^4 - 1$ **c** $y = (x + 3)^4 + 2$
d $y = (x - 2)^4 + 5$ **e** $y = (x + 2)^4 - 5$

4 Sketch the graphs of the following functions:

a $y = 2x^4 + 3$ **b** $y = 2(x - 3)^4 + 2$ **c** $y = x^4 - 16$
d $y = 16 - x^4$ **e** $y = (3 - x)^4$ **f** $y = -2(x + 1)^4 + 1$

5 Use a graphics calculator to compare the graphs of higher degree power functions.

6.2 Division of polynomials

Not all cubics can be written in the form $y = a(x - h)^3 + k$.

When sketching the graphs of cubics which are not of the form $y = a(x - h)^3 + k$ begin by finding the x-axis intercepts. All cubics will have at least one x-axis intercept. Some will have two and others will have three.

As with quadratics, finding x-axis intercepts can be done by factorising and solving the resulting equation using the null factor law.

We shall first look at the techniques required for factorising cubics.

Reviewing the process of long division, for example $172 \div 13$, gives

$$
\begin{array}{r}
13 \\
13\overline{)172} \\
\underline{13} \\
42 \\
\underline{39} \\
3
\end{array}
$$

$\therefore \dfrac{172}{13} = 13$ and 3 remainder, i.e $13\frac{3}{13}$

We note that as there is a remainder, it can be seen that 13 is not a factor of 172.

The process of dividing a polynomial by a linear factor follows very similar steps.

For example, $x^2 + 7x + 11 \div (x - 2)$ gives

$$
\begin{array}{rl}
x + 9 & \\
x - 2\overline{)x^2 + 7x + 11} & \text{Divide } x^2 \text{ by } x. \\
\underline{x^2 - 2x} & \text{Multiply } (x - 2) \text{ by } x \text{ and subtract from } x^2 + 7x + 11. \\
9x + 11 & \text{This leaves } 9x + 11,\ x \text{ into } 9x \text{ goes 9 times.} \\
\underline{9x - 18} & \text{Multiply } (x - 2) \text{ by 9 and subtract from } 9x + 11. \\
29 & \text{This leaves 29 remainder.}
\end{array}
$$

Thus $x^2 + 7x + 11 \div (x - 2) = x + 9$ with remainder 29.

$\therefore \dfrac{x^2 + 7x + 11}{x - 2} = x + 9 + \dfrac{29}{x - 2}$

We can see in this example that $x - 2$ is *not* a factor of $x^2 + 7x + 11$.

Example 3

Divide $x^3 + x^2 - 14x - 24$ by $x + 2$.

Solution

$$
\begin{array}{r}
x^2 - x - 12 \\
x + 2\overline{)x^3 + x^2 - 14x - 24} \\
\underline{x^3 + 2x^2} \\
-x^2 - 14x \\
\underline{-x^2 - 2x} \\
-12x - 24 \\
\underline{-12x - 24} \\
0
\end{array}
$$

In this example we see that $x + 2$ *is* a factor of $x^3 + x^2 - 14x - 24$, as the remainder is zero.

Thus $x^3 + x^2 - 14x - 24 \div (x + 2) = x^2 - x - 12$ with zero remainder.

$\therefore \dfrac{x^3 + x^2 - 14x - 12}{x + 2} = x^2 - x - 12$

Example 4

Divide $3x^3 + x - 3$ by $x - 2$.

Solution

Note: Here there is no term in x^2, however we can rewrite the polynomial as $3x^3 + 0x^2 + x - 3$.

$$
\begin{array}{r}
3x^2 + 6x + 13 \\
x - 2 \overline{)\,3x^3 + 0x^2 + x - 3} \\
\underline{3x^3 - 6x^2} \\
6x^2 + x \\
\underline{6x^2 - 12x} \\
13x - 3 \\
\underline{13x - 26} \\
23
\end{array}
$$

Thus $\dfrac{3x^3 + x - 3}{x - 2} = 3x^2 + 6x + 13$ with a remainder of 23

$$= 3x^2 + 6x + 13 + \frac{23}{x - 2}$$

Example 5

Divide $3x^3 + 2x^2 - x - 2$ by $2x + 1$.

Solution

$$
\begin{array}{r}
\frac{3}{2}x^2 + \frac{1}{4}x - \frac{5}{8} \\
2x + 1 \overline{)\,3x^3 + 2x^2 - x - 2} \\
\underline{3x^3 + \frac{3}{2}x^2} \\
\frac{1}{2}x^2 - x \\
\underline{\frac{1}{2}x^2 + \frac{1}{4}x} \\
-\frac{5}{4}x - 2 \\
\underline{-\frac{5}{4}x - \frac{5}{8}} \\
-1\frac{3}{8}
\end{array}
$$

$$\therefore \frac{3x^3 + 2x^2 - x - 2}{2x + 1} = \frac{3x^2}{2} + \frac{x}{4} - \frac{5}{8} - \frac{11}{8(2x + 1)}$$

Exercise 6B

1 For each of the following divide the polynomial by the accompanying linear expression:

Example 3 Example 4

a $x^3 + x^2 - 2x + 3, x - 1$ **b** $2x^3 + x^2 - 4x + 3, x + 1$

c $3x^3 - 4x^2 + 2x + 1, x + 2$ **d** $x^3 + 3x - 4, x + 1$

e $2x^3 - 3x^2 + x - 2, x - 3$ **f** $2x^3 + 3x^2 + 17x + 15, x + 4$

g $x^3 + 4x^2 + 3x + 2, x + 3$

Example 5 **2** For each of the following divide the polynomial by the accompanying linear expression:

a $x^3 + 6x^2 + 8x + 11, 2x + 5$ **b** $2x^3 + 5x^2 - 4x - 5, 2x + 1$

c $x^3 - 3x^2 + 1, 3x - 1$ **d** $x^3 - 3x^2 + 6x + 5, x - 2$

e $2x^3 + 3x^2 - 32x + 15, 2x - 1$ **f** $x^3 + 2x^2 - 1, 2x + 1$

3 a Write $\dfrac{x^3 + 2x^2 + 5x + 1}{x - 1}$ in the form $P(x) + \dfrac{a}{x - 1}$, where $P(x)$ is a quadratic expression and a is a real number.

b Write $\dfrac{2x^3 - 2x^2 + 5x + 3}{2x - 1}$ in the form $P(x) + \dfrac{a}{2x - 1}$, where $P(x)$ is a quadratic expression and a is a real number.

6.3 Factorisation of polynomials

Remainder theorem

Example 6

Let $P(x) = x^3 + 3x^2 + 2x + 1$.

a i Divide $P(x)$ by $(x - 1)$. **ii** Evaluate $P(1)$.

b i Divide $P(x)$ by $(x - 2)$. **ii** Evaluate $P(2)$.

Solution

a i
$$\begin{array}{r} x^2 + 4x + 6 \\ x - 1 \overline{)\, x^3 + 3x^2 + 2x + 1} \\ \underline{x^3 - x^2} \\ 4x^2 + 2x \\ \underline{4x^2 - 4x} \\ 6x + 1 \\ \underline{6x - 6} \\ 7 \end{array}$$

Thus $\dfrac{x^3 + 3x^2 + 2x + 1}{x - 1} = x^2 + 4x + 6 + \dfrac{7}{x - 1}$

ii $P(1) = 1^3 + 3(1)^2 + 2 + 1$
$= 7$

b i
$$\begin{array}{r} x^2 + 5x + 12 \\ x - 2\overline{)\,x^3 + 3x^2 + 2x + 1} \\ \underline{x^3 - 2x^2} \\ 5x^2 + 2x \\ \underline{5x^2 - 10x} \\ 12x + 1 \\ \underline{12x - 24} \\ 25 \end{array}$$

Thus $\dfrac{x^3 + 3x^2 + 2x + 1}{x - 2} = x^2 + 5x + 12 + \dfrac{25}{x - 2}$

ii $P(2) = (2)^3 + 3(2)^2 + 2(2) + 1$
$= 8 + 12 + 4 + 1$
$= 25$

The results indicate that, when $P(x)$ is divided by $(x - a)$, the remainder is equal to $P(a)$. This is in fact true, and the result is called **remainder theorem**.

It is proved as follows. Suppose that, when the polynomial $P(x)$ is divided by $(x - a)$, the quotient is $Q(x)$ and the remainder is R, then

$$P(x) = (x - a)Q(x) + R$$

Now, as the two expressions are equal for all values of x, they are equal for $x = a$.

$$\therefore P(a) = (a - a)Q(a) + R \qquad \therefore R = P(a)$$

i.e. the remainder when $P(x)$ is divided by $(x - a)$ is equal to $P(a)$. We therefore have

$$P(x) = (x - a)Q(x) + P(a)$$

More generally:

When $P(x)$ is divided by $ax + b$ the remainder is $P\left(-\dfrac{b}{a}\right)$.

Example 7

Use the remainder theorem to find the value of the remainder when $P(x) = x^3 - 2x + 4$ is divided by $2x + 1$.

Solution

$$P\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right) + 4 = -\frac{1}{8} + 1 + 4 = \frac{39}{8}$$

The remainder is $\dfrac{39}{8}$.

Example 8

When $P(x) = x^3 + 2x + a$ is divided by $x - 2$ the remainder is 4. Find the value of a.

Solution

$$P(2) = 8 + 4 + a = 4.$$

Therefore $a = -8$.

Exercise 6C

Example 7

1 Without dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 - x^2 - 3x + 1, x - 1$
b $x^3 - 3x^2 + 4x - 1, x + 2$
c $2x^3 - 2x^2 + 3x + 1, x - 2$
d $x^3 - 2x + 3, x + 1$
e $x^3 + 2x - 5, x - 2$
f $2x^3 + 3x^2 + 3x - 2, x + 2$
g $6 - 5x + 9x^2 + 10x^3, 2x + 3$
h $10x^3 - 3x^2 + 4x - 1, 2x + 1$
i $108x^3 - 27x^2 - 1, 3x + 1$

Example 8

2 Find the values of a in the expressions below when the following conditions are satisfied:

a $x^3 + ax^2 + 3x - 5$ has remainder -3 when divided by $x - 2$.
b $x^3 + x^2 - 2ax + a^2$ has remainder 8 when divided by $x - 2$.
c $x^3 - 3x^2 + ax + 5$ has remainder 17 when divided by $x - 3$.
d $x^3 + x^2 + ax + 8$ has remainder 0 when divided by $x - 1$.

6.4 Factor theorem

Now, in order for $(x - a)$ to be a factor of the polynomial $P(x)$, the remainder must be zero. We state this result as the **factor theorem**.

> If for a polynomial, $P(x)$, $P(a) = 0$ then $x - a$ is a factor.
> Conversely, if $x - a$ is a factor of $P(x)$ then $P(a) = 0$.

More generally:

> If $ax + b$ is a factor of $P(x)$ then $P\left(-\frac{b}{a}\right) = 0$.
> Conversely, if $P\left(-\frac{b}{a}\right) = 0$ then $ax + b$ is a factor of $P(x)$.

Example 9

Show that $(x + 1)$ is a factor of $x^3 - 4x^2 + x + 6$ and hence find the other linear factors.

Solution

Let $P(x) = x^3 - 4x^2 + x + 6$

$P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6$

$= 0$

$\therefore$ From the factor theorem $(x - (-1)) = (x + 1)$ is a factor.

Using division of polynomials, the other factor can be found. This will be a quadratic factor.

$$
\begin{array}{r}
x^2 - 5x + 6 \\
x + 1 \overline{)\, x^3 - 4x^2 + x + 6} \\
\underline{x^3 + x^2} \\
-5x^2 + x \\
\underline{-5x^2 - 5x} \\
6x + 6 \\
\underline{6x + 6} \\
0
\end{array}
$$

$\therefore x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$

Now factorising the quadratic factor we have

$$P(x) = (x + 1)(x - 3)(x - 2)$$

$\therefore$ the linear factors of $x^3 - 4x^2 + x + 6$ are $(x + 1)$, $(x - 3)$ and $(x - 2)$.

Example 10

Factorise $x^3 - 2x^2 - 5x + 6$.

Solution

Let us assume there are three linear factors

i.e. $x^3 - 2x^2 - 5x + 6 = (x - a)(x - b)(x - c)$

$= x^3 - (a + b + c)x^2 + (ab + bc + ac)x - abc$

By considering the constant term it can be seen that $abc = -6$.

Thus only the factors of -6 need be considered (i.e. ± 1, ± 2, ± 3, ± 6).

Try these in turn until a value for a makes $P(a) = 0$:

$$P(1) = 1 - 2(1) - 5 + 6 = 0$$

$\therefore (x - 1)$ is a factor.

Now divide to find the other factors.

$$\begin{array}{r} x^2 - x - 6 \\ x - 1 \overline{)\, x^3 - 2x^2 - 5x + 6} \\ \underline{x^3 - x^2} \\ -x^2 + 5x \\ \underline{-x^2 + x} \\ -6x + 6 \\ \underline{-6x + 6} \\ 0 \end{array}$$

$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$

$= (x - 1)(x - 3)(x + 2)$

$\therefore$ the factors of $x^3 - 2x^2 - 5x + 6$ are $(x - 1)$, $(x - 3)$ and $(x + 2)$.

Special cases: sums and differences of cubes

Example 11

Factorise $x^3 - 27$.

Solution

Let $P(x) = x^3 - 27$

$P(3) = 27 - 27 = 0$

$\therefore (x - 3)$ is a factor.

$$\begin{array}{r} x^2 + 3x + 9 \\ x - 3 \overline{)\, x^3 + 0x^2 + 0x - 27} \\ \underline{x^3 - 3x^2} \\ 3x^2 \\ \underline{3x^2 - 9x} \\ 9x - 27 \\ \underline{9x - 27} \\ 0 \end{array}$$

$\therefore x^3 - 27 = (x - 3)(x^2 + 3x + 9)$

In general, if $P(x) = x^3 - a^3$ then $(x - a)$ is a factor and by division

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

If a is replaced by $-a$ then

$$x^3 - (-a)^3 = (x - (-a))(x^2 + (-a)x + (-a)^2)$$

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Example 12

Factorise $8x^3 + 64$.

Solution

$$\begin{aligned} 8x^3 + 64 &= (2x)^3 + (4)^3 \\ &= (2x + 4)(4x^2 - 8x + 16) \end{aligned}$$

Exercise 6D

Example 9

1 Without dividing, show that the first polynomial is exactly divisible by the second polynomial:

a $x^3 - x^2 + x - 1, x - 1$ **b** $x^3 + 3x^2 - x - 3, x - 1$

c $2x^3 - 3x^2 - 11x + 6, x + 2$ **d** $2x^3 - 13x^2 + 27x - 18, 2x - 3$

2 Find the value of m if the first polynomial is exactly divisible by the second:

a $x^3 - 4x^2 + x + m, x - 3$ **b** $2x^3 - 3x^2 - (m + 1)x - 30, x - 5$

c $x^3 - (m + 1)x^2 - x + 30, x + 3$

Examples 9, 10

3 Factorise each of the following:

a $2x^3 + x^2 - 2x - 1$ **b** $x^3 + 3x^2 + 3x + 1$ **c** $6x^3 - 13x^2 + 13x - 6$

d $x^3 - 21x + 20$ **e** $2x^3 + 3x^2 - 1$ **f** $x^3 - x^2 - x + 1$

g $4x^3 + 3x - 38$ **h** $4x^3 + 4x^2 - 11x - 6$

Examples 11, 12

4 Factorise each of the following:

a $x^3 - 1$ **b** $x^3 + 64$ **c** $27x^3 - 1$

d $64x^3 - 125$ **e** $1 - 125x^3$ **f** $8 + 27x^3$

g $64m^3 - 27n^3$ **h** $27b^3 + 8a^3$

5 Factorise each of the following:

a $x^3 + x^2 - x + 2$ **b** $3x^3 - 7x^2 + 4$

c $x^3 - 4x^2 + x + 6$ **d** $6x^3 + 17x^2 - 4x - 3$

6 Find the values of a and b and factorise the polynomial $P(x) = x^3 + ax^2 - x + b$, given that $P(x)$ is divisible by $x - 1$ and $x + 3$.

7 **a** Show that, for any constant a and any natural number n, $x - a$ is a factor of $x^n - a^n$.

b Find conditions (if any) on n that are required in order that:

i $x + a$ is a factor of $x^n + a^n$ **ii** $x + a$ is a factor of $x^n - a^n$

8 The polynomial $P(x)$ has a remainder of 2 when divided by $x - 1$ and a remainder of 3 when divided by $x - 2$. The remainder when $P(x)$ is divided by $(x - 1)(x - 2)$ is $ax + b$, i.e. $P(x)$ can be written as $P(x) = (x - 1)(x - 2)Q(x) + ax + b$.

a Find the values of a and b.

b **i** Given that $P(x)$ is a cubic polynomial with coefficient of x^3 being 1, and -1 is a solution of the equation $P(x) = 0$, find $P(x)$.

ii Show that the equation $P(x) = 0$ has no other real roots.

6.5 Solving cubic equations

Example 13

Solve $(x - 2)(x + 1)(x + 3) = 0$.

Solution

By the null factor law $(x - 2)(x + 1)(x + 3) = 0$

$$\text{implies } x - 2 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x + 3 = 0.$$

Thus the solutions are $x = 2, -1$ and -3.

Example 14

Solve $x^3 - 4x^2 - 11x + 30 = 0$.

Solution

Let
$$\begin{aligned} P(x) &= x^3 - 4x^2 - 11x + 30 \\ P(1) &= 1 - 4 - 11 + 30 \neq 0 \\ P(-1) &= -1 - 4 + 11 + 30 \neq 0 \\ P(2) &= 8 - 16 - 22 + 30 = 0 \end{aligned}$$

$\therefore (x - 2)$ is a factor.

By division

$$\begin{aligned} x^3 - 4x^2 - 11x + 30 &= (x - 2)(x^2 - 2x - 15) \\ &= (x - 2)(x - 5)(x + 3) \end{aligned}$$

$\therefore (x - 2)(x - 5)(x + 3) = 0$

$\therefore x - 2 = 0$ or $x - 5 = 0$ or $x + 3 = 0$

$\therefore x = 2, 5$ or -3.

Example 15

CAS APPENDIX

Solve $2x^3 - 5x^2 + x + 2 = 0$.

Solution

Let $P(x) = 2x^3 - 5x^2 + x + 2$

$P(1) = 2 - 5 + 1 + 2 = 0$

$\therefore (x - 1)$ is a factor

By division

$$2x^3 - 5x^2 + x + 2 = (x - 1)(2x^2 - 3x - 2)$$
$$= (x - 1)(2x + 1)(x - 2)$$

$\therefore (x - 1)(2x + 1)(x - 2) = 0$

$\therefore\ x = 1, -\frac{1}{2},$ or 2

Example 16

Solve each of the following equations for x:

a $2x^3 - x^2 - x = 0$ **b** $x^3 + 2x^2 - 10x = 0$

Solution

a $2x^3 - x^2 - x = 0$

$x(2x^2 - x - 1) = 0$

$x(2x + 1)(x - 1) = 0$

$\therefore\ x = 0$ or $x = -\frac{1}{2}$ or $x = 1$

b $x^3 + 2x^2 - 10x = 0$

$x(x^2 + 2x - 10) = 0$

Completing the square gives

$x(x^2 + 2x + 1 - 10 - 1) = 0$

$x(x^2 + 2x + 1 - (\sqrt{11})^2) = 0$

$x((x + 1)^2 - (\sqrt{11})^2) = 0$

$x(x + 1 - \sqrt{11})(x + 1 + \sqrt{11}) = 0$

$x = 0$ or $x + 1 = \sqrt{11}$ or $x + 1 = -\sqrt{11}$

$\therefore\ x = 0$ or $x = -1 + \sqrt{11}$ or $x = -1 - \sqrt{11}$

Example 17

Solve each of the following equations for x:

a $x^3 - 4x^2 - 11x + 44 = 0$ **b** $x^3 - ax^2 - 11x + 11a = 0$

Solution

Grouping of terms sometimes facilitates factorisation.

a $x^3 - 4x^2 - 11x + 44 = 0$

$x^2(x - 4) - 11(x - 4) = 0$

Therefore $(x - 4)(x^2 - 11) = 0$

And $x = 4$ or $x = \pm\sqrt{11}$

b $x^3 - ax^2 - 11x + 11a = 0$

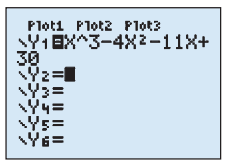
$x^2(x - a) - 11(x - a) = 0$

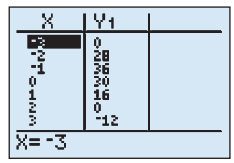
Therefore $(x - a)(x^2 - 11) = 0$

And $x = a$ or $x = \pm\sqrt{11}$

Using a graphics calculator

The use of **Table** on the graphics calculator can aid finding factors and solving equations. For example, enter $Y_1 = X\hat{}3 - 4X\hat{}2 - 11X + 30$ in the **Y=** window.





Scroll down in the table menu to see that $X = 5$ also has a zero value for Y_1.

Exercise 6E

1 Solve each of the following. Remember to check your solutions in the original equations.

Example 13 Example 14 Example 15

a $(x-1)(x+2)(x-4) = 0$ **b** $(x-4)(x-4)(x-6) = 0$

c $(2x-1)(x-3)(3x+2) = 0$ **d** $x^3 + 2x^2 - x - 2 = 0$

e $x^3 - 19x + 30 = 0$ **f** $3x^3 - 4x^2 - 13x - 6 = 0$

g $x^3 - x^2 - 2x + 2 = 0$ **h** $5x^3 + 12x^2 - 36x - 16 = 0$

i $6x^3 - 5x^2 - 2x + 1 = 0$ **j** $2x^3 - 3x^2 - 29x - 30 = 0$

2 Solve each of the following for x:

a $x^3 + x^2 - 24x + 36 = 0$ **b** $6x^3 + 13x^2 - 4 = 0$

c $x^3 - x^2 - 2x - 12 = 0$ **d** $2x^3 + 3x^2 + 7x + 6 = 0$

e $x^3 - x^2 - 5x - 3 = 0$ **f** $x^3 + x^2 - 11x - 3 = 0$

Example 16 **3** Solve each of the following equations for x:

a $x^3 - 2x^2 - 8x = 0$ **b** $x^3 + 2x^2 - 11x = 0$

c $x^3 - 3x^2 - 40x = 0$ **d** $x^3 + 2x^2 - 16x = 0$

4 Solve each of the following equations for x:

a $2x^3 = 16x$ **b** $2(x-1)^3 = 32$ **c** $x^3 + 8 = 0$

d $2x^3 + 250 = 0$ **e** $1000 = \dfrac{1}{x^3}$

Example 17 **5** Use grouping to solve each of the following:

a $x^3 - x^2 + x - 1 = 0$ **b** $x^3 + x^2 + x + 1 = 0$

c $x^3 - 5x^2 - 10x + 50 = 0$ **d** $x^3 - ax^2 - 16x + 16a = 0$

6 Factorise each of the following cubic expressions, using a graphics calculator to help find at least one linear factor:

a $2x^3 - 22x^2 - 250x + 2574$ **b** $2x^3 + 27x^2 + 52x - 33$

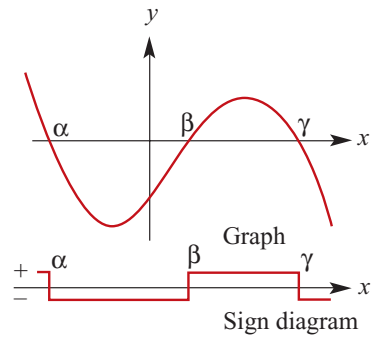
c $2x^3 - 9x^2 - 242x + 1089$ **d** $2x^3 + 51x^2 + 304x - 165$

6.6 Graphs of cubic functions

We will look at a technique which assists in sketching graphs of cubic functions.

A sign diagram is a number line diagram which shows when an expression is positive or negative. Shown is a sign diagram for the cubic function, the graph of which is also shown.

The factorisation method requires that factors, and hence the x-axis intercepts, be found. The y-axis intercept and sign diagram can then be used to complete the graph.



Example 18

Sketch the graph of $y = x^3 + 2x^2 - 5x - 6$.

Solution

Let $P(x) = x^3 + 2x^2 - 5x - 6$

$P(1) = 1 + 2 - 5 - 6 \neq 0$

$P(-1) = -1 + 2 + 5 - 6 = 0$

$\therefore (x + 1)$ is a factor.

By division

$y = (x + 1)(x - 2)(x + 3)$

For $(x + 1)(x - 2)(x + 3) = 0$

$x + 1 = 0$ or $x - 2 = 0$ or $x + 3 = 0$

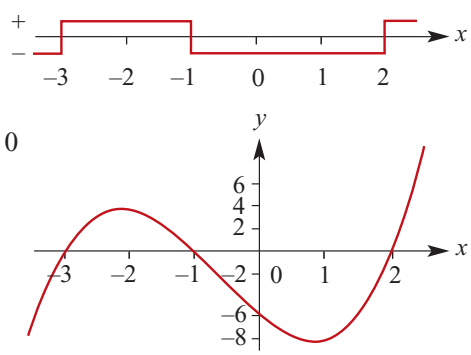
$\therefore x = -1, 2,$ or -3

When $x < -3$, y is negative

$-3 < x < -1$, y is positive

$-1 < x < 2$, y is negative

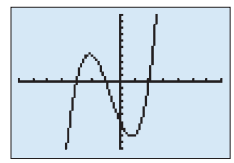
giving the sign diagram.



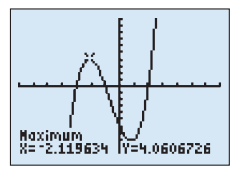
At this stage the location of the turning points is unspecified. It is important, however, to note that, unlike quadratic graphs, the turning points are not symmetrically located between x-axis intercepts. How to determine the exact values of the coordinates of the turning point will be shown later in the course.

Using a graphics calculator

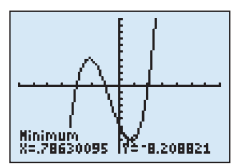
In order to provide more detail, the coordinates of the turning points can be found on a graphics calculator. Enter $Y_1 = X\hat{}3 + 2X\hat{}2 - 5X - 6$ in the **Y=** screen and choose a suitable window. Graph the function.



Choose **4:maximum** from the **CALC** menu. In the Graph screen respond to the **Left Bound** prompt by typing -3 then press ENTER and the **Right Bound** prompt by typing -1. Press ENTER twice.



The minimum is found by selecting **3:minimum** from the **CALC** menu and following a similar procedure.



Example 19

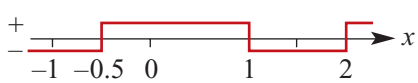
Sketch the graph of $y = 2x^3 - 5x^2 + x + 2$.

Solution

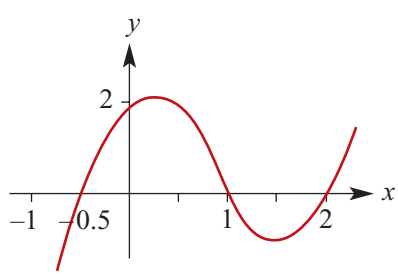
From Example 15 it can be seen that

$$2x^3 - 5x^2 + x + 2 = (x - 1)(2x + 1)(x - 2).$$

The y-axis intercept is $(0, 2)$. The sign diagram and graph are shown.



Sign diagram



Example 20

Sketch the graph of $y = (x - 1)(x + 2)(x + 1)$. Do not give coordinates of turning points.

Solution

The cubic has been given in factorised form, so the x-axis intercepts can be easily found.

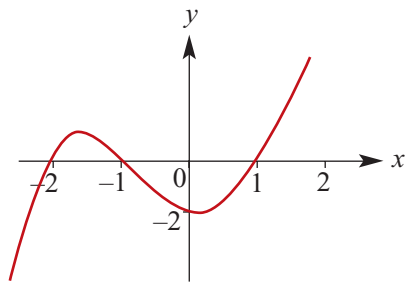
Let $y = 0$

$$0 = (x - 1)(x + 2)(x + 1)$$

Using the null factor law, the x-axis intercepts are 1, -1, and -2.

To find the y-axis intercept, let $x = 0$

$$y = (0 - 1)(0 + 2)(0 + 1)$$
$$= -2$$



It is noted that the implied domain of all cubics is $\boldsymbol{R}$ and that the range is also $\boldsymbol{R}$.

If the factorised cubic has a repeated factor there are only two x-axis intercepts and the repeated factor corresponds to one of the turning points.

Example 21

Sketch the graph of $y = x^2(x - 1)$.

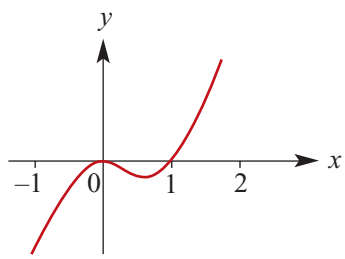
Solution

To find the x-axis intercepts, let $y = 0$.

Then $x^2(x - 1) = 0$

x-axis intercepts are at $x = 0$ and 1 and, because the repeated factor is x^2, there is also a turning point at $x = 0$.

y-axis intercept (letting $x = 0$) is at $y = 0$.



Some cubics will only have one x-axis intercept. This is because, when they are factorised, they are found to have only one linear factor, with the remaining quadratic factor unable to be factorised further.

Example 22

Sketch the graph of $y = -(x - 1)(x^2 + 4x + 5)$.

Solution

To find the x-axis intercept, let $y = 0$.

First, we note that the factor of $x^2 + 4x + 5$ cannot be factorised further.

$$\Delta = b^2 - 4ac$$
$$\Delta = 4^2 - 4(1)(5)$$
$$= -4$$

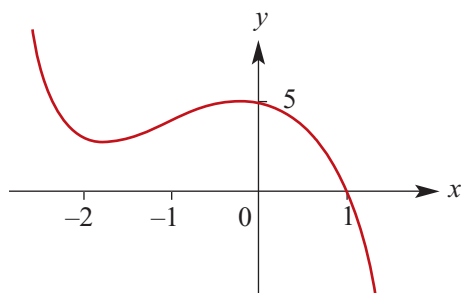
$\therefore$ there are no linear factors.

Hence, when solving the equation $-(x - 1)(x^2 + 4x + 5) = 0$, there is only one solution.

$\therefore$ x-axis intercept is $x = 1$.

To find the y-axis intercept, let $x = 0$.

$$-((0) - 1)((0)^2 + 4(0) + 5) = 5$$



On a graphics calculator it is found that the turning points are at (0, 5) and (−1.82, 2.91), where the values for the coordinates of the second point are given to 2 decimal places.

Exercise 6F

Examples 18–22

1 Sketch the graphs for each of the following and draw a sign diagram. Label your sketch graph showing the points of intersection with the axes. (Do not determine coordinates of turning points.)

a $y = x(x - 1)(x - 3)$
b $y = (x - 1)(x + 1)(x + 2)$
c $y = (x - 1)(x - 2)(x - 3)$
d $y = (2x - 1)(x - 2)(x + 3)$
e $y = x^3 - 9x$
f $y = x^3 + x^2$
g $y = x^3 - 5x^2 + 7x - 3$
h $y = x^3 - 4x^2 - 3x + 18$
i $y = -x^3 + x^2 + 3x - 3$
j $y = 3x^3 - 4x^2 - 13x - 6$
k $y = 6x^3 - 5x^2 - 2x + 1$

2 Sketch the graphs of each of the following, using a graphics calculator to find the coordinates of axes intercepts and local maximum and local minimum values:

a $y = -4x^3 - 12x^2 + 37x - 15$
b $y = -4x^3 + 19x - 15$
c $y = -4x^3 + 0.8x^2 + 19.8x - 18$
d $y = 2x^3 + 11x^2 + 15x$
e $y = 2x^3 + 6x^2$
f $y = 2x^3 + 6x^2 + 6$

3 Show that the graph of f, where $f(x) = x^3 - x^2 - 5x - 3$, cuts the x-axis at one point and touches it at another. Find the values of x at these points.

6.7 Solving cubic inequations

As was done with quadratic inequations, cubic inequations can be solved by considering the graph of the corresponding equation and determining the solution of the inequation from the graph.

Example 23

Find $\{x: x^3 + x^2 - 5x + 3 \leq 0\}$

Solution

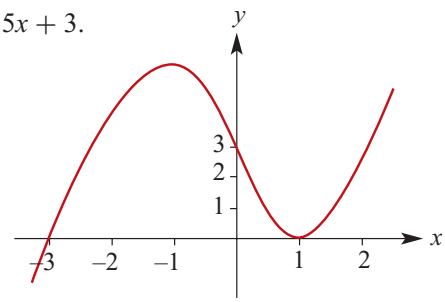
Start by sketching the graph of $y = x^3 + x^2 - 5x + 3$.

Let $P(x) = x^3 + x^2 - 5x + 3$

$P(1) = 1 + 1 - 5 + 3 = 0$

$\therefore$ $(x - 1)$ is a factor.

By division $y = (x - 1)^2(x + 3)$



There are only two x-axis intercepts, $(1, 0)$ and $(-3, 0)$.

The y-axis intercept is $(0, 3)$.

From the graph we can see that $y \leq 0$ when x is ≤ -3 or when $x = 1$.

$\therefore \{x: x^3 + x^2 - 5x + 3 \leq 0\} = \{x: x \leq -3\} \cup \{x: x = 1\}$

Exercise 6G

Example 23 Solve the following cubic inequations:

a $(x - 1)(x + 2)(x - 3) \leq 0$

b $(x + 1)(x + 2)(x - 4) \geq 0$

c $(x - 1)(x - 2)^2 < 0$

d $x(x + 2)(x - 3) > 0$

e $(x - 1)^3 + 8 \leq 0$

f $x^3 - 1 \geq 0$

g $x^2(x - 4) > 0$

h $(x + 3)(x^2 + 2x + 5) \leq 0$

6.8 Finding equations for given cubic graphs

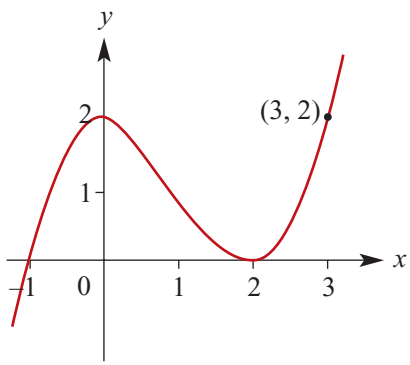
Example 24

Excel

Determine the cubic rule for graphs **a** and **b**.

Solution

a



The x-axis intercepts are -1 and 2 and the graph touches the x-axis at 2, therefore the form of the rule appears to be $y = a(x + 1)(x - 2)^2$.

Put (3, 2) in the equation:

$$2 = a(4)(1)$$
$$\frac{1}{2} = a$$
$$\therefore y = \frac{1}{2}(x + 1)(x - 2)^2 \text{ is the rule.}$$

Alternatively:

Using the general form of a cubic equation $y = ax^3 + bx^2 + cx + d$.

From the graph $d = 2$. Putting each of the ordered pairs $(-1, 0)$, $(2, 0)$ and $(3, 2)$ in the general equation will give three simultaneous equations in three unknowns:

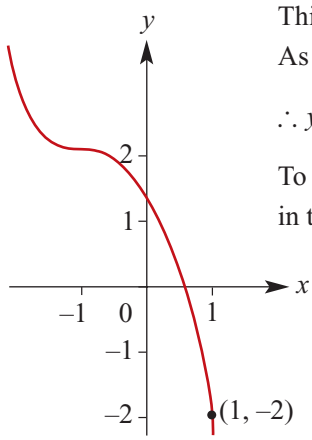
$$-2 = -a + b - c \qquad (1)$$
$$-1 = 4a + 2b + c \qquad (2)$$
$$0 = 9a + 3b + c \qquad (3)$$

Solving these gives the values

$$a = \frac{1}{2}, \quad b = -1\frac{1}{2} \text{ and } c = 0$$
$$\therefore y = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 2$$

This is equivalent to $y = \frac{1}{2}(x + 1)(x - 2)^2$.

b



This graph appears to be of the form $y = a(x - h)^3 + k$.

As can be seen from the graph, $k = 2$, $h = -1$.

$$\therefore y - 2 = a(x + 1)^3$$

To determine a, put one of the known points, say $(1, -2)$, in the equation.

$$-2 - 2 = a(2)^3$$
$$-4 = 8a$$
$$-\frac{1}{2} = a$$

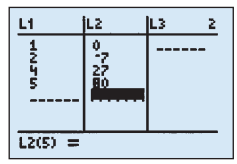
$$\therefore \text{The rule is } y - 2 = \frac{-1}{2}(x + 1)^3$$

Using a graphics calculator

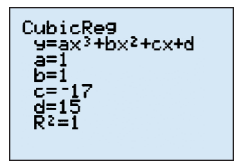
Cubic regression

It is known that the points with coordinates (1, 0), (2, −7), (4, 27), (5, 80) lie on a curve with equation $y = f(x)$, where $f(x)$ is a cubic. The equation can be found by using **6:CubicReg** from the **CALC** submenu of **STAT.**

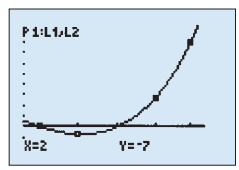
- Enter the data in **L1** and **L2** as shown in the screen. The lists are obtained by pressing STAT and selecting **EDIT**.



- Select **6:CubicReg** from the **CALC** submenu of **STAT**, and complete in the Home screen as **CubicReg L1**, **L2, Y1**. Press ENTER. The result is as shown in the middle screen. The rule for $f(x)$ is stored in the Y_1 position of the **Y=** screen.



- **Plot 1** is activated with the default settings. Select **9:Zoomstat** from the **ZOOM** menu to produce the plot shown.

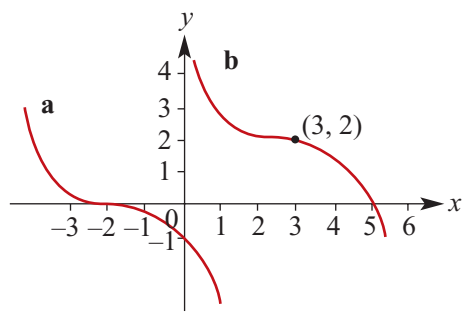


Skillsheet TEACHER

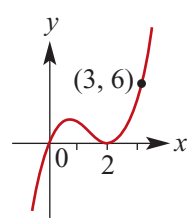
Exercise 6H

Example 24

1 The graphs shown are similar to the basic curve $y = -x^3$. Find possible cubic functions which define each of the curves.



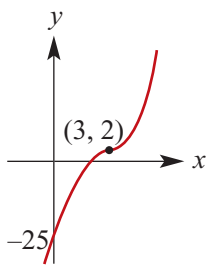
2 Find the equation of the cubic function for which the graph is shown.

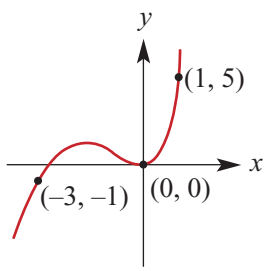


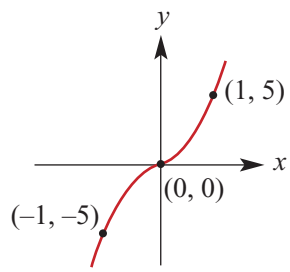
3 Find a cubic function whose graph touches the x-axis at $x = -4$, cuts it at the origin, and has a value 6 when $x = -3$.

4 The graphs below have equations of the form shown. In each case, determine the equation.

a $y = a(x - h)^3 + k$ **b** $y = ax^3 + bx^2$ **c** $y = ax^3$

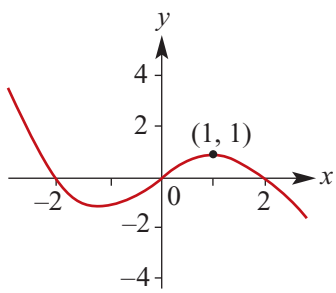




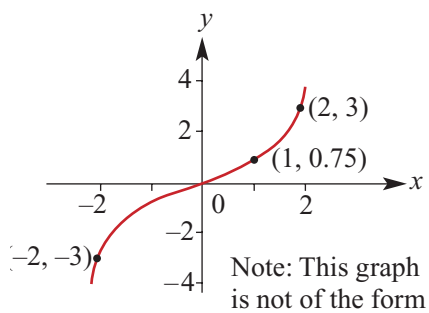


5 Find the expressions which define the following cubic curves:

a



b



Note: This graph is not of the form $y = a(x - h)^3 + k$.

6 For each of the following, use a graphics calculator to find the values of a, b, c, d of the cubic equation $y = ax^3 + bx^2 + cx + d$, given that the following points lie on its graph:

a (0, 270) (1, 312) (2, 230) (3, 0)
b (−2, −406) (0, 26) (1, 50) (2, −22)
c (−2, −32) (2, 8) (3, 23) (8, 428)
d (1, −1) (2, 10) (3, 45) (4, 116)
e (−3, −74) (−2, −23) (−1, −2) (1, −2)
f (−3, −47) (−2, −15) (1, −3) (2, −7)
g (−4, 25) (−3, 7) (−2, 1) (1, −5)

6.9 Graphs of quartic functions

The techniques for graphing quartic functions are very similar to those employed for cubic functions. In this section solving simple quartic equations is first considered. A graphics calculator is to be used in the graphing of these functions. Great care needs to be taken in this process as it is easy to miss key points on the graph using these techniques.

Example 25

Solve each of the following equations for x:

a $x^4 - 8x = 0$ **b** $2x^4 - 8x^2 = 0$ **c** $x^4 - 2x^3 - 24x^2 = 0$

Solution

a $x^4 - 8x = 0$

Factorise to obtain

$x(x^3 - 8) = 0$

$\therefore x = 0$ or $x^3 - 8 = 0$

Thus $x = 2$ or $x = 0$.

b $2x^4 - 8x^2 = 0$

Factorise to obtain

$2x^2(x^2 - 4) = 0$

$\therefore 2x^2 = 0$ or $x^2 - 4 = 0$

Thus $x = 2$ or $x = -2$ or $x = 0$.

c $x^4 - 2x^3 - 24x^2 = 0$

Factorise to obtain

$x^2(x^2 - 2x - 24) = 0$

$\therefore x^2 = 0$ or $x^2 - 2x - 24 = 0$

i.e. $x = 0$ or $(x - 6)(x + 4) = 0$

Thus $x = 0$ or $x = 6$ or $x = -4$.

Quartic equations in general can be solved by techniques similar to those used for solving cubic functions, but only equations which can be solved by 'simple' techniques are dealt with in this course.

Using a graphics calculator

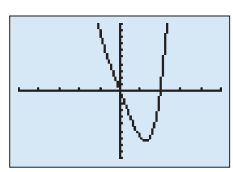
Example 26

Graph each of the following quartic equations. Use a graphics calculator to help.

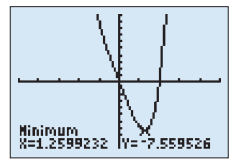
a $f(x) = x^4 - 8x$ **b** $f(x) = 2x^4 - 8x^2$ **c** $f(x) = \frac{1}{2}(x^4 - 2x^3 - 24x^2)$

Solutions

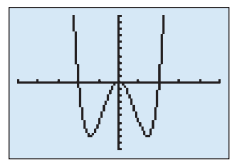
a $f(x) = x^4 - 8x$. The x-axis intercepts have been found in the previous example. The graph is as shown. There are only two x-axis intercepts, one at the origin and the other at $x = 2$. Approximate values of the coordinates of the turning point can be found. This is done by selecting **3:minimum** from the **CALC** menu.

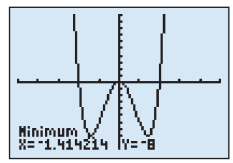


It can be seen that the turning point has coordinates (1.26, −7.56), with values given correct to 2 decimal places.
It should be noted that, unlike parabolas, this graph has no vertical line of symmetry.

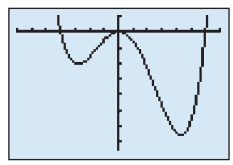


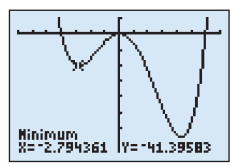
b $f(x) = 2x^4 - 8x^2$. The x-axis intercepts have been found in the previous example.
The graph is as shown. There are three x-axis intercepts, one at the origin and the others at $x = 2$ and $x = -2$. Approximate values of the coordinates of the turning points can be found. This is done by selecting **3:minimum** from the **CALC** menu.
Note that $f(x) = f(-x)$ for this function.
Hence the coordinates of the turning points (with values given to 2 decimal places) are (−1.41, −8) and (1.41, −8).

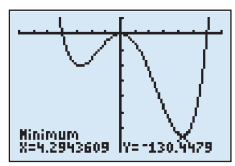




c $f(x) = \frac{1}{2}(x^4 - 2x^3 - 24x^2)$. The x-axis intercepts have been found in the previous example.
The graph is as shown. There are three x-axis intercepts, one at the origin and the others at $x = 6$ and $x = -4$. Approximate values of the coordinates of the turning points can be found by selecting **3:minimum** from the **CALC** menu.
The coordinates of the turning points are (−2.79, −41.4) and (4.29, −130.45), with values given correct to 2 decimal places.







Exercise 6I

Example 25

1 Solve each of the following equations for x:

a $x^4 - 27x = 0$
b $(x^2 - x - 2)(x^2 - 2x - 15) = 0$
c $x^4 + 8x = 0$
d $x^4 - 6x^3 = 0$
e $x^4 - 9x^2 = 0$
f $81 - x^4 = 0$
g $x^4 - 16x^2 = 0$
h $x^4 - 7x^3 + 12x^2 = 0$
i $x^4 - 9x^3 + 20x^2 = 0$
j $(x^2 - 4)(x^2 - 9) = 0$
k $(x - 4)(x^2 + 2x + 8) = 0$
l $(x + 4)(x^2 + 2x - 8) = 0$

Example 26

2 Use a graphics calculator to help draw the graphs of each of the following. Give x-axis intercepts and coordinates of turning points. (Values of coordinates of turning points to be given correct to 2 decimal places.)

a $y = x^4 - 125x$
b $y = (x^2 - x - 20)(x^2 - 2x - 24)$
c $y = x^4 + 27x$
d $y = x^4 - 4x^3$
e $y = x^4 - 25x^2$
f $y = 16 - x^4$
g $y = x^4 - 81x^2$
h $y = x^4 - 7x^3 + 12x^2$
i $y = x^4 - 9x^3 + 20x^2$
j $y = (x^2 - 16)(x^2 - 25)$
k $y = (x - 2)(x^2 + 2x + 10)$
l $y = (x + 4)(x^2 + 2x - 35)$

6.10 Finite differences for sequences generated by polynomials

The following are examples of sequences:

A 1, 3, 5, 7, . . .
B 1, 4, 9, 16, . . .
C 1, 3, 6, 10, . . .
D 1, 5, 14, 30, 55, 9, . . .

Sequences can be defined by a function $f: N \to R$, where N is the set of natural numbers.

For sequence A: $f(n) = 2n - 1$ (linear function)

For sequence B: $f(n) = n^2$ (quadratic function)

In this section we consider those sequences that correspond to polynomial functions of degree less than or equal to 3 which have domain N.

For sequence C: $f(n) = \dfrac{n(n+1)}{2}$ (quadratic function)

For sequence D: $f(n) = \dfrac{n(n+1)(2n+1)}{6}$ (cubic function)

$f(n)$ is called the $\boldsymbol{n^{\text{th}}}$ term of the sequence.

In the study of these functions we will assume that the pattern continues infinitely. For example, for sequence A the sequence consists of all odd numbers and for sequence B the sequence continues to give all the square numbers.

We can construct a difference table for the sequences, A, B, and D given above. The difference table is continued until a constant is obtained. We will use the following notations for differences:

$$\Delta_1^1 = f(2) - f(1); \quad \Delta_1^2 = f(3) - f(2); \quad \Delta_1^n = f(n+1) - f(n)$$

$$\Delta_2^1 = \Delta_1^2 - \Delta_1^1; \quad \Delta_2^2 = \Delta_1^3 - \Delta_1^2; \quad \Delta_2^n = \Delta_1^{n+1} - \Delta_1^n$$

and in general $\Delta_{n+1}^m = \Delta_n^{m+1} - \Delta_n^m$

Difference table A

n	$f(n)$	Δ_1
1	1	2 (= 3 – 1)
2	3	2 (= 5 – 3)
3	5	2 (= 7 – 5)
4	7	2 (= 9 – 7)
5	9	

Difference table B

n	$f(n)$	Δ_1	Δ_2
1	1	3 (= 4 – 1)	2 (= 5 – 3)
2	4	5 (= 9 – 4)	2 (= 7 – 5)
3	9	7 (= 16 – 9)	2 (= 9 – 7)
4	16	9 (= 25 – 16)	
5	25		

Difference table C

n	$f(n)$	Δ_1	Δ_2	Δ_3
1	1	4	5	2
2	5	9	7	2
3	14	16	9	2
4	30	25	11	
5	55	36		
6	91			

From these tables we conjecture, but do not prove, that for a sequence generated by a polynomial function:

If the column Δ_1 is constant and non zero the function is linear.
If the column Δ_2 is constant and non zero the function is quadratic.
If the column Δ_3 is constant and non zero the function is cubic.

The converse result also holds:

i.e.
If f is linear $\quad \Delta_1^n = c$, a constant, for all n.
If f is quadratic $\quad \Delta_2^n = k$, a constant, for all n.
If f is cubic $\quad \Delta_3^n = l$, a constant, for all n.

The following three difference tables are for the general linear, quadratic and cubic functions. We will use these tables throughout the remainder of this section.

Linear: $f(n) = an + b$

n	$f(n)$	Δ_1
1	$a + b$	
		a
2	$2a + b$	
		a
3	$3a + b$	

Quadratic: $f(n) = an^2 + bn + c$

n	$f(n)$	Δ_1	Δ_2
1	$a + b + c$		
		$3a + b$	
2	$4a + 2b + c$		$2a$
		$5a + b$	
3	$9a + 3b + c$		$2a$
		$7a + b$	
4	$16a + 4b + c$		$2a$
		$9a + b$	
5	$25a + 5b + c$		

Cubic: $f(n) = an^3 + bn^2 + cn + d$

n	$f(n)$	Δ_1	Δ_2	Δ_3
1	$a + b + c + d$			
		$7a + 3b + c$		
2	$8a + 4b + 2c + d$		$12a + 2b$	
		$19a + 5b + c$		$6a$
3	$27a + 9b + 3c + d$		$18a + 2b$	
		$37a + 7b + c$		$6a$
4	$64a + 16b + 4c + d$		$24a + 2b$	
		$61a + 9b + c$		$6a$
5	$125a + 25b + 5c + d$		$30a + 2b$	
		$91a + 11b + c$		
6	$216a + 36b + 6c + d$			

These tables may be used to determine the rules for a given sequence. This is illustrated by the following example. We note that they do not constitute a proof that the rule is the one for the given sequence.

Example 27

Find the rule for each of the following sequences, using finite difference tables:

a 5, 11, 19, 29, 41, . . . **b** 6, 26, 64, 126, 218, 346 . . .

Solution

a

n	$f(n)$	Δ_1	Δ_2
1	5		
		6	
2	11		2
		8	
3	19		2
		10	
4	29		2
		12	
5	41		

The function is a quadratic of the form $f(n) = an^2 + bn + c$.
From the difference table for the quadratic

$2a = 2$ and therefore $a = 1$ (using column Δ_2)
Also $3a + b = 6$ and therefore $b = 3$ (using column Δ_1)
Finally $a + b + c = 5$ and therefore $c = 1$ (using column $f(n)$).
$\therefore$ $f(n) = n^2 + 3n + 1$

b

n	$f(n)$	Δ_1	Δ_2	Δ_3
1	6			
		20		
2	26		18	
		38		6
3	64		24	
		62		6
4	126		30	
		92		6
5	218		36	
		128		
6	346			

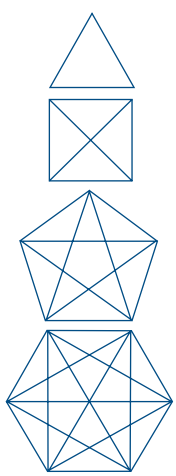
Therefore $f(n) = an^3 + bn^2 + cn + d$.

$\therefore$ $6a = 6$ $\therefore$ $a = 1$ (from column Δ_3)
$12a + 2b = 18$ $\therefore$ $b = 3$ (from column Δ_2)
$7a + 3b + c = 20$ $\therefore$ $c = 4$ (from column Δ_1)
$a + b + c + d = 6$ $\therefore$ $d = -2$ (from column $f(n)$)
$\therefore$ $f(n) = n^3 + 3n^2 + 4n - 2$

Example 28

How many diagonals are there in a convex polygon with n sides?

Solution



For a triangle, i.e. $n = 3$, there are no diagonals.

For a quadrilateral, i.e. $n = 4$, there are two diagonals.

For a pentagon, i.e. $n = 5$, there are five diagonals.

For a hexagon, i.e. $n = 6$, there are nine diagonals.

This gives a difference table:

n	$f(n)$	Δ_1	Δ_2
3	0		
		2	
4	2		1
		3	
5	5		1
		4	
6	9		

$\therefore$ The rule is given by a quadratic of the form $f(n) = an^2 + bn + c$,

where n is the number of sides.

We refer to the general difference table for a quadratic:

$$2a = 1 \quad \text{and} \quad \therefore a = \frac{1}{2}$$

$$7a + b = 2 \quad \text{and} \quad \therefore b = -\frac{3}{2}$$

and $\quad 9a + 3b + c = 0$

which implies $c = 0$

$$\therefore \quad f(n) = \frac{1}{2}n^2 - \frac{3}{2}n = \frac{n}{2}(n - 3)$$

Example 29

Find the sum of n terms of $2 + 6 + 10 + 14 + 18 \ldots$

Solution

Consider the sequence of sums

$$S_1 = 2, S_2 = 8, S_3 = 18, S_4 = 32, S_5 = 50$$

where S_i is the sum of the first i terms. Consider the difference table

i	S_i	1	2
1	2		
		6	
2	8		4
		10	
3	18		4
		14	
4	32		4
		18	
5	50		

The rule for S_n is given by a quadratic of the form $S_n = an^2 + bn + c$.

The general difference table for a quadratic now gives

$2a = 4$ i.e. $a = 2$
$3a + b = 6$ i.e. $b = 0$
$a + b + c = 2$ i.e. $c = 0$
and $S_n = 2n^2$

Exercise 6J

Example 27

1 Find an expression to determine the n^{th} term of each of the following sequences:

a 4, 7, 12, 19, 28, . . .
b 3, 3, 5, 9, 15, . . .
c 1, 4, 10, 20, 35, 56, . . .
d 1, 5, 14, 30, 55, 91, . . .
e -3, 11, 49, 123, 245, 427, 681, . . .

Example 29

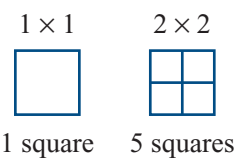
2 Find the sum to n terms of:

a $1 + 3 + 5 + 7 + \ldots$
b $2 + 4 + 6 + 8 + \ldots$
c $1^2 + 2^2 + 3^2 + 4^2 + \ldots$
d $1^2 + 3^2 + 5^2 + 7^2 + \ldots$
e $1 \times 3 + 2 \times 4 + 3 \times 5 + 4 \times 6 + \ldots$
f $2 \times 3 + 4 \times 5 + 6 \times 7 + 8 \times 9 + \ldots$

3 Given two points on a circle, one chord can be drawn. How many different chords can be drawn when there are n points?

4 A large square can be divided into smaller squares as shown. How many squares are there in an $n \times n$ square?

5 How many rectangles are there in an $n \times n$ square?

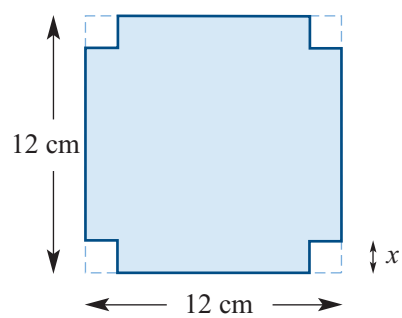


6.11 Applications of polynomial functions

Using a graphics calculator

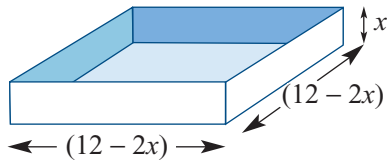
Example 30

A square sheet of tin measures 12 cm × 12 cm. Four equal squares of edge x cm are cut out of the corners and the sides are turned up to form an open rectangular box.
Find:
a the values of x for which the volume is 100 cm^3
b the maximum volume



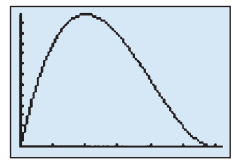
Solution

The figure shows how it is possible to form many open rectangular boxes with dimensions $12 - 2x$, $12 - 2x$ and x.

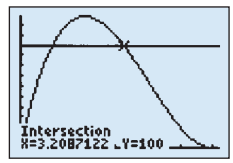


Volume of the box is $V = x(12 - 2x)^2$, $0 < x < 6$, a cubic model.

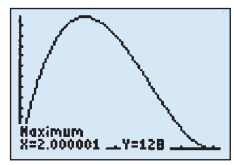
The graph may be plotted with a graphics calculator.



a The values of x for which $V = 100$ may be found by plotting the graph of $V = 100$ on the same screen and using **intersection** from the **CALC** menu.



b The maximum volume of the box may be found by using **maximum** from the **CALC** menu.

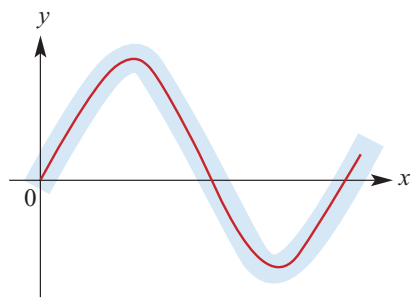


Example 31

It is found that 250 metres of the path of a stream can be modelled by a cubic function. The cubic passes throught the points $(0, 0)$, $(100, 22)$, $(150, -10)$, $(200, -20)$.

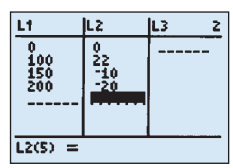
a Find the equation of the cubic function.

b Find the maximum deviation of the graph from the x-axis for $x \in [0, 250]$.

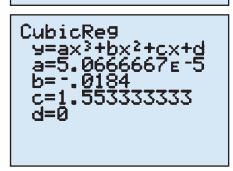


Solution

a Enter the x-coordinates in **L1** and the y-coordinates in **L2** as shown.



Enter **CubicReg L1, L2, Y1** in the Home screen and press ENTER. The result is as shown.



The equation is

$$y = \frac{1}{10\,000}\left(\frac{38}{75}x^3 - 184x^2 + \frac{46\,600}{3}x\right)$$

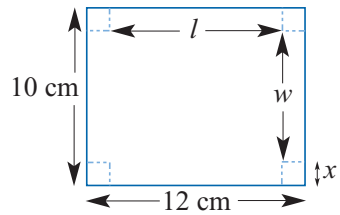
(In order to obtain this, multiply each of the coefficients by 10 000 and use **Frac** to change to fraction form.)

b Use **maximum** and **minimum** from the **CALC** menu to determine the maximum deviation from the x-axis.

This is 38.20 metres.

Exercise 6K

1 A rectangular sheet of metal measuring 10 cm × 12 cm is to be used to construct an open rectangular tray. The tray will be constructed by cutting out four equal squares from each corner of the sheet as shown in the diagram.



a If the edge of each cut-out square is x cm, express l and w in terms of x.

b Write down a rule for the volume, V cm^3, of the open tray in terms of x.

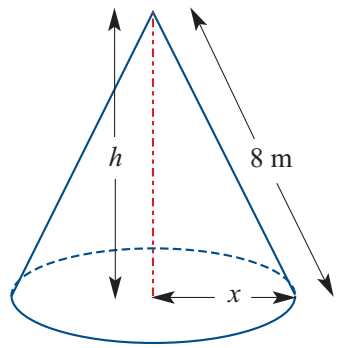
c Use a graphics calculator to help draw the graph of V against x for suitable values of x.

d Find the value of V when $x = 1$.

e Find the values of x for which $V = 50$.

f Find the maximum volume of the box and the value of x for which this occurs.

2 The diagram shows a conical heap of gravel. The slant height of the heap is 8 m, the radius of the base x m, and the height h m.



a Express x in terms of h.

b Construct a function which expresses V, the volume of the heap in m^3, in terms of h.

c Use a graphics calculator to help draw the graph of V against h.

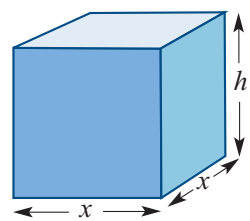
d State the domain for the function.

e Find the value of V when $h = 4$.

f Find the values of h for which $V = 150$.

g Find the maximum volume of the cone and the corresponding value of h.

3 The figure shows a rectangular prism with a square cross-section.



a If the sum of the dimensions, length plus width plus height, is 160 cm, express the height, h, in terms of x.

b Write down an expression for the volume, V cm^3, for the prism in terms of x.
State the domain.

c Use a graphics calculator to help draw the graph of V against x.

d Find the value(s) of x for which $V = 50\,000$.

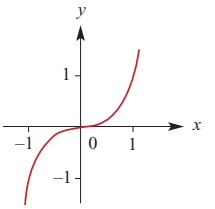
e Find the maximum volume of the box.

MC TEST D&D TEST

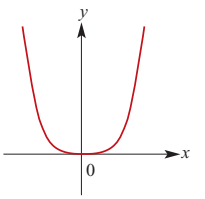
Review

Chapter summary

- The basic shape of the curve defined by $y = x^3$ is shown in the graph.



- The graph of $y - k = a(x - h)^3$ has the same shape as $y = ax^3$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction. (h and k are positive constants.)
- The basic shape of the curve defined by $y = x^4$ is shown in the graph.



- The graph of $y - k = a(x - h)^4$ has the same shape as $y = ax^4$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction. (h and k are positive constants.)
  The turning point is at (h, k).
  When sketching quartic graphs which are of the form $y = a(x - h)^4 + k$, first identify the turning point. To add further detail to the graph the x- and y-axis intercepts are found.
- **Factorisation of polynomials**
  - **a** Remainder theorem
    When $P(x)$ is divided by $(x - a)$ then the remainder is equal to $P(a)$.
    e.g. If $P(x) = x^3 + 3x^2 + 2x + 1$ is divided by $x - 2$ then the remainder is
    $$P(2) = (2)^3 + 3(2)^2 + 2(2) + 1$$
    $$= 25$$
  - **b** Factor theorem
    If for a polynomial $P(x)$, $P(a) = 0$ then $x - a$ is a factor and, conversely, if $x - a$ is a factor of $P(x)$ then $P(a) = 0$.
    e.g. If $P(x) = x^3 - 4x^2 + x + 6$
    $$P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$$
    then $(x + 1)$ is a factor or $P(x)$.
  - **c** Sums and differences of cubes
    $$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$
    e.g. $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$
    $$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$
    e.g. $8x^3 + 64 = (2x)^3 + 4^3$
    $$= (2x + 4)(4x^2 - 8x + 16)$$

■ **Solving cubic equations**

The following are the steps in the process of solving cubic equations:

i Determine factors by using the factor theorem and dividing.

ii Use the null factor law to determine solutions.

e.g. Solving $x^3 - 4x^2 - 11x + 30 = 0$

$P(2) = 8 - 16 - 22 + 30 = 0 \quad \therefore x - 2$ is a factor

Dividing $x - 2$ into $x^3 - 4x^2 - 11x + 30$ gives

$$x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15)$$
$$= (x - 2)(x - 5)(x + 3)$$
$$\therefore (x - 2)(x - 5)(x + 3) = 0$$
$$\therefore x = 2, 5 \text{ and } -3$$

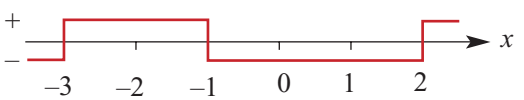
■ Sign diagrams assist in sketching graphs of cubic functions.

e.g. $y = x^3 + 2x^2 - 5x - 6$

$= (x + 1)(x - 2)(x + 3)$

When $x < -3$, y is negative.
When $-3 < x < -1$, y is positive.
When $-1 < x < 2$, y is negative.
When $x > 2$, y is positive.

Sign diagram



■ Sketching the graph of cubic functions $y = ax^3 + bx^2 + cx + d$

The steps in the process are:

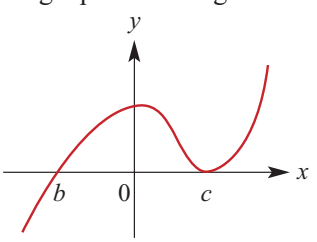
i Use the factor theorem and division to determine the x-axis intercepts.

ii d gives the y-axis intercept.

iii Draw a sign diagram.

■ Finding equations for given cubic graphs. The following may assist:

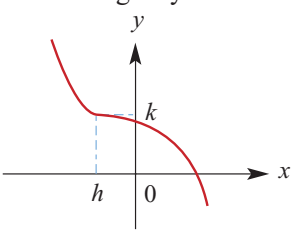
i



Form: $y = a(x - b)(x - c)^2$

Assume b and c are known, substitute another known point to calculate a.

ii



Form: $y - k = a(x - h)^3$

Substitute known values to determine a.

Alternatively, use the general form $y = ax^3 + bx^2 + cx + d$ and the known points to determine a, b, c and d.

■ Finite difference tables can be used to find the polynomial rule for a polynomial generated sequence.

Review

Multiple-choice questions

1 If $P(x) = x^3 + 3x^2 + x - 3$, then $P(-2) =$

A 1 **B** -1 **C** -25 **D** 3 **E** -5

2 For $a > b > c$ and $P(x) = (x - a)^2(x - b)(x - c)$, $P(x) < 0$ for $x \in$

A $(-\infty, a)$ **B** $(-\infty, b)$ **C** $(-\infty, c)$ **D** (c, b) **E** (b, a)

3 The image of the graph of $y = x^3$ under a dilation of factor 2 from the y-axis followed by a reflection in the y-axis and then a translation of 4 units in the negative direction of the y-axis is

A $y = -\frac{x^3}{8} - 4$ **B** $y = -\frac{x^3}{2} - 4$ **C** $y = -8x^3 - 4$

D $y = -\frac{x^3}{2} + 4$ **E** $y = \frac{x^3}{8} + 4$

4 The equation $x^3 + 5x - 10 = 0$ has only one solution. This solution lies between:

A -2 and -1 **B** -1 and 0 **C** 0 and 1 **D** 1 and 2 **E** 2 and 8

5 The polynomial $P(x) = x^4 + ax^2 - 4$ has zeros at $x = \sqrt{2}$ and $x = -\sqrt{2}$. The value of a is

A 0 **B** 2 **C** -2 **D** -3 **E** 3

6 The polynomial $P(x) = x^3 + ax^2 + bx - 9$ has zeros at $x = 1$ and $x = -3$. The values of a and b are

A $a = 1, b = -3$ **B** $a = -1, b = 3$ **C** $a = 5, b = 3$

D $a = -5, b = -3$ **E** $a = 0, b = 0$

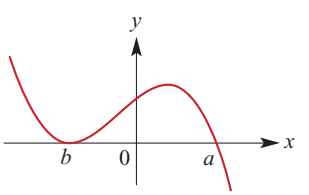
7 If $ax^3 + 2x^2 + 5$ is exactly divisible by $x + 1$, the value of a is

A 1 **B** 7 **C** -1 **D** 3 **E** -7

8 When the polynomial $P(x) = x^3 + 2x^2 - 5x + d$ is divided by $x - 2$, the remainder is 10. The value of d is

A 10 **B** 4 **C** -10 **D** -4 **E** 3

9 The diagram shows part of the graph of a polynomial function.

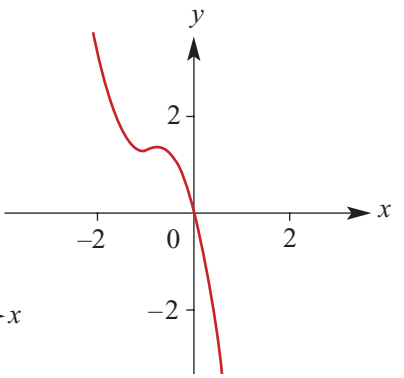


A possible equation for the rule of the function is

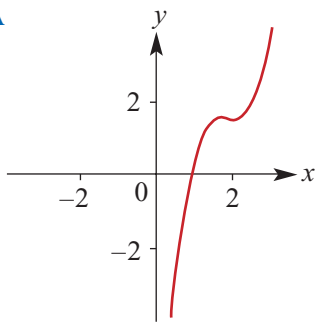
A $y = (x - b)^2(x - a)$ **B** $y = (x - a)^2(x - b)$ **C** $y = -(x + b)^2(x - a)$

D $y = (x - b)^2(a - x)$ **E** $y = (x + b)^2(a - x)$

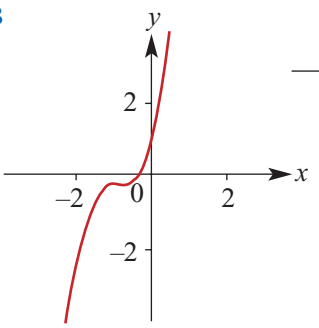
10 The function with rule $y = f(x)$ is shown:
Which one of the following could be the graph of the function with rule $y = 1 - f(x)$?



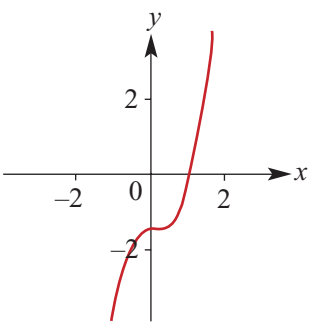
A



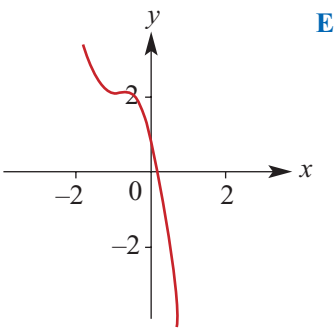
B



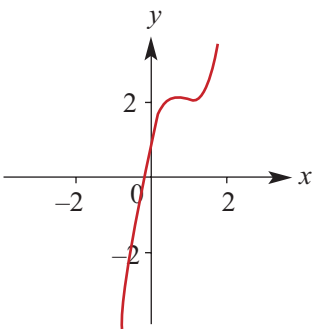
C



D



E



Short-answer questions (technology-free)

1 Sketch the graphs of each of the following:

a $y = (x - 1)^3 - 2$ **b** $y = (2x - 1)^3 + 1$ **c** $y = 3(x - 1)^3 - 1$

d $y = -3x^3$ **e** $y = -3x^3 + 1$ **f** $y = -3(x - 2)^3 + 1$

g $y = 4(x + 2)^3 - 3$ **h** $y = 1 - 3(x + 2)^3$

2 Sketch the graphs of each of the following:

a $y = (x - 1)^4$ **b** $y = (2x - 1)^4 + 1$ **c** $y = (x - 1)^4 - 1$

d $y = -2x^4$ **e** $y = -3x^4 + 1$ **f** $y = -(x - 2)^4 + 1$

g $y = 2(x + 1)^4 - 3$ **h** $y = 1 - 2(x + 2)^4$

3 **a** Show by use of the factor theorem that $2x - 3$ and $x + 2$ are factors of $6x^3 + 5x^2 - 17x - 6$. Find the other factor.

b Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$.

c Solve the equation $x^3 + x^2 - 11x - 3 = 8$.

d **i** Show that $3x - 1$ is a factor of $3x^3 + 2x^2 - 19x + 6$.

ii Find the factors of $3x^3 + 2x^2 - 19x + 6$.

4 Let $f(x) = x^3 - kx^2 + 2kx - k - 1$.

a Show that $f(x)$ is divisible by $x - 1$. **b** Factorise $f(x)$.

Review

5 Find the values of a and b for which $x^3 + ax^2 - 10x + b$ is divisible by $x^2 + x - 12$.

6 Draw a sign diagram for each of the following and hence sketch the graph:

a $y = (x + 2)(3 - x)(x + 4)$ **b** $y = (x - 2)(x + 3)(x - 4)$

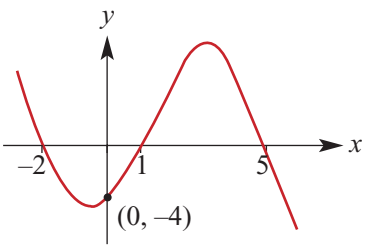
c $y = 6x^3 + 13x^2 - 4$ **d** $y = x^3 + x^2 - 24x + 36$

7 Without actually dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 + 4x^2 - 5x + 1, x + 6$ **b** $2x^3 - 3x^2 + 2x + 4, x - 2$

c $3x^3 + 2x + 4, 3x - 1$

8 Find the rule of the cubic for which the graph is shown.



9 Find a cubic function whose graph touches the x-axis at $x = -4$, passes through the origin and has a value of 10 when $x = 5$.

10 The function $f(x) = 2x^3 + ax^2 - bx + 3$. When $f(x)$ is divided by $x - 2$ the remainder is 15 and $f(1) = 0$.

a Calculate the values of a and b. **b** Find the other two factors of $f(x)$.

11 For the function f with rule $f(x) = x^3$, find the equation for the graph of the image under each of the following transformations:

a a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

b a dilation of factor 2 from the x-axis

c a reflection in the x-axis

d a reflection in the y-axis

e a dilation of factor 3 from the y-axis

12 For the function f with rule $f(x) = x^4$, find the equation for the graph of the image under each of the following transformations:

a a reflection in the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

b a dilation of factor 2 from the x-axis followed by a reflection in the y-axis

c a reflection in the x-axis followed by a translation of 2 units in the negative direction of the x-axis and 3 units in the positive direction of the y-axis

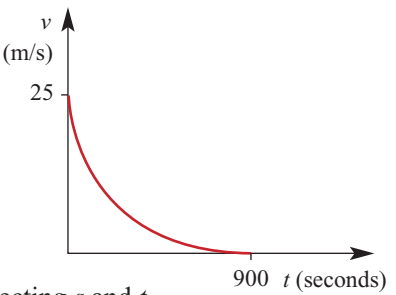
13 For each of the following find a sequence of transformations that take the graph of $y = x^3$ to the graph of:

a $y = 2(x - 1)^3 + 3$ **b** $y = -(x + 1)^3 + 2$ **c** $y = (2x + 1)^3 - 2$

Extended-response questions

1 There is a proposal to provide a quicker, more efficient and more environmentally 'friendly' system of inner-city public transport by using electric taxis. The proposal necessitates the installation of power sources at various locations as the taxis can only be driven for a limited time before requiring recharging.

The graph shows the speed, v m/s, which the taxi will maintain if it is driven at constant speed in such a way that it uses all its energy up in t seconds. The curve is a section of a parabola which touches the t-axis at $t = 900$. When $t = 0$, $v = 25$.



a Construct a rule for v in terms of t.

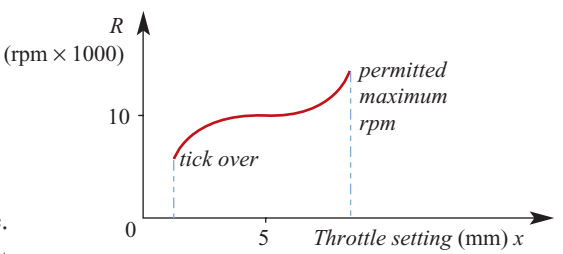
b If s metres is the distance that a taxi can travel before running out of electrical energy, write down a rule connecting s and t.

c Use a graphics calculator to help draw the graph of s against t.

d Originally the power sources were to be located at 2 km intervals. However there is a further proposal to place them at 3.5 km intervals. Is this new distance feasible?

e With the power sources at 2 km intervals, use your graph to determine approximately both the maximum and minimum speeds recommended for drivers. Explain your answer.

2 The figure shows part of a cubic graph that represents the relationship between the engine speed, R rpm, and the throttle setting, x mm from the closed position, for a new engine. It can be seen from the graph that the engine has a 'flat spot' where an increase in x has very little effect on R.



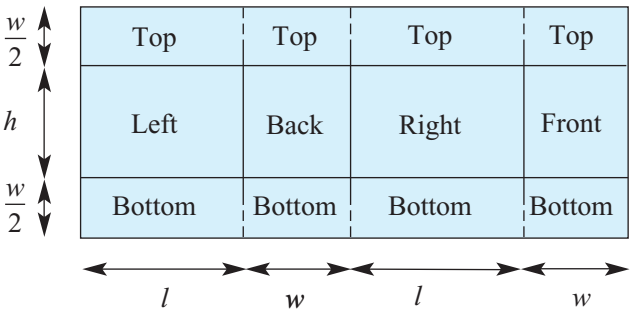
a Develop a cubic expression for R in terms of x of the form $R - k = a(x - h)^3$.

b Find a, if when the graph is extended it passes through the origin.

c In a proposed modification to the design, the 'flat spot' will occur when $x = 7$ mm. The speed of the engine in this case will be 12 000 rpm when $x = 7$ mm. Assuming that a cubic model still applies and that $R = 0$ when $x = 0$, write down an expression for R as a function of x.

Review

3 A net for making a cardboard box with overlapping flaps is shown in the figure. The dotted lines represent cuts and the solid lines represent lines along which the cardboard is folded.



a If $l = 35$ cm, $w = 20$ cm and $h = 23$ cm, calculate the area of the net.

b If the area of the net is to remain constant at the value calculated in **a** and $l = h$, write down an expression for V, the volume of the box in cm^3, as a function of l. (The maximum volume of the box will occur when $l = h$.)

Graphics Calculator **c** Use a graphics calculator to help draw the graph of V against l.

d Find the value of l when the volume of the box is:

i 14 000 cm^3 **ii** 1 litre = 10 000 cm^3

e Find the maximum volume of the box and the value of l for which this occurs.

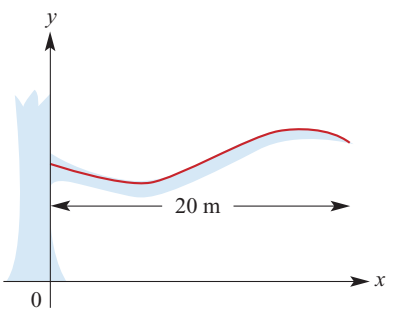
4 It is found that the shape of a branch of a eucalyptus tree can be modelled by a cubic function. The coordinates of several points on the branch are (0, 15.8), (10, 14.5), (15, 15.6), (20, 15).

Graphics Calculator **a** The rule for the function is of the form $y = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.

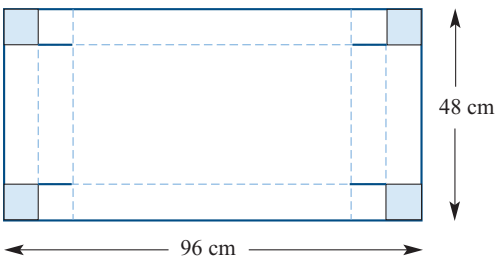
b Find the coordinates of the point on the branch that is:

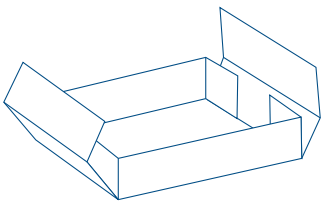
i closest to the ground

ii furthest from the ground



5 A reinforced box is made by cutting congruent squares of side length x cm from the four corners of a rectangular piece of cardboard that measures 48 cm by 96 cm. The flaps are folded up.





a Find an expression for V, the volume of the box formed.

b Plot a graph of V against x on your graphics calculator:

i What is the domain of the function V?

ii Using your graphics calculator, find the maximum volume of the box and the value of x for which this occurs (approximate values required).

c Find the volume of the box when $x = 10$.

d It is decided that $0 \leq x \leq 5$. Find the maximum volume possible.

e If $5 \leq x \leq 15$, what is the minimum volume of the box?

CHAPTER

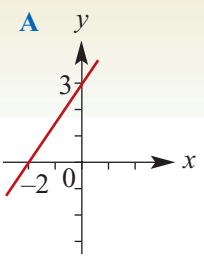
7

Revision

Revision of Chapters 2–6

7.1 Multiple-choice questions

1 The graph of $3x - 2y = -6$ is

A


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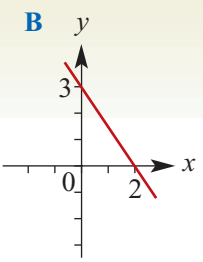

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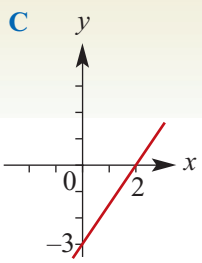
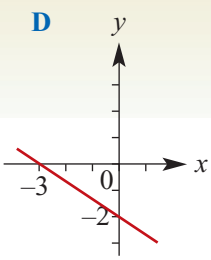
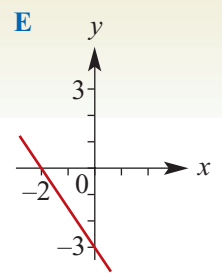


D


E


2 Which of the following points lie on both the line with equation $y = 3x - 1$ and the line with equation $4x + 2y = 8$?

A (0, 0) **B** (0, 4) **C** (2, 0) **D** (1, 2) **E** (2, 1)

3 The factors of $x^3 - 8$ are

A $(x + 2)(x^2 - 2x + 4)$ **B** $(x - 2)^3$ **C** $(x^2 - 4)(x + 2)$
D $(x - 2)(x^2 + 2x + 4)$ **E** $(x^2 + 4)(x - 2)$

4 The linear factors of $2x^2 - 5x - 12$ are

A $(2x + 1)$ and $(x - 12)$ **B** $(2x - 1)$ and $(x + 12)$ **C** $(2x + 3)$ and $(x - 4)$
D $2(2x - 3)$ and $(x + 2)$ **E** $2\left(x - \frac{1}{2}\right)$ and $(x + 6)$

5 The remainder when $4x^3 - 5x + 5$ is divided by $2x + 3$ is

A 4 **B** 11 **C** -1 **D** 1 **E** 2

6 The equation $x^2 + y^2 + 6x - 2y + 6 = 0$ corresponds to a circle of radius

A 4 **B** $\sqrt{6}$ **C** 2 **D** 6 **E** 10

7 The gradient of the line with equation $2x + 4y - 6 = 0$ is

A $-\frac{1}{2}$ **B** 2 **C** 4 **D** -2 **E** 1

8 The equation of the line perpendicular to the line with equation $2x + 4y = 3$ and containing the point (1, 2) is

A $y = -\frac{1}{2}x + \frac{5}{2}$ **B** $2y = x + 2$ **C** $2y = x$ **D** $y = 2x - 4$ **E** $y = 2x$

9 If $x - 3$ is a factor of $x^3 + ax^2 - x - 6$ then a is equal to

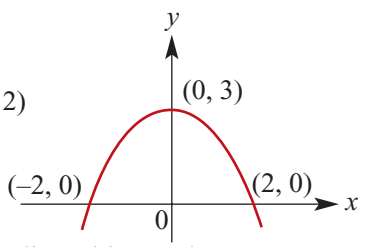
A 2 **B** -2 **C** 1 **D** -1 **E** -3

10 The linear factors of $x^3 + 8x^2 + 9x - 18$ are

A $x - 1, x + 3, x + 6$ **B** $x - 1, x - 3, x - 6$ **C** $x + 1, x - 3, x + 6$
D $x + 1, x + 3, x - 1$ **E** $x + 1, x - 3, x - 6$

11 The parabola shown has equation

A $y = 2(x - 2)(x + 2)$ **B** $y = -2(x - 2)^2$
C $3y = 4(x + 2)(x - 2)$ **D** $3y = 2(x + 2)(x - 2)$
E $4y = -3(x + 2)(x - 2)$



12 The line with equation $y = mx + c$ is perpendicular to the line with equation $y = 4 - 3x$. The value of m is

A 3 **B** $\frac{1}{3}$ **C** -3 **D** $-\frac{1}{3}$ **E** -1

13 If $f(x) = x^2 - 1$ then $f(x - 1)$ is equal to

A 0 **B** 1 **C** $x - 1$ **D** $x^2 - 2x$ **E** $x^2 - 2x - 2$

14 The graph of $y = x^2 + kx + k + 8$ touches the x-axis. The value of k is

A -4 **B** 8 **C** 12 **D** -4 or 8 **E** -8 or 12

15 Let $P(x) = 3x^3 - 4x - k$. If $P(x)$ is divisible by $x - k$, the remainder when $P(x)$ is divided by $x + k$ is

A $2k$ **B** k **C** 0 **D** $-k$ **E** $-2k$

16 The turning point of the graph of the quadratic function whose rule is $y = a(x - b)^2 + c$ is the point with coordinates

A $\left(\frac{b}{a}, c\right)$ **B** (b, c) **C** (c, b) **D** $(-b, c)$ **E** $\left(\frac{b}{a}, \frac{c}{a}\right)$

Revision

17 If the graphs of $y = 3 + 4x - x^2$ and $y = k$ have only one common point, then k is

A -1 **B** 1 **C** 4 **D** 7 **E** 2

18 The coordinates of the midpoint of the line segment with the end points (12, 7) and $(-3, 5)$ are

A $\left(7\frac{1}{2}, 6\right)$ **B** (9, 12) **C** (15, 2) **D** $\left(7\frac{1}{2}, 1\right)$ **E** $\left(4\frac{1}{2}, 6\right)$

19 M is the midpoint of XY. The coordinates of M and Y are $(7, -3)$ and (5, 4) respectively. The coordinates of X are

A $\left(6, \frac{1}{2}\right)$ **B** $(4, -14)$ **C** $(1, \infty)$ **D** $(9, -10)$ **E** $(6, -5)$

20 The range of the function represented by the set of ordered pairs $\{(x, y): y = x^2 + 1, x \in [-2, 1]\}$ is

A $[-3, 1]$ **B** $[1, 5]$ **C** $(1, \infty)$ **D** $[12, 5]$ **E** R

21 The equation $x^3 + 2x - 8 = 0$ has only one solution. This solution lies between

A -2 and -1 **B** -1 and 0 **C** 0 and 1 **D** 1 and 2 **E** 2 and 8

22 If $f: R \to R$ where $f(x) = x(x - 2)$ then $f(-3)$ is equal to

A -8 **B** 2 **C** 8 **D** 15 **E** -15

23 The circle $x^2 + y^2 - 11x - 10y + 24 = 0$ cuts the y-axis at M and N. The distance between M and N is

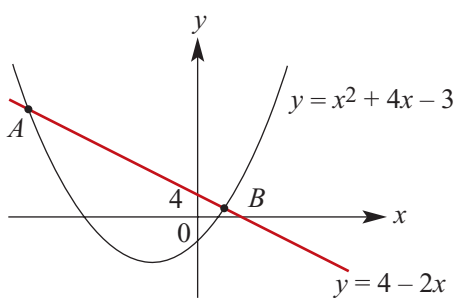
A 2 **B** 5 **C** 10 **D** 11 **E** 24

24 The length of the line segment joining the points with coordinates $(-4, -3)$ and $(-5, -10)$ is

A $2\sqrt{5}$ **B** $5\sqrt{2}$ **C** $5\sqrt{10}$ **D** $25\sqrt{2}$ **E** $15\sqrt{2}$

25 The straight line intersects the parabola at points A and B. The length of line segment AB is

A $\sqrt{48}$ **B** $\sqrt{60}$
C $\sqrt{180}$ **D** $\sqrt{320}$
E $\sqrt{360}$



26 Which one of the following ordered pairs **is not** a member of the relation $\{(x, y): y \leq 2x + 3\}$?

A $(1, 4)$ **B** $(-1, 1)$ **C** $\left(\frac{1}{2}, 3\frac{1}{2}\right)$ **D** $\left(-\frac{1}{2}, 2\frac{1}{2}\right)$ **E** $(2, 5)$

27 If the graph of $y = k + 2x - x^2$ touches the x-axis, the value of k is

A -2 **B** -1 **C** 0 **D** 1 **E** 2

28 If the line with equation $kx + y - 4 = 0$ is perpendicular to the line with equation $x - 2y + 3 = 0$, then k equals

A -2 **B** $\frac{1}{2}$ **C** 2 **D** $-\frac{1}{2}$ **E** $\frac{3}{4}$

29 The simultaneous equations $y = x^2 + k$ and $y = x$ have only one solution. The value of k is

A $\frac{1}{4}$ **B** $-\frac{1}{4}$ **C** 4 **D** -4 **E** 1

30 A circle has its centre at $(-4, 2)$ and touches the y-axis, Its equation is

A $x^2 + y^2 = 20$ **B** $x^2 + y^2 - 8x - 4y = 0$ **C** $x^2 + y^2 + 8x - 4y + 4 = 0$
D $x^2 + y^2 + 8x - 4y + 16 = 0$ **E** $x^2 + y^2 - 8x + 4y + 4 = 0$

31 If two lines $2x - y + 3 = 0$ and $ax + 3y - 1 = 0$ are parallel then the value of a is

A -6 **B** -2 **C** 2 **D** 3 **E** 6

32 The maximum domain of the function with rule $f(x) = \sqrt{4 - x^2}$ is

A $[0, 2]$ **B** $[-2, 2]$ **C** $(2, \infty)$ **D** $(-\infty, 2)$ **E** $(-2, 2)$

33 The range of the function $f: R \to R, f(x) = 2x^2 + 3x + 4$ is

A $(-\infty, 4]$ **B** $\left[\frac{4}{3}, \infty\right)$ **C** $\left[\frac{23}{8}, \infty\right)$ **D** $\left[\frac{3}{3}, \infty\right)$ **E** $[4, \infty)$

34 When $x^3 - kx^2 - 10kx + 25$ is divided by $x - 2$, the remainder is 9. The value of k is

A 25 **B** $-\frac{1}{2}$ **C** $\frac{7}{4}$ **D** 1 **E** $-\frac{13}{8}$

35 If $f(x) = x^2 - 7x + k$ and $f(k) = -9$, then $f(-1)$ equals

A -9 **B** -3 **C** 3 **D** 5 **E** 11

36 $2xy - x^2 - y^2$ is equal to

A $(x - y)^2$ **B** $(-x - y)^2$ **C** $(-x + y)^2$
D $-(x + y)^2$ **E** $-(x - y)^2$

37 $\{x: x^2 - x - 12 \leq 0\}$ is equal to

A $[3, 4]$ **B** $(-\infty, 3] \cup [4, \infty)$ **C** $[-3, 4]$
D $(-\infty, -4] \cup [-3, \infty)$ **E** $[-4, 3]$

Revision

38 If $f(x) = \frac{1}{2}x(x-1)$, then $f(x) - f(x+1)$ is equal to

A 1 **B** -2 **C** $-x$ **D** $-2x$ **E** $-\frac{x}{2}$

39 The set $\{x: 2x^2 - 2 \leq 0\}$ is equal to

A $(-\infty, 1]$ **B** $[-1, \infty)$ **C** $[-1, 1]$
D $[1, \infty) \cup (-\infty, -1]$ **E** $(-\infty, 1) \cup [1, \infty)$

40 The range of the function $f(x) = -2\left[\left(x - \frac{1}{2}\right)^2 - 3\right]$ is

A $(-\infty, 6]$ **B** $(-\infty, 3]$ **C** $(-3, \infty)$ **D** $[6, \infty]$ **E** $[-6, \infty]$

7.2 Extended-response questions

1 A particular plastic plate manufactured at a factory sells at \$11.50. The cost of production consists of an initial cost of \$3500 and then \$10.50 a plate. Let x be the number of plates produced.

a Let \$$C$ be the cost of production of x plates. Write an expression for C in terms of x.
b Let I be the income from selling x plates. Write an expression for I in terms of x.
c On the one set of axes sketch the graphs of I against x and C against x.
d How many plates must be sold for the income to equal the cost of production?
e Let $P = I - C$. Sketch the graph of P against x. What does P represent?
f How many plates must be sold for a profit of \$2000 to be made?

2 A swimming pool initially contains 45 000 litres of water. At noon, an inlet valve is opened, letting 40 litres of water per minute into the pool. Assume no water evaporates.

a Find an expression for the volume, V litres, in the pool m minutes after noon.
b When will the pool reach its maximum capacity of 55 000 litres?
c Sketch the graph of V against m (use the appropriate domain).

3 A tank of capacity 1000 litres is initially empty. Water flows into the tank at 20 litres/min for 10 minutes and then the rate is decreased to 15 litres/min. The water continues to flow in at this rate until the tank is filled.

a How much water is in the tank after 10 minutes?

The volume of water in the tank can be described by the rule

$$V = \begin{cases} at & 0 \leq t \leq 10 \\ bt + c & 10 < t \leq d \end{cases}$$

b Find the values of a, b, c and d. **c** Sketch the graph of V against t.

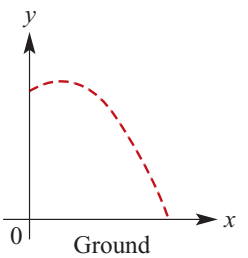
4 From a piece of wire 42 cm long, a length $10x$ cm is cut off and bent into a rectangle whose length is one and a half times its width. The remainder is bent to form a square.

a State the area of the rectangle in terms of x.
b State the area of the square in terms of x.
c State the possible values for x.
d Find an expression for the total area, A cm^2, of the square and the rectangle.
e Sketch the graph of A against x.
f What is the maximum possible total area.
g If the combined area of the rectangle and the square is 63 cm^2 find their dimensions.

5 An object is projected from a balcony as shown.
The path is described by the equation

$$y = -\frac{1}{10}(x + 10)(x - 20) \quad x \geq 0$$

where y metres is the height above the ground when the stone has travelled x metres horizontally.

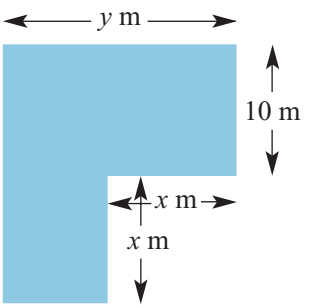


a What is the height of the stone at the point of the projection?
b What is the horizontal distance travelled before the stone hits the ground?
c What is the maximum height reached by the stone?

6 A rectangular block is 2 cm wider than it is high and twice as long as it is wide. Let x cm be the height of the block.

a Find an expression for the total surface area, A cm^2, in terms of x.
b Find A if: **i** $x = 1$ **ii** $x = 2$
c Find the value of x if $A = 190$.
d Sketch the graph of A against x for suitable values of x.
e Find an expression for the volume, V cm^3, of the block in terms of x.
f If the volume of the block is 150 cm^3 find the value of x.
g If the volume of the block is 1000 cm^3 find the value of x.

7 A region is enclosed as shown.



a Find expressions for:
i A (cm^2), the area of the region in terms of x and y
ii P (m), the perimeter of the region in terms of x and y
b **i** If the perimeter is 100 m find A in terms of x.
ii What is the maximum area possible?
iii State the possible values for x.
iv Sketch the graph of A against x for these values.

8 The diagram shows a window with six equal square panes divided by equally thick timber framing.

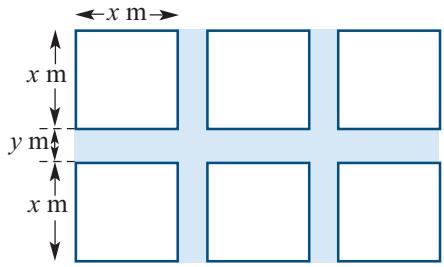
a Write down the total area of the whole window in terms of x and y.

Revision

b Show that the total area of the dividing wood is $7xy + 2y^2$.

c The total area of glass is 1.5 m^2 and the total area of the dividing wood is 1 m^2.

i Find x. **ii** Find y.



9 A fountain is 5 metres high. Water is coming vertically out of the fountain. The function which describes the height of a drop of water above the spout t seconds after leaving the spout of the fountain is given by $h(t) = -4.9t^2 + 30t + 5$.

a How high is the drop of water after 3 seconds?

b When will the drop be back at the height of the spout?

c Sketch the graph of the height of the drop above the fountain against time.

d If there is a little wind, the drop will not travel along a vertical line but its time in the air will not be affected and h will still describe its height above the ground. How long will it take for a drop of water to hit the ground?

10 An open rectangular cardboard box is 7 cm high and its length is 5 cm greater than its breadth, which is x cm.

a Find the length of the box in terms of x.

b Find the volume (V cm^3) of the box in terms of x.

c Find the surface area (S cm^2) in terms of x.

d Sketch the graphs of S against x and V against x for a suitable domain on the one set of axes.

e For what value of x is $V = S$?

f If the total surface area is 500 cm^2, find x.

11 Two points A and C have coordinates (1, 3) and (7, 7).

a Find the equation of the perpendicular bisector of AC.

b B is a point on the y-axis which is equidistant from A and C and $ABCD$ is a rhombus. Find: **i** the coordinates of B **ii** the coordinates of D

c Find the area of the rhombus $ABCD$.

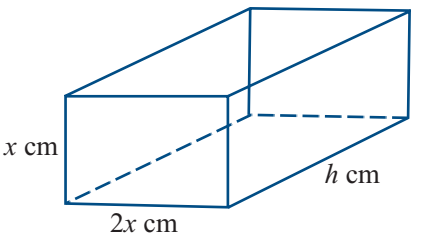
d Calculate the perpendicular distance of A from BC.

12 **a** A train travels 300 km at a constant speed of V km/h. If the train had travelled 5 km/h faster, the journey would have taken two hours less. Find the speed of the train travelling at the slower speed.

b A tank can be filled by two taps A and B in $33\frac{1}{3}$ minutes (33 minutes 20 seconds) when they are running together. Tap A running by itself fills the tank in 15 minutes less than tap B. Find the time taken for each tap running by itself to fill the tank.

c A hall can be paved with 200 square tiles of a certain size. If each tile were 1 cm longer and wider it would take 128 tiles. Find the length of each tile.

13 A piece of wire 400 cm long is used to make the 12 edges of a cuboid with dimensions as shown.



a Find h in terms of x

b Find the volume, V cm^3, in terms of x.

c State the possible values for x.

d Plot the graph of V against x on a graphics calculator for the domain determined in part c.

e State the values of x (correct to 3 decimal places) which will result in a volume of:

i 30 000 cm^3 **ii** 20 000 cm^3

f State the maximum volume, correct to 3 decimal places, and the corresponding value of x.

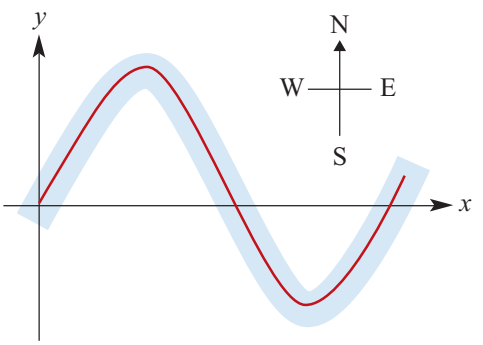
g The cuboid is covered in paper.

i Find the surface area, S cm^2, of the cuboid in terms of x.

ii Find the maximum value of S and the value of x for which this occurs.

h Find the values of x for which $S = V$.

14 A section of a path along the edge of a river can be modelled by a cubic function for $x \in [0, 250]$. Measurements are in metres. The following are the coordinates of points on the path: (0, 0), (100, 33), (50, 57) and (150, −15).

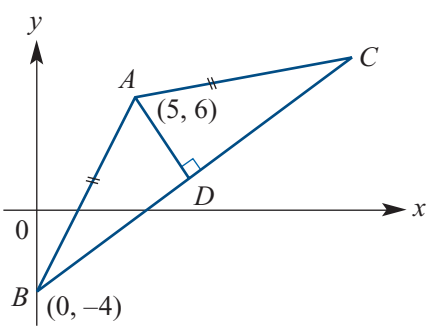


a Find the equation of the graph of the cubic function which passes through these points.

b The bank of the river is 5 metres to the north of the centre of the path. Find the equation of the cubic function which models the river bank.

c Find the largest deviation of the path from the x-axis.

15 ABC is an isosceles triangle. The coordinates of A and B are (5, 6) and (0, −4) respectively. Given that the gradient of BC is $\frac{3}{4}$ and D is the midpoint of BC find:



a the equation of BC

b the equation of AD

c the coordinates of D

d the length of the perpendicular AD

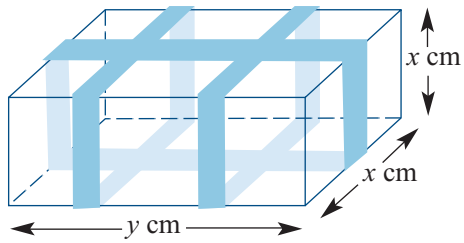
e the area of triangle ABC

Revision

16 A parcel in the form of a rectangular block is held together by three pieces of tape as shown.

The parcel has square ends with side length x cm. The length of the parcel is y cm.

The total length of tape is 500 cm.



Graphics CALCULATOR

a **i** Find y in terms of x.

ii Find V, the volume of the parcel in cm^3, in terms of x.

b Draw the graph of V against x for suitable values of x.

c State the domain of this function.

d Find the values of x for which $V = 25\,000$.

e Find the maximum volume of the parcel and the corresponding values of x and y.

CHAPTER

8

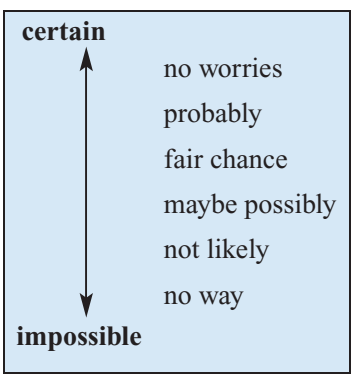
Probability

Objectives

- To understand the basic rules and notation of **set theory**.
- To understand the basic concepts and rules of **probability**.
- To use **Venn diagrams, tree diagrams**, and **Karnaugh maps** to determine probabilities for compound events.
- To understand and be able to apply the **addition rule**.
- To introduce the ideas of **conditional probability** and **independent events**.
- To use the **law of total probability** to calculate probabilities.

Uncertainty is involved in much of the reasoning we undertake every day of our lives. We make decisions based on the chances of this or that happening. More formally, chances are called **probability**. Some things which occur in the world can be predicted from our present store of knowledge, such as the time of the next high tide. Other things such as whether a head or tail will show when a coin is tossed, or the sex of a new baby are unpredictable. But even though the particular outcome is uncertain, the results are not entirely haphazard. There is a pattern that emerges in the long run which enables us to assign a numerical probability to each outcome, even though the individual outcome is unknown.

Probability has an everyday usage, giving some sort of rough gradation between the two extremes of *impossible* and *certain*, as indicated in the diagram.



8.1 Random experiments and events

In general, a **random experiment** is one in which the outcome of a single trial is uncertain, but is nonetheless an observable outcome. For example, consider the random experiment of tossing a coin and noting whether a head (H) or a tail (T) appears uppermost. The outcome observed must be either H or T, but on a particular toss it is not known which of H or T will be observed. The outcome observed must be one of a known set of possible outcomes, and the set of possible outcomes is called the **sample space** for the experiment. Set notation will be used in listing all the elements in the sample space in set brackets with each element separated by a comma. A single outcome of an experiment is often referred to as a sample point. For example, the sample space for the tossing of a coin would be written as

$$\{H, T\}$$

where H indicates head and T indicates tail. Throughout this chapter the letter ε will be used to denote the sample space.

For example, the following table lists the sample spaces for each of the random experiments described.

Random experiment	*Sample space*
The number of brown eggs in a carton of 12 eggs	$\varepsilon = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$
The result when two coins are tossed	$\varepsilon = \{HH, HT, TH, TT\}$
The number of calls to your phone in the next two hours	$\varepsilon = \{0, 1, 2, 3, 4. \ldots\}$
The time, in hours, it takes to complete your homework	$\varepsilon = \{t : t \geq 0\}$
The actual quantity of lemonade in a 1 litre bottle	$\varepsilon = \{q : 0 \leq q \leq 1\}$

An **event** may consist of a single outcome, or it may be several outcomes. For example, when rolling a die, the event of interest may be 'getting a six'. In this case, since the event consists of just one outcome, it is called a **simple event**. However, the event 'getting an odd number' can be achieved by rolling a 1, a 3 or a 5. As there is more than one outcome which defines this event, it is called a **compound event**.

It is sometimes convenient to use set notation to list the elements of the event. In general we use capital letters, A, B, C, . . . to denote sets. Naturally, the set which is associated with an event will be a subset of the sample space. That is, every element of the set is also contained in the sample space.

For example, the following table lists the experiments shown earlier and describes the sample space and an event for each one.

Sample space	*An event*
■ The number of brown eggs in a carton of 12 eggs The experiment is for a randomly chosen carton. $\varepsilon = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$	'More than half brown' $= \{7, 8, 9, 10, 11, 12\}$
■ The result when two coins are tossed $\varepsilon = \{HH, HT, TH, TT\}$	'Two heads' $= \{HH\}$
■ The number of calls to your phone in the next two hours $\varepsilon = \{0, 1, 2, 3, 4, \ldots\}$	'Fewer than two phone calls' $= \{0, 1\}$
■ The time in hours it takes to complete your homework $\varepsilon = \{t : t \geq 0\}$	'More than two hours' $= \{t : t > 2\}$
■ The actual quantity of lemonade in a 1 litre bottle $\varepsilon = \{q : 0 \leq q \leq 1\}$	'Less than half full' $= \{q : 0 \leq q < 0.5\}$

Example 1

A bag contains 7 marbles numbered from 1 to 7 and a marble is withdrawn.

a Give the sample space for this experiment.

b List the sample points (elements) of the event 'a marble with an odd number is withdrawn'.

Solution

a $\{1, 2, 3, 4, 5, 6, 7\}$ **b** $\{1, 3, 5, 7\}$

Some of the experiments given in this example are **multi-stage experiments**. That is, they are concerned with the experiments which could be considered to take place in more than one stage. For example, when considering the outcomes from tossing two coins we should consider the possible outcomes in two stages: the outcome from coin 1, followed by the outcome from coin 2. In such cases, it is helpful to list the elements of the sample space systematically by means of a tree diagram.

Example 2

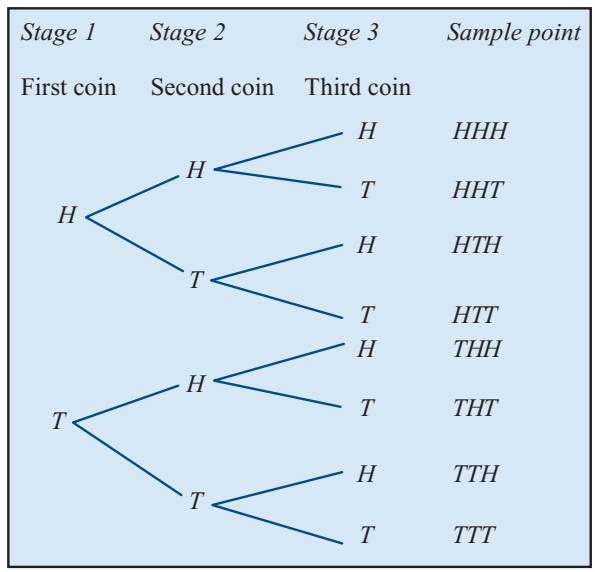
Three coins are tossed and the outcomes noted. List the sample space for this experiment.

Solution

A tree diagram is constructed to list the elements of the sample space.

Each path along the branches of the tree gives a sample point.

Thus the required sample space is $\varepsilon = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.



Exercise 8A

Example 1

1 List the sample space for the toss of a coin.

2 List the sample space for the outcomes when a die is rolled.

3 Answer the following for a normal deck of playing cards:

- **a** How many cards are there?
- **b** How many suits are there?
- **c** What are the suits called?
- **d** Which suits are red and which suits are black?
- **e** How many cards are there in each suit?
- **f** Which cards are known as the 'picture cards'?
- **g** How many aces are there in the deck?
- **h** How many 'picture cards' cards are there in the deck?

Example 2

4 List the sample spaces for the following experiments:

- **a** Two balls are chosen from a bag containing two black and two red balls.
- **b** A coin is tossed and then a die is rolled.
- **c** Three students are chosen for a committee from a class of 10 male and 10 female students, and the sex of each student noted.

5 List the sample spaces for the following experiments:

a the number of picture cards in a hand of five cards

b the number of female children in a family with six children

c the number of female students on a committee of three students chosen from a class of 10 male and 10 female students

6 List the sample spaces for the following experiments:

a the number of cars which pass through a particular intersection in a day

b the number of people on board a bus licensed to carry 40 passengers

c the number of times a die is rolled before a six is observed

7 List the outcomes (sample points) associated with the following events:

a 'an even number' when a die is rolled

b 'more than two female students' when three students are chosen for a committee from a class of 10 male and 10 female students

c 'more than four aces' when five cards are dealt from a standard pack of 52 cards

8 An experiment is defined as follows: A coin is tossed and the outcome observed. If the coin shows a head it is tossed again, but if it shows a tail a die is rolled. Use a tree diagram to list the sample space for the experiment.

9 A spinner is divided into four equal parts numbered 1, 2, 3 and 4. The spinner is spun twice.

a Use a tree diagram to list the sample space for the experiment.

b Circle the outcomes associated with the event 'the sum of the numbers is equal to 6'.

10 Spinner A is divided into three equal parts which are numbered 1, 2 and 3. Spinner B is divided into four equal parts which are numbered 1, 2, 3 and 4. Spinner A is spun, and the result noted. Then spinner B is spun and the result noted.

a Use a tree diagram to list the sample space for the experiment.

b List the outcomes associated with the event 'same number shows on both spinners'.

11 A die is rolled twice and the number uppermost recorded.

a Use a tree diagram to list the sample space for the experiment.

b List the outcomes associated with the event 'the first number is odd and the second is even'.

12 Michael has a drawer containing three red, two black and seven white socks. He decides to choose socks with his eyes closed until he has a pair of socks.

a Use a tree diagram to list the sample space for the experiment.

b What is the minimum number of socks he must draw in order to ensure he has a pair?

13 Cassie deals cards from a pack of 52 until she has two of the same suit.

a Use a tree diagram to illustrate the event 'a spade on the first and a heart on the second'.

b What is the minimum number of cards she must deal in order to ensure she has two of the same suit?

8.2 Determining empirical probabilities

In theory, every event has a true and constant probability or chance of occurring. The problem is that strategies are needed to help to determine the value of that probability. Sometimes, the probability is assigned a value just on the basis of experiences. For example, a newspaper sports journalist suggesting that Essendon has a 60% chance of winning its next game is relying purely on his or her own experiences. Another journalist might well assign this probability an entirely different value. Such probabilities are called **subjective probabilities**.

In practice, many probabilities are estimated by **experimentation**, by performing the random experiment leading to the outcome of interest many times and recording the results. This information can then be used to estimate the chances of it happening again in the future. Such probabilities are called **empirical probabilities**.

For example, consider the event 'obtaining a head' when a coin is tossed. Suppose one tosses the coin many times and counts the number of times a head is observed. The proportion of trials that resulted in a head is called the **relative frequency** of that event. That is:

$$\text{relative frequency of a head occurring} = \frac{\text{number of heads observed}}{\text{number of trials}}$$

The mathematics of probability recognises that in the long run the relative frequency of the occurrence of a particular event will settle down to a constant value, the true value of the probability. That is, if the number of trials, n, is sufficiently large, then the observed **relative frequency** becomes close to the **probability** $\Pr(A)$, or

$$\Pr(A) \approx \frac{\text{number of times the event occurs}}{\text{number of trials}} \quad \text{for large } n$$

This rule can be illustrated by repeating a simple experiment, such as the tossing of a coin, many times. On one historical occasion in about 1900, English statistician Karl Pearson tossed a coin 24 000 times, resulting in 12 012 heads and a relative frequency of 0.5005!

Example 3

Suppose that an actual observation of 100 rolls of two dice gave the following results:

Event	*Same numbers on the dice*	*Different numbers on the dice*
Number of occurrences	19	81

Use these results to estimate the probability of the same number appearing on both dice, and different numbers appearing on both dice.

Solution

We can calculate the proportion of the number of occurrences of each outcome to the number of tosses. In this example we divide each of the results by 100:

Event	*Same numbers on the dice*	*Different numbers on the dice*
Relative frequency	0.19	0.81

These proportions, or relative frequencies, are **estimates** of the chances of getting a particular number of 'doubles' on two rolls of the dice. If this experiment was repeated, it would generally be found that the results were slightly different. One might conclude that relative frequency is not a very good way of estimating probability. Sometimes, however, experiments are the **only** way to get at an unknown probability. One of the most valuable lessons to be learnt is that such estimates are not exact. Our best estimates of the probabilities will result from using as many trials as possible.

Example 4

In order to investigate the probability that a drawing pin lands point up, Katia decides to toss it 50 times and to count the number of favourable outcomes, which turns out to be 33. Mikki repeats the experiment, but she tosses the same drawing pin 100 times and counts 62 favourable outcomes. What is Katia's estimate of the probability of the drawing pin landing point up? What is Mikki's estimate? Which of these is the preferred estimate of the probability from these experiments? Based on the information available, what would be the best estimate of the probability?

Solution

From Katia's information $\Pr(\text{point up}) \approx \frac{33}{50} = 0.66$

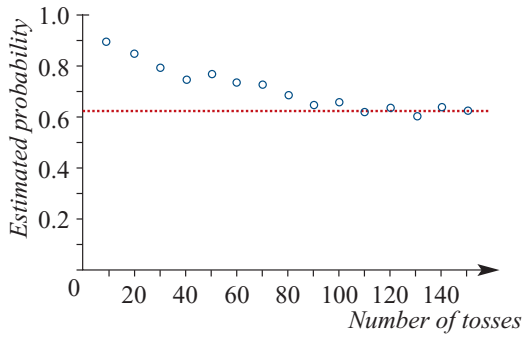
From Mikki's information $\Pr(\text{point up}) \approx \frac{62}{100} = 0.62$

Since Mikki has estimated the probability from a larger number of trials, her estimate would be preferred to Katia's.

Based on the information available, the best estimate of the probability would be found by combining the data from both experiments, and so maximising the number of trials. In total, 95 favourable outcomes were observed in 150 tosses, and this gives a 'best' estimate of the probability of $\frac{95}{150} = 0.63$.

Consider how the estimated probability of a drawing pin landing with the point up actually alters as the number of tosses used to estimate it is changed. The following graph shows the results of tossing a drawing pin 150 times.

The probability of the drawing pin landing point up was estimated every 10 throws. From the graph it may be seen that as the number of trials increases the estimated probability converges to a value and then stays fairly stable.



Thus, one interpretation of probability is as the proportion of times that event will occur in the long run. This interpretation also defines the minimum and maximum values of probability as 0 (the event never occurs) and 1 (the event always occurs). On analysing the frequency concept of probability, we see that other conditions must also hold which form the basis of our definition of probability. These are:

- $\Pr(A) \geq 0$. Since the relative frequency of occurrence of any event must be greater than or equal to zero, then so too must the probability associated with that event.
- $\Pr(\varepsilon) = 1$. Since the relative frequency of the whole sample space must be unity then so too is the corresponding probability.
- The sum of the probabilities of all the outcomes of a random experiment must be equal to 1.

If the probability of an event is zero, then this event is said to be **impossible**, as it cannot occur. Conversely, an event associated with a probability of 1 must occur and so is said to be **certain**.

Example 5

A random experiment may result in 1, 2, 3 or 4. If $\Pr(1) = \frac{1}{13}$, $\Pr(2) = \frac{2}{13}$, $\Pr(3) = \frac{3}{13}$ find the probability of obtaining a 4.

Solution

The sum of the probabilities is 1, therefore

$$\Pr(4) = 1 - \left(\frac{1}{13} + \frac{2}{13} + \frac{3}{13}\right)$$
$$= 1 - \frac{6}{13} = \frac{7}{13}$$

Example 6

A random experiment may result in A, B, C, D, E, where A, B, C, D are equally likely and E is twice as likely as A. Find $\Pr(E)$.

Solution

Let $\Pr(A) = \Pr(B) = \Pr(C) = \Pr(D) = x$. Then $\Pr(E) = 2x$.

The sum of the probabilities is 1, therefore

$$x + x + x + x + 2x = 1$$
$$6x = 1$$
$$x = \frac{1}{6}$$

Thus: $\Pr(E) = 2x = \frac{1}{3}$

Example 7

Consider the drawing pin tossing experiment in Example 4. What is the probability of the drawing pin landing point down?

Solution

Let U represent the event that the drawing pin lands point up, and D represent the event that the pin lands point down.

When two events U and D are mutually exclusive, and such that together they make up the entire sample space, they are said to be **complementary** events and

$$\Pr(U) + \Pr(D) = 1$$

Thus: $\Pr(D) = 1 - \Pr(U) = 1 - 0.63 = 0.37$

In general, the complement of any event A is denoted A' and $\Pr(A') = 1 - \Pr(A)$.

Exercise 8B

Example 3

1 Estimate the probability of the event specified occurring, use the data given:

- **a** Pr(head) if a coin is tossed 100 times and 34 heads observed
- **b** Pr(ten) if a spinner is spun 200 times and lands on the 'ten' 20 times
- **c** Pr(two heads) of two coins are tossed 150 times and two heads are observed on 40 occasions
- **d** Pr(three sixes) if three dice are rolled 200 times and three sixes observed only once

Example 4

2 A student decides to toss two coins and notes the results.

- **a** Do you think relative frequencies obtained from 20 trials would make for a good estimate of the probabilities?
- **b** Perform the experiment 20 times and estimate Pr(two heads), Pr(one head), and Pr(no heads).
- **c** Combine your results with those of your friends, so that you have results from at least 100 trials. Use these results to again estimate the probabilities.
- **d** Do you think the 100 trials data give better estimates of the probabilities?
- **e** How many trials would you need to find the probabilities exactly?

3 A student decides to roll two dice and record their sum.

- **a** Do you think the relative frequencies obtained from 20 trials would provide a good estimate of the probabilities?
- **b** Perform the experiment 20 times and estimate Pr(total = 4), Pr(total = 7), and Pr(total = 10).
- **c** Combine your results with those of your friends, or carry out four more sets of twenty trials, so that you have results from at least 100 trials of the experiment. Use these results to again estimate the probabilities.
- **d** Do you think the 100 trials data give better estimates of the probabilities?
- **e** How many trials would you need to find the probabilities exactly?

4 Two misshapen six-sided dice were used for the following experiment. The first die was thrown 500 times and 78 sixes were observed. The second die was thrown 700 times and 102 sixes were observed. If you wished to throw a six, which die would you choose to throw, and why?

5 A coin is known to be biased in favour of heads. To get some idea of the probability of the coin showing heads it is tossed 2000 times. The following figures, which were the only records kept, give the number of heads after various numbers of tosses:

Number of tosses	100	500	1000	2000
Number of heads	62	361	712	1404

- **a** Based on the 2000 tosses, what would be your best guess for the probability of the coin coming up heads?
- **b** How would this guess change if the coin had been tossed only 500 times?
- **c** Suppose that you found out that immediately after the 100^{th} toss a chip had come off the coin. Explain what effect this might have.
- **d** Use the above data to obtain the best estimate of the probability of the coin coming up heads after it has been chipped.

Example 5 **6** A random experiment results in 1, 2, 3, 4, 5, or 6. If $\Pr(1) = \frac{1}{12}$, $\Pr(2) = \frac{1}{6}$, $\Pr(3) = \frac{1}{8}$, $\Pr(5) = \frac{1}{6}$, $\Pr(6) = \frac{1}{8}$, find the probability of obtaining a 4.

Example 6 **7** When a biased six-sided die was tossed a large number of times, it was observed that the numbers 2, 3, 4 and 5 were equally likely to occur. The number 6 was noted to occur twice as often as the number 2, whereas the number 1 was noted to occur half as often as the number 2. Find the probability of each of the possible outcomes.

8 A target is divided into five regions labelled A, B, C, D, E. The probability of hitting the regions A, B, C and D are equal. If $\Pr(E) = 0.1$, find the probability of hitting region A, $\Pr(A)$.

Example 7 **9** For the target described in Question 8, find the probability of not hitting region A, $\Pr(A')$.

8.3 Determining probabilities by symmetry

Performing an experiment to estimate the probability of an event occurring is a time-consuming exercise. There must be other ways of predicting the probability of an event, using our knowledge of the situation – that is, to predict the probability of an outcome **before** the event takes place. This is certainly true when we are rolling a perfectly fair die. Since six outcomes are possible, it seems reasonable to claim that the probability of any one of them occurring on a roll is equal. That is:

$$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \Pr(6) = \frac{1}{6}$$

This claim is made on the assumption that the outcomes from rolling the die are all equally likely to occur.

Whenever an experiment has equally likely outcomes, we may define the probability of an event E as:

$$\Pr(E) = \frac{\text{number of outcomes favourable to } E}{\text{total number of possible outcomes}} = \frac{n(E)}{n(\varepsilon)}$$

where the notation $n(E)$ is used to represent the number of elements in set E.

Recall that the set of possible outcomes from a random experiment is called the sample space for the experiment, and is written as a list of items separated by commas and surrounded by curly brackets, thus $\{a, b, c\}$. In the two dice examples discussed in the previous section, there is the possibility of $\{1, 2, 3, 4, 5, 6\}$ on die 1, and $\{1, 2, 3, 4, 5, 6\}$ on die 2. So the sample space for this experiment looks like this:

Die 1 \ Die 2	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The events discussed in the previous section correspond to subsets of the sample space.

For example,

$$\text{'double'} = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

Thus, the probability of rolling a double can be determined:

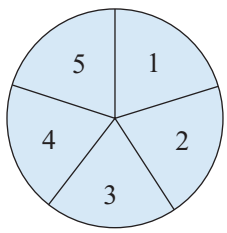
$$\Pr(\text{double}) = \frac{6}{36} = \frac{1}{6} = 0.167$$

From the previous section you can see that the experimental results are quite close to these, but not exactly the same. The probabilities determined by symmetry predict what would happen to the experimental probabilities in the long run. Like the experimental approach to probability, this definition defines the minimum and maximum values of probability as 0 (the event never occurs) and 1 (the event always occurs), and it can be readily verified that the other rules for probability still hold here where probabilities are calculated using symmetry.

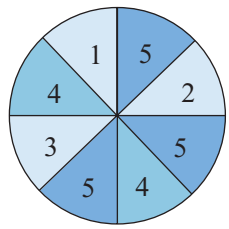
Example 8

Find the probability that each of the possible outcomes is observed for the following spinners.

a



b



Solution

a On spinner **a** there are five equally likely outcomes, so

$$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = 0.2.$$

b On spinner **b** the areas allotted to each of the outcomes are not all equal, so the associated probabilities will not all be the same. The probabilities will be equal to the fraction of the whole circle that each outcome defines.
Thus:

$$\Pr(1) = \Pr(2) = \Pr(3) = \frac{1}{8} = 0.125$$
$$\Pr(4) = \frac{2}{8} = \frac{1}{4} = 0.25$$
$$\Pr(5) = \frac{3}{8} = 0.375$$

Note that in both these cases $\Pr(1) + \Pr(2) + \Pr(3) + \Pr(4) + \Pr(5) = 1$.

If the sample space for an experiment contains n elements, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each of these points. The probability of any event A, which contains m of these sample points, occurring is then the ratio of the number of elements in A, $n(A)$, to the number of elements in ε, $n(\varepsilon)$. That is:

$$\Pr(A) = \frac{n(A)}{n(\varepsilon)} = \frac{m}{n}$$

Symmetry can also be used with areas of regions.

Example 9

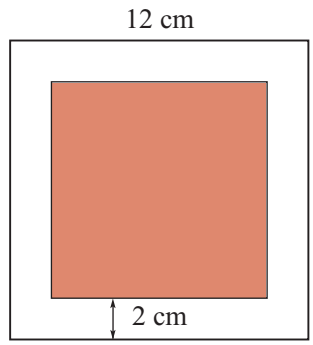
A square with side length 12 cm is drawn on a table.

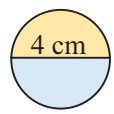
a A circular disc of radius 2 cm is placed on the table so that it lies completely inside the square. Draw a diagram of the square and indicate the region in which the centre of the disc can lie if the disc must be inside the square. (Touching an edge is accepted.)

b A game consists of throwing the disc described above onto the table. A point is scored if the disc lies completely within the square. (Touching an edge is accepted.) If a player always gets the centre of the disc into the square, what is the probability of scoring a point?

Solution

a The red square indicates the region in which the centre of the disc can lie if the disc is to lie within the square.





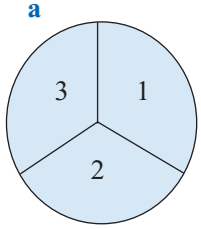
b There is a 2 cm wide border surrounding the red square. The area of the initial square is 144 cm^2. The area of the red region is 64 cm^2. Therefore the probability of the disc lying entirely within the square is $\frac{64}{144} = \frac{4}{9}$.

Exercise 8C

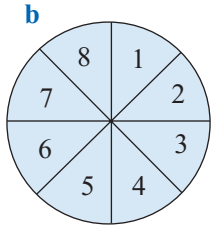
Example 8

1 Consider the following spinners. In each case, what is the chance of the pointer stopping in region 1?

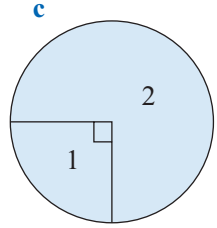
a



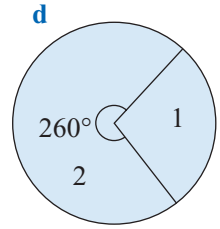
b



c



d



2 In 1989 the number of university enrolments in Victoria, by location, were as follows:

University	*Number enrolled*
Deakin	8262
La Trobe	13491
Monash	14847
Melbourne	21819

a What is the probability that a randomly selected university student is enrolled at Deakin University?
b What is the probability that a randomly selected university student is enrolled at Melbourne or Monash universities?
c What is the probability that a randomly selected university student is not enrolled at La Trobe University?

3 Suppose that in a certain city the same number of voters were born on each of the 365 days of the year, and that nobody was born on 29 February. Find the probability that the birthday of a voter selected at random:

a is 29 November
b is in November
c falls between 15 January and 15 February, not including either day
d is in the first three months of the year
e is not on 15 April
f is not in July

4 A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability that the card is:

a a club
b red
c a picture card (ace, king, queen or jack)
d not a diamond

5 A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability that the card is:

a less than 10
b less than or equal to 10
c an even number
d an ace

6 Two regular dice are rolled. List the sample space for this experiment and from it find the probability that the sum of the numbers showing is:

a even
b 3
c less than 6

7 Two regular dice are rolled. List the sample space for this experiment and from it find the probability that the sum of the numbers showing is:

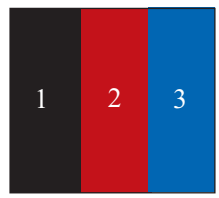
a equal to 10
b odd
c less than or equal to 7

8 An experiment consists of rolling a die and tossing a coin. Use a tree diagram to list the sample space for the experiment. Find the probability of obtaining a head and an even number.

9 A child is given a circle, a rectangle and a square to colour in, and a choice of three colours: red, blue and green. For each shape the child chooses a colour at random with which to colour it.

a Draw a tree diagram to show all the possible outcomes.
b Find the probability that:
i all the shapes are coloured red
ii the shapes are all of different colours
iii the circle is red
iv the rectangle is green and the square is not green.

10 A flag is made up of three sections, as shown in the diagram. Four colours are available for the flag: red, yellow, black and blue. However, each segment of the flag must be of a different colour.



a Draw a tree diagram to show all the possible outcomes, when colouring sections 1, 2 and 3 in succession.

b Find the probability that:

i red is not chosen **ii** red and black are chosen

iii section 1 is black and section 2 is red.

11 A bag contains five red balls, two blue balls and six white balls. A ball is chosen at random and its colour noted. Find the probability that the ball chosen is:

a red **b** not blue

12 A bag contains five balls, numbered 1 to 5. A ball is chosen at random, the number noted and the ball replaced. A second ball is then chosen at random and its number noted.

a Draw up a table of ordered pairs to show the sample space for the experiment.

b Find the probability that:

i the sum of the two numbers is 5 **ii** the two numbers are different

iii the second number is two more than the first.

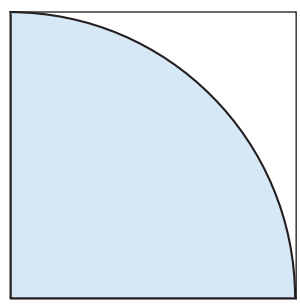
Example 9

13 A square with side length 5 cm is drawn on a table.

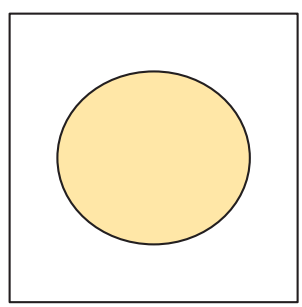
a A circular disc of radius 1 cm is placed on the table so that it lies completely inside the square. Draw a diagram of the square and indicate the region in which the centre of the disc can lie if the disc must be inside the square. (Touching an edge is accepted.)

b A game consists of throwing the disc described above onto the table. A point is scored if the disc lies completely within the square. (Touching an edge is accepted.) If a player always gets the centre of the disc into the square, what is the probability of scoring a point?

14 The square has side length 1 metre. A dart is thrown at the square. The blue region is a one-quarter of a circular disc centred at the bottom left vertex of the square. The dart is equally likely to hit any part of the square and it hits the square every time. Find the probability of hitting the blue region.



15 In a sideshow at a fete a dart is thrown at a square with side length 1 metre. The circle shown has a radius of 0.4 m. The dart is equally likely to hit any point on the square. Find the probability that the dart will hit:



a the shaded part of the square

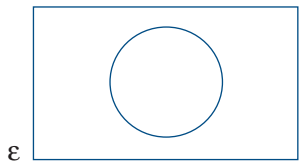
b the unshaded part of the square.

8.4 The addition rule

Before proceeding with the discussion of probability a review of sets and set notation is necessary.

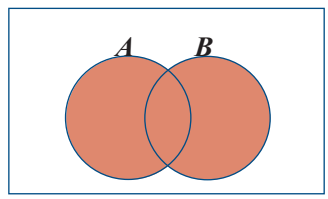
The **null** or **empty set**, denoted by Ø, is the set consisting of no elements. This is different from $\{0\}$, which is a set containing one element, 0.

Sets, and the relationships between sets, can be illustrated clearly by using **Venn diagrams**. ε is usually shown as a rectangle, and a subset of ε as a circle.



If A and B are any two sets, then the **union** of A and B, denoted $A \cup B$, is the set of all elements in A or B or both.

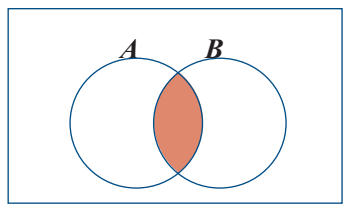
For example, if A is the set of students in a school who play hockey, and B the set of students who play tennis, then the union of A and B (shown on the Venn diagram by shading both sets A and B), would represent the set of students who play either hockey or tennis or both.



Union

The **intersection** of A and B, denoted by $A \cap B$, is the set of elements that are in both A and B.

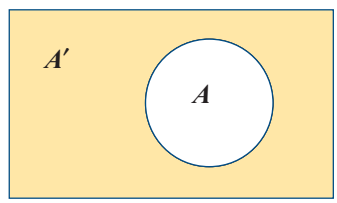
For example, the intersection of the sets previously described would represent the set of students who play both hockey and tennis, and is shown on a Venn diagram by shading only the area contained in both A and B.



Intersection

As previously, note that the **complement** of A, denoted A', is the set of points that are in ε but not in A.

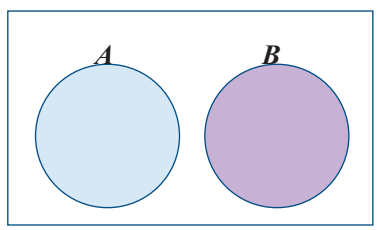
The complement of the set of students who play hockey in a school would represent the set of students who do not play hockey and is shown on a Venn diagram by shading only the area not contained in A.



Complement

Two sets, A and B, are said to be **disjoint** or **mutually exclusive** if they have no elements in common.

Thus, if A is the set of girls who play hockey in a school, and B is the set of boys who play hockey, then A and B are mutually exclusive, as no student can belong to both sets. The Venn diagram illustrates that the two sets are mutually exclusive. That is, $A \cap B = \emptyset$.



Disjoint sets

Finally, the number of elements in a set A is usually denoted $n(A)$. For example, if $A = \{a_1, a_2, a_3\}$ then $n(A) = 3$.

Venn diagrams can be used to help us solve practical problems involving sets.

Example 10

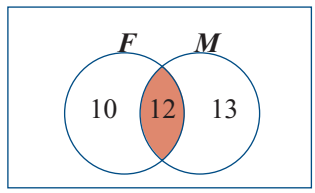
Fifty teenagers were asked what they did on the weekends. A total of 35 said they went to football matches, the movies or both. Of the 22 who went to football matches, 12 said they also went to the movies. Show this information on a Venn diagram. How many teenagers went to the movies but not to football matches? How many did not go to either of these events?

Solution

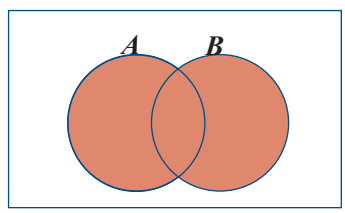
Let F denote the set of teenagers who attend football matches and M denote the set of teenagers who attend movies.

Hence from the information given
$n(F \cup M) = 35$, $n(F) = 22$ and $n(F \cap M) = 12$.

Students who go to the movies but not to football matches are found in the region $F' \cap M$, and from the diagram $n(F' \cap M) = 13$. Those who attend neither event are found in the region $F' \cap M'$, and from the diagram.
$n(F' \cap M') = 15$



Venn diagrams can be used to illustrate a very important rule that will enable us to calculate probabilities for more complex events. If A and B are two events in a sample space ε, and $A \cap B \neq \varnothing$, then the relationship between them can be represented by a Venn diagram, as shown.



From the Venn diagram we can see that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

(As the intersection has been counted twice, in both $n(A)$ and in $n(B)$, we must subtract it.)

Dividing through by $n(\varepsilon)$ gives:

$$\frac{n(A \cup B)}{n(\varepsilon)} = \frac{n(A)}{n(\varepsilon)} + \frac{n(B)}{n(\varepsilon)} - \frac{n(A \cap B)}{n(\varepsilon)}$$

Now, if each of the outcomes in ε is equally likely to occur, then each term in this expression is equal to the probability of that event occurring. This can be rewritten as:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

So the probability of A or B or both occurring can be calculated using

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

This is called the **addition rule** for combining probabilities.

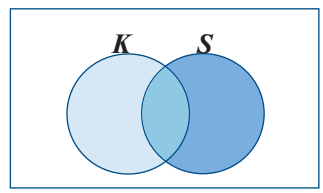
This rule can be used to help solve more complex problems in probability.

Example 11

If one card is chosen at random from a well-shuffled deck, what is the probability that the card is either a king or a spade?

Solution

Let event K be 'a king'. Then $K = \{\text{king of spades, king of hearts, king of diamonds, king of clubs}\}$, and $n(K) = 4$.



Let event S be 'a spade'. Then $S = \{\text{ace of spades, king of spades, queen of spades, . . .}\}$, and $n(S) = 13$.

The event 'a king or a spade' corresponds to the union of sets K and S.

$$\Pr(K) = \frac{4}{52}$$

$$\Pr(S) = \frac{13}{52}$$

$$\Pr(K \cap S) = \frac{1}{52}$$

and so, using the addition rule, we find

$$\Pr(K \cup S) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = 0.3077$$ (correct to 4 decimal places)

Skillsheet TEACHER Example 10

Exercise 8D

1 $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$. Show these sets on a Venn diagram and use your diagram to find:

a $A \cup B$ **b** $A \cap B$ **c** A'
d $A \cap B'$ **e** $(A \cap B)'$ **f** $(A \cup B)'$

2 $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{\text{multiples of four}\}$, $B = \{\text{even numbers}\}$. Show these sets on a Venn diagram and use your diagram to find:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

3 $\varepsilon = \{\text{different letters of the word MATHEMATICS}\}$

$A = \{\text{different letters of the word ATTIC}\}$

$B = \{\text{different letters of the word TASTE}\}$

Show ε, A and B on a Venn diagram, entering all the elements. Hence list the sets:

a A' **b** B' **c** $A \cup B$
d $(A \cup B)'$ **e** $A' \cup B'$ **f** $A' \cap B'$

4 In a survey of 100 university students, a market research company found that 70 students owned CD players, 45 owned cars and 35 owned CD players and cars. Use a Venn diagram to help you answer the following questions:

a How many students owned neither a car nor a CD player?

b How many students owned either a car or a CD player, but not both?

5 A swimming team consists of 18 members. Each member performs in at least one of the three events, freestyle (F), backstroke (B) and diving (D). Every diver also races. Eleven of the team swim freestyle and ten swim backstroke. Two of the team swim backstroke and dive, but do not swim freestyle. Five of the team swim freestyle only, and seven of those who swim backstroke do not dive. Draw a Venn diagram and find:

a $n(D)$ **b** $n(F \cap D \cap B)$ **c** $n(F \cup B)$ **d** $n(F \cap B \cap D')$

6 If $\varepsilon = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 4, 6\}$ and $B = \{3\}$ find:

a $\Pr(A \cup B)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A')$ **d** $\Pr(B')$

Example 11

7 Let $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

If A is the event 'an even number' and B is the event 'a multiple of three', find:

a $\Pr(A)$ **b** $\Pr(B)$ **c** $\Pr(A \cap B)$ and hence $\Pr(A \cup B)$

8 A letter is drawn at random from the word PROBABILITY. Let

A = the event 'drawing a vowel'

B = the event 'drawing a consonant'

C = the event 'drawing an I'

D = the event 'drawing a B'.

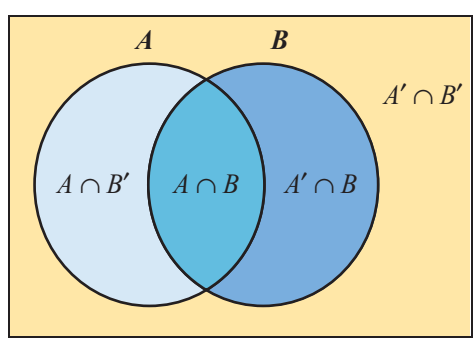
Find the probability of:

a $A \cup B$ **b** $A \cup C$ **c** $B \cup C$ **d** $A \cup D$ **e** $B \cup D$ **f** $C \cup D$

8.5 Probability tables and Karnaugh maps

The **probability table** is an alternative to a Venn diagram when illustrating a probability problem diagrammatically. Consider the Venn diagram which illustrates two intersecting sets A and B.

From the Venn diagram it can be seen that the sample space is divided by the sets into four disjoint regions: $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$. These regions may be represented in a table as follows. Such a table is sometimes referred to as a **Karnaugh map**:



	B	B'
A	$A \cap B$	$A \cap B'$
A'	$A' \cap B$	$A' \cap B'$

In a probability table, the entries in each cell give the probabilities of each of these events occurring.

		Column 1	Column 2
		B	B'
Row 1	A	$\Pr(A \cap B)$	$\Pr(A \cap B')$
Row 2	A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$

Further, from the Venn diagram we can see that set A consists of the union of the section of set A that intersects with set B, and the section of set A which does not intersect with set B. That is:

$$A = (A \cap B) \cup (A \cap B')$$

Sets $A \cap B$ and $A \cap B'$ are mutually exclusive, therefore:

$$\Pr(A \cap B) + \Pr(A \cap B') = \Pr(A)$$

and thus summing the probabilities in row 1 gives $\Pr(A)$.

Similarly, $\Pr(A \cap B) + \Pr(A' \cap B) = \Pr(B)$

$\Pr(A' \cap B) + \Pr(A' \cap B') = \Pr(A')$

$\Pr(A \cap B') + \Pr(A' \cap B') = \Pr(B')$

Finally, since

$$\Pr(A) + \Pr(A') = 1 \qquad \text{and} \qquad \Pr(B) + \Pr(B') = 1$$

the totals for both column 3 and row 3 are equal to 1. Thus, the completed table becomes:

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
Row 2	A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
Row 3		$\Pr(B)$	$\Pr(B')$	1

These tables can be useful when solving problems involving probability, as shown in the next three examples.

Example 12

If A and B are events such that $\Pr(A) = 0.7$, $\Pr(A \cap B) = 0.4$ and $\Pr(A' \cap B) = 0.2$, find:

a $\Pr(A \cap B')$ **b** $\Pr(B)$ **c** $\Pr(A' \cap B')$ **d** $\Pr(A \cup B)$

Solution

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$\Pr(A \cap B) = 0.4$	$\Pr(A \cap B')$	$\Pr(A) = 0.7$
Row 2	A'	$\Pr(A' \cap B) = 0.2$	$\Pr(A' \cap B')$	$\Pr(A')$
Row 3		$\Pr(B)$	$\Pr(B')$	1

The given information has been entered in the table in red.

a From row 1: $\Pr(A \cap B') = \Pr(A) - \Pr(A \cap B) = 0.7 - 0.4 = 0.3$

b From column 1: $\Pr(B) = \Pr(A' \cap B) + \Pr(A \cap B) = 0.2 + 0.4 = 0.6$

c From column 3: $\Pr(A') = 1 - \Pr(A) = 1 - 0.7 = 0.3$

From row 2: $\Pr(A' \cap B') = 0.3 - 0.2 = 0.1$

d The addition rule gives $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$= 0.7 + 0.6 - 0.4$

$= 0.9$

The completed table is shown below:

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$\Pr(A \cap B) = 0.4$	$\Pr(A \cap B') = 0.3$	$\Pr(A) = 0.7$
Row 2	A'	$\Pr(A' \cap B) = 0.2$	$\Pr(A' \cap B') = 0.1$	$\Pr(A') = 0.3$
Row 3		$\Pr(B) = 0.6$	$\Pr(B') = 0.4$	1

Example 13

Records indicate that, in Australia, 65% of secondary students participate in sport, and 71% of secondary students are Australian by birth. They also show that 53% of students are Australian by birth and participate in sport. Use this information to find:

a the probability that a person selected at random is not Australian by birth

b the probability that a person selected at random is not Australian by birth, and does not participate in sport.

Solution

The information in the question may be entered into a table as shown. (We will use A to represent 'Australian' and S to represent 'participation in sport'.)

	S	S'	
A	0.53		0.71
A'			
	0.65		1

All the empty cells in the table may now be filled in by subtraction.
In column 1: $\Pr(A' \cap S) = 0.65 - 0.53 = 0.12$
In column 3: $\Pr(A') = 1 - 0.71 = 0.29$
In row 1: $\Pr(A \cap S') = 0.71 - 0.53 = 0.18$
In row 3: $\Pr(S') = 1 - 0.65 = 0.35$
In row 2: $\Pr(A' \cap S') = 0.35 - 0.18 = 0.17$

The completed table is as follows:

	S	S'	
A	0.53	0.18	0.71
A'	0.12	0.17	0.29
	0.65	0.35	1

The probability that a person selected at random is not Australian by birth is given by $\Pr(A') = 0.29$.

The probability that a person selected at random is not Australian by birth and does not participate in sport is given by $\Pr(A' \cap S') = 0.17$.

Example 14

Suppose that of the people in Australia approximately 6% are colourblind. Suppose also that 45% of Australians are male and, further, that 5% of males are colourblind. Use this information to find:

a the probability that a person selected at random is female and colourblind

b the probability that a person selected at random is male and not colourblind.

Solution

This information is summarised in the following table. (We will use M to represent 'male' and B to represent 'colourblind'.)

	M	M'	
B	0.05		0.06
B'			
	0.45		1

Filling the empty cells by subtraction gives:

	M	M'	
B	0.05	0.01	0.06
B'	0.40	0.54	0.94
	0.45	0.55	1

Thus, reading from the table, the probability that a person selected at random is female (not male) and colourblind is $\Pr(M' \cap B) = 0.01$.

The probability that a person selected a random is male and not colourblind is given by $\Pr(M \cap B') = 0.40$.

Example 12

Exercise 8E

1 If A and B are events such that $\Pr(A) = 0.6$, $\Pr(A \cap B) = 0.4$ and $\Pr(A' \cap B) = 0.1$, find:

a $\Pr(A \cap B')$ **b** $\Pr(B)$ **c** $\Pr(A' \cap B')$ **d** $\Pr(A \cup B)$

2 If A and B are events such that $\Pr(A') = 0.25$, $\Pr(A' \cap B) = 0.12$ and $\Pr(B) = 0.52$, find:

a $\Pr(A)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A \cup B)$ **d** $\Pr(B')$

3 If C and D are events such that $\Pr(C \cup D) = 0.85$, $\Pr(C) = 0.45$ and $\Pr(D') = 0.37$, find:

a $\Pr(D)$ **b** $\Pr(C \cap D)$ **c** $\Pr(C \cap D')$ **d** $\Pr(C' \cup D')$

4 If E and F are events such that $\Pr(E \cup F) = 0.7$, $\Pr(E \cap F) = 0.15$ and $\Pr(E') = 0.55$, find:

a $\Pr(E)$ **b** $\Pr(F)$ **c** $\Pr(E' \cap F)$ **d** $\Pr(E' \cup F)$

5 If A and B are events such that $\Pr(A) = 0.8$, $\Pr(B) = 0.7$ and $\Pr(A' \cap B') = 0.1$, find:

a $\Pr(A \cup B)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A' \cap B)$ **d** $\Pr(A \cup B')$

Example 13

6 In a recent survey of senior citizens, it was found that 85% favoured giving greater powers of arrest to police, 60% favoured longer sentences for convicted persons, and 50% favoured both propositions.

a What percentage favoured at least one of the two propositions?

b What percentage favoured neither proposition?

7 Suppose a card is selected at random from an ordinary deck of 52 playing cards. Let

A = event a picture card is selected (i.e. jack, queen, king or ace)

C = event a heart is selected.

a List the event spaces corresponding to events A and C.

b Determine the following probabilities and express your results in words:

i $\Pr(A)$ **ii** $\Pr(C)$ **iii** $\Pr(A \cap C)$ **iv** $\Pr(A \cup C)$ **v** $\Pr(A \cup C')$

8 The following information applies to a particular class:

- The probability that a student's name begins with M and that the student studies French is $\frac{1}{6}$.
- The probability that a student's name begins with M is $\frac{3}{10}$.
- The probability that a student does not study French is $\frac{7}{15}$.

Find the probability that a student chosen at random from this class:

a studies French

b has a name which does not begin with M

c has a name which does begin with M, but does not study French

d has a name which does not begin with M and does not study French.

9 A frame is chosen at random from a shop where picture frames are sold. It is known that in this shop:

- the probability that the frame is made of wood is 0.72
- the probability that the frame is freestanding is 0.65
- the probability that the frame is not made of wood and is not freestanding is 0.2.

Find the probability that the randomly chosen frame is:

a made of wood or is freestanding

b made of wood and is freestanding

c not made of wood

d not made of wood but is freestanding.

10 A book is chosen at random from a bookshop. It is known that in this bookshop:

- the probability that the book is a hardback but not a novel is 0.05
- the probability that the book is not hardback but is a novel is 0.12
- the probability that the book is not a novel is 0.19.

Find the probability that the randomly chosen book is:

a a novel
b a hardback novel
c a hardback
d a novel or a hardback

11 At a school camp consisting of 60 students, sailing was offered as an activity one morning, and bushwalking in the afternoon. Every student attended at least one activity. 32 students went sailing and 40 students went bushwalking, find the probability that a student chosen at random:

a undertook neither of these activities
b has sailed or bushwalked
c has sailed and bushwalked
d has sailed but not bushwalked

12 At a barbecue attended by 50 people, hamburgers and sausages were available. It was found that 35 hamburgers and 38 sausages were eaten, and six people were noted to have eaten neither a hamburger nor a sausage. If no person ate more than one hamburger or one sausage, find the probability that a person chosen at random ate:

a a hamburger or a sausage
b a hamburger and a sausage
c had only one serve of food
d had only a hamburger

8.6 Conditional probability and the multiplication rule

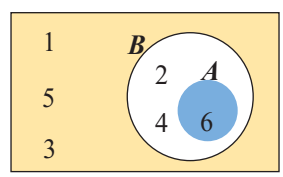
The probability of an event A occurring when it is known that some event B has occurred is called **conditional probability** and is written $\Pr(A|B)$. This is usually read as 'the probability of A given B', and can be thought of as a means of adjusting probability in the light of new information.

Example 15

Suppose we roll a fair die and define event A as 'rolling a six' and event B as 'rolling an even number'. What is the probability of rolling a six given the information that an even number was rolled?

Solution

The events A and B can be shown on a Venn diagram thus:



Since we know that event B has already occurred, we know that the outcome was 2, 4 or 6.

Thus

$$\Pr(\text{a 6 is rolled given an even number is rolled}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

$$= \frac{n(A)}{n(A \cup B)}$$

$$= \frac{1}{3}$$

Example 16

1000 drivers were questioned and classified according to age and number of accidents in the last year. The results are shown in the table.

	Age < 30	*Age* ≥ 30	*Total*
At most one accident	130	170	300
More than one accident	470	230	700
Total	600	400	1000

A driver is selected at random from the 1000 drivers. Given that the selected driver is less that 30 years of age, what is the probability that this person has had more than one accident?

Solution

It is known that the driver selected is less than 30 years of age, so the driver in question is one of the 600 drivers in this age group. Of these, 470 have had more than one accident. Thus, the probability is $\frac{470}{600}$.

Thus: $\Pr(\text{more than one accident} | \text{age} < 30) = \frac{470}{600} = \frac{47}{60}$

It may be seen from the table that the same answer is obtained from the quotient

$$\frac{\Pr(\text{more than one accident} \cap \text{age} < 30)}{\Pr(\text{age} < 30)} = \frac{{}^{470}/_{1000}}{{}^{600}/_{1000}} = \frac{470}{600} = \frac{47}{60}$$

That is:

$$\Pr(\text{more than one accident} | \text{age} < 30) = \frac{\Pr(\text{more than one accident} \cap \text{age} < 30)}{\Pr(\text{age} < 30)}$$

It can be shown that this relationship between the conditional probability of event A given that event B has already occurred, the probability of the intersection of events A and B, and the probability of event B is true for all events A and B, so long as the probability of event B is not equal to zero.

In general, the conditional probability of an event A, given that event B has already occurred, is given by

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad \text{if } \Pr(B) \neq 0$$

This formula may be rearranged to give the **multiplication rule for probability:**

$$\Pr(A \cap B) = \Pr(A|B) \times \Pr(B)$$

Example 17

Given that for two events A and B, $\Pr(A) = 0.7$, $\Pr(B) = 0.3$ and $\Pr(B|A) = 0.4$, find:

a $\Pr(A \cap B)$ **b** $\Pr(A|B)$

Solution

a $\Pr(A \cap B) = \Pr(B|A) \times \Pr(A) = 0.4 \times 0.7 = 0.28$

b $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{0.28}{0.3} = \dfrac{14}{15}$

Example 18

In a particular school 55% of the students are male and 45% are female. Of the male students 13% say mathematics is their favourite subject, while of the female students 18% prefer mathematics. Find the probability that:

a a student chosen at random prefers mathematics and is female

b a student chosen at random prefers mathematics and is male.

Solution

Let us use M to represent male, F for female, and P for prefer mathematics.

Thus: $\Pr(M) = 0.55$, $\Pr(F) = 0.45$, $\Pr(P|M) = 0.13$ and $\Pr(P|F) = 0.18$

We can use the multiplication rule to find the required probabilities:

a The event 'prefers mathematics' and 'is female' is represented by $P \cap F$.

Thus: $\Pr(P \cap F) = \Pr(P|F) \times \Pr(F) = 0.18 \times 0.45 = 0.081$

b The event 'prefers mathematics' and 'is male' is represented by $P \cap M$.

Thus: $\Pr(P \cap M) = \Pr(P|M) \times \Pr(M) = 0.13 \times 0.55 = 0.0715$

As has already been seen, the tree diagram is an efficient way of listing a multi-stage sample space. If the probabilities associated with each stage are also added to the tree diagram, it becomes a very useful way of calculating the probability for each sample outcome. The probabilities at each stage are conditional probabilities that the particular path will be followed, and the multiplication rule says that the probability of reaching the end of a given branch is the product of the probabilities associated with each segment of that branch.

Example 19

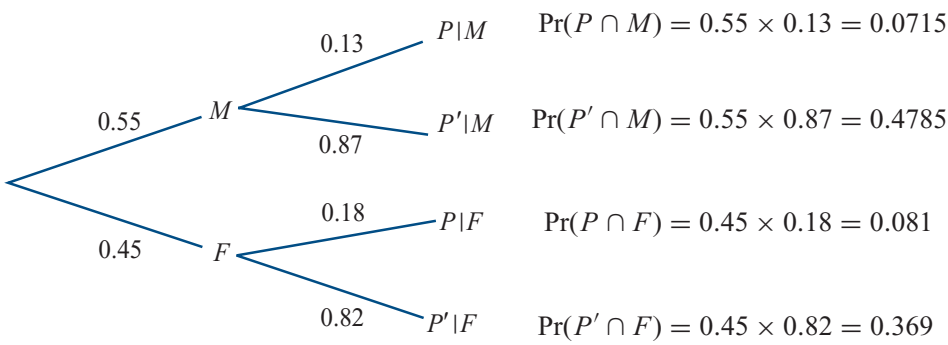
Using the information for Example 18, construct a tree diagram and use it to determine:

a the probability that a student selected is female and does not prefer mathematics

b the overall percentage of students who prefer mathematics.

Solution

The situation described can be represented by a tree diagram as follows:



a To find the probability that a student is female and does not prefer mathematics we multiply along the appropriate branches thus:

$$\Pr(F \cap P') = \Pr(F) \times \Pr(P'|F) = 0.45 \times 0.82 = 0.369$$

b Now, to find the overall percentage of students who prefer mathematics we recall that:

$$P = (P \cap F) \cup (P \cap M)$$

Since $P \cap F$ and $P \cap M$ are mutually exclusive:

$$\Pr(P) = \Pr(P \cap F) + \Pr(P \cap M) = 0.081 + 0.0715 = 0.1525$$

Thus 15.25% of all students prefer mathematics.

The solution to part b of Example 19 is an application of a rule known as the **law of total probability**. This can be expressed in general terms as follows:

In general, the **law of total probability** states that in the case of two events, A and B:

$$\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|B')\Pr(B')$$

A further example of the use of the law of total probability is given in the following example.

Example 20

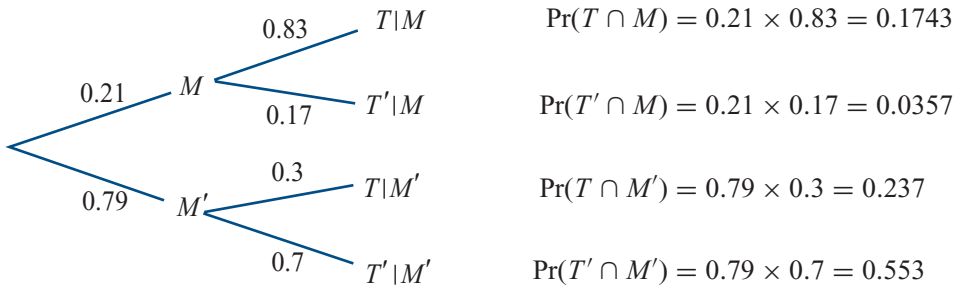
In a certain town, the probability that it rains on any Monday is 0.21. If it rains on Monday, then the probability that it rains on Tuesday is 0.83. If it does not rain on Monday, then the probability of rain on Tuesday is 0.3. Find the probability that it rains:

a on both days **b** on Tuesdays

Solution

Let M represents the event 'rain on Monday' and T represent the event 'rain on Tuesday'.

The situation described in the question can be represented by a tree diagram. You can check that the probabilities are correct by seeing if they add to 1.



a The probability that it rains on both Monday and Tuesday is given by

$\Pr(T \cap M) = 0.21 \times 0.83 = 0.1743$

b The probability that it rains on Tuesdays is given by

$\Pr(T) = \Pr(T \cap M) + \Pr(T \cap M') = 0.1743 + 0.237 = 0.4113$

Exercise 8F

Example 15

1 Suppose that a fair die is rolled, and event A is defined as 'rolling a six' and event B as 'rolling a number greater than 2'. Find $\Pr(A|B)$.

Example 16

2 The following data was derived from accident records on a highway noted for its above-average accident rate.

Type of accident	*Probable cause*				*Total*
	Speed	*Alcohol*	*Reckless driving*	*Other*	
Fatal	42	61	22	12	137
Non fatal	88	185	98	60	431
Total	130	246	120	72	568

Use the table to estimate:

a the probability that speed is the cause of the accident
b the probability that the accident is fatal
c the probability that the accident is fatal, given that speed is the cause
d the probability that the accident is fatal, given that alcohol is the cause.

Example 17

3 Given that for two events A and B, $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(B|A) = 0.1$, find:

a $\Pr(A \cap B)$ **b** $\Pr(A|B)$

4 For events A and B:

a $\Pr(A) = 0.7$, $\Pr(A \cap B) = 0.4$, find $\Pr(B|A)$

b $\Pr(A|B) = 0.6$, $\Pr(B) = 0.5$, find $\Pr(A \cap B)$

c $\Pr(A|B) = 0.44$, $\Pr(A \cap B) = 0.3$, find $\Pr(B)$

5 In a random experiment $\Pr(A) = 0.5$, $\Pr(B) = 0.4$, and $\Pr(A \cup B) = 0.7$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A|B)$ **c** $\Pr(B|A)$

6 In a random experiment $\Pr(A) = 0.6$, $\Pr(B) = 0.54$, and $\Pr(A \cap B') = 0.4$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A|B)$ **c** $\Pr(B|A)$

7 In a random experiment $\Pr(A) = 0.4$, $\Pr(A|B) = 0.6$, and $\Pr(B) = 0.5$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(B|A)$

8 A fair coin is tossed twice. Let A be the event 'the first toss is a head', B be the event 'the second toss is a head' and C be the event 'at least one toss is a head'. Find the following probabilities:

a $\Pr(B)$ **b** $\Pr(C)$ **c** $\Pr(B|A)$

d $\Pr(C|A)$ **e** $\Pr(B|C)$ **f** $\Pr(C|B')$

9 The current football fixture has the local team playing at home for 60% of its matches. When it plays at home, the team wins 80% of the time. When it plays away, the team wins only 40% of the time. What percentage of its games does the team play away and win?

10 The probability that a car will need an oil change is 0.15, the probability that it needs a new oil filter is 0.08, and the probability that both the oil and the filter need changing is 0.03. Given that the oil has to be changed, what is the probability that a new oil filter is also needed?

11 A card is selected from a pack of 52 playing cards. The card is replaced and a second card chosen. Find the probability that:

a both cards are hearts **b** both cards are aces

c the first card is red and the second is black **d** both cards are picture cards

12 A card is selected from a pack of 52 playing cards, and **not** replaced. Then a second card chosen. Find the probability that:

a both cards are hearts **b** both cards are aces

c the first card is red and the second is black **d** both cards are picture cards

13 A person is chosen at random from the employees of a large company. Let W be the event that the person chosen is a woman, and A be the event that the person chosen is 25 years or older. Suppose the probability of selecting a woman $\Pr(W) = 0.652$ and the probability of a woman being 25 years or older is $\Pr(A|W) = 0.354$. Find the probability that a randomly chosen employee is a woman aged 25 years or older.

14 In a class of 28 students there are 15 girls. Of the students in the class, six girls and eight boys play basketball. A student is chosen at random from the class. If G represents the event that a girl student is chosen and B represents the event that the student chosen plays basketball, find:

a $\Pr(G)$ **b** $\Pr(B)$ **c** $\Pr(B')$ **d** $\Pr(B|G)$
e $\Pr(G|B)$ **f** $\Pr(B|G')$ **g** $\Pr(B' \cap G')$ **h** $\Pr(B \cap G)$

15 In a recent survey it was found that 85% of the population eats red meat. Of those who eat red meat, 60% preferred lamb. A person is chosen at random from the population. If R represents the event that the person eats red meat, and L represents the event that the person prefers lamb, find:

a $\Pr(R)$ **b** $\Pr(L|R)$ **c** $\Pr(L \cap R)$ **d** $\Pr(L)$

16 In a senior college, 25% of the Year 11 students and 40% of the Year 12 students would prefer not to wear school uniform. This particular college has 320 Year 11 students and 280 Year 12 students. Find the probability that a randomly chosen student is in Year 11 and is in favour of wearing school uniform. What is the overall percentage of students who are in favour of wearing school uniform?

17 At a certain school it was found that 35% of the 500 boys and 40% of the 400 girls enjoyed bushwalking. One student from the school is chosen at random. Let G represent the event that the student is a girl, and B represent the event that the student enjoys bushwalking.

a Find, correct to 2 decimal places:
i $\Pr(G)$ **ii** $\Pr(B|G)$ **iii** $\Pr(B|G')$ **iv** $\Pr(B \cap G)$ **v** $\Pr(B \cap G')$
b Find $\Pr(B)$.
c Hence find:
i $\Pr(G|B)$ **ii** $\Pr(G|B')$

18 In a factory two machines produce a particular circuit board. The older machine produces 480 boards every day, of which an average of 12% are defective. The newer machine produces 620 boards each day, of which an average of 5% are defective. A board is chosen at random and checked. Let N represent the event that the board comes from the newer machine, and D represent the event that the board is defective.

a Find, correct to 2 decimal places:
i $\Pr(N)$ **ii** $\Pr(D|N)$ **iii** $\Pr(D|N')$ **iv** $\Pr(D \cap N)$ **v** $\Pr(D \cap N')$
b Find $\Pr(D)$.
c Hence find Pr(N|D), correct to 2 decimal places.

19 Jane has three bags of lollies. In bag 1 there are three mints and three toffees, in bag 2 there are three mints and two toffees and in bag 3 there are two mints and one toffee. Jane selects a bag at random, and then selects a lolly at random. Find:

a the probability she chooses a mint from bag 1
b the probability she chooses a mint
c the probability that Jane chose bag 1, given that she selects a mint.

20 Describe the relationship between non-empty events A and B if:

a $\Pr(A|B) = 1$ **b** $\Pr(A|B) = 0$ **c** $\Pr(A|B) = \dfrac{\Pr(A)}{\Pr(B)}$

8.7 Independent events

Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other. That is, if

$$\Pr(A|B) = \Pr(A)$$

If $\Pr(B) \neq 0$ then:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad \text{(multiplication rule of probability)}$$

Thus, equating the two expressions for $\Pr(A|B)$:

$$\Pr(A) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad \text{or} \quad \Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

When A and B are independent events:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

Compare this with the multiplication rule:

$$\Pr(A \cap B) = \Pr(A|B) \times \Pr(B)$$

Example 21

Consider the information given in Example 16 in the previous section. Is the number of accidents independent of the driver's age?

Solution

From the table in Example 16:

$$\Pr(\text{more than one accident}|\text{age} < 30) = \frac{470}{600} = 0.783$$

$$\Pr(\text{more than one accident}) = \frac{700}{1000} = 0.7$$

Therefore, as $\Pr(\text{more than one accident}|\text{age} < 30) \neq \Pr(\text{more than one accident})$, the two events are not independent.

The concept of mathematical independence is sometimes confused with physical independence. If two events are physically independent, then they are also mathematically independent, but the converse is not necessarily true. The following example illustrates this.

Example 22

Suppose we roll a die twice and define the following events:

A = the first roll shows a 4

B = the second roll shows a 4

C = the sum of the numbers showing is at least 10

Are A and B independent events? What about A and C?

Solution

Since A and B are physically independent they are also mathematically independent.

Checking gives $\Pr(A) = \frac{1}{6}$

$\Pr(B) = \frac{1}{6}$

If we write the sample space as ordered pairs, in which the first element is the result of the first throw, and the second the result of the second throw, then

$$\varepsilon = \{(1, 1), (1, 2), (1, 3), \ldots (6, 5), (6, 6)\} \text{ and}$$
$$n(\varepsilon) = 36$$

The event $A \cap B$ corresponds to the outcome (4, 4) and so

$$n(A \cap B) = 1$$

Thus: $\Pr(A \cap B) = \frac{1}{36} = \Pr(A) \times \Pr(B)$

and so A and B are independent.

Now consider event $C = \{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$

and so $n(C) = 6$

Hence $\Pr(C) = \frac{1}{6}$

Also $A \cap C = \{(4, 6)\}$ and $n(A \cap C) = 1$

So $\Pr(A \cap C) = \frac{1}{36}$

Thus: $\Pr(A) \times \Pr(C) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

This means that A and C are also independent events.

Knowing that events are independent means that we can determine the probability of their intersection by multiplying together individual probabilities. This is illustrated in the following example.

Example 23

Suppose that the probability that a family in a certain town owns a television set (T) is 0.75, and the probability that a family owns a station wagon (S) is 0.25. If these events are independent, find the following probabilities:

a A family chosen at random owns both a television set and a station wagon.

b A family chosen at random owns at least one of these items.

Solution

a The event 'owns a television set' and 'owns a station wagon' is represented by $T \cap S$. Thus:

$$\begin{aligned} \Pr(T \cap S) &= \Pr(\text{T}) \times \Pr(\text{S}) && \text{(T and S are independent)} \\ &= 0.75 \times 0.25 = 0.1875 \end{aligned}$$

b The event 'owns at least one of these items' is represented by $T \cup S$. Thus:

$$\begin{aligned} \Pr(T \cup S) &= \Pr(\text{T}) + \Pr(\text{S}) - \Pr(T \cap S) && \text{(from the addition rule)} \\ &= 0.75 + 0.25 - 0.75 \times 0.25 && \text{(T and S are independent)} \\ &= 0.8125 \end{aligned}$$

Confusion often arises between independent and mutually exclusive events. That two events A and B are mutually exclusive means that $A \cap B = \emptyset$ and hence that $\Pr(A \cap B) = 0$. Thus, if two events are independent, they cannot also be mutually exclusive, unless one or other (or both) are empty.

Exercise 8G

Example 21

1 An experiment consists of drawing a number at random from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5, 7, 9, 11\}$ and $C = \{4, 6, 8, 9\}$.

a Are A and B independent? **b** Are A and C independent?

c Are B and C independent?

2 If A and B are independent events with $\Pr(A) = 0.5$ and $\Pr(B) = 0.2$ find $\Pr(A \cup B)$.

3 A die is thrown and the number uppermost is recorded. A and B are events defined by an even number and a square number respectively. Show that A and B are independent.

4 Two events A and B are such that $\Pr(A) = 0.3$, $\Pr(B) = 0.1$, and $\Pr(A \cap B) = 0.1$. Are A and B independent?

5 A and B are independent events, and $\Pr(A) = 0.6$, $\Pr(B) = 0.7$. Find:

a $\Pr(A|B)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A \cup B)$

Example 22

6 A man and a woman decide to marry. Assume that the probability that each will have a specific blood group is as follows:

Blood group	O	A	B	AB
Probability	0.5	0.35	0.1	0.05

If the blood group of the husband is independent of that of his wife, find the probability that:

a the husband is group A **b** the husband is group A and his wife is group B
c both are group A **d** the wife is group AB and her husband is group O

7 The 165 subjects volunteering for a medical study are classified by sex and blood pressure (high (H), normal (N) and low (L)).

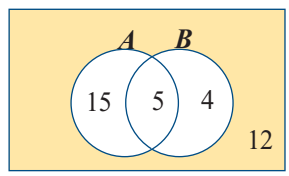
	H	N	L
M	88	22	10
F	11	22	12

If a subject is selected at random, find:

a $\Pr(N)$ **b** $\Pr(F \cap H)$ **c** $\Pr(F \cup H)$
d $\Pr(F|L)$ **e** $\Pr(L|F)$

Are F and L independent? Explain.

8 Events A and B are as shown in the Venn diagram. Show that A and B are independent.



9 The probability that a married woman watches a certain television show is 0.4, and the probability that her husband watches the show is 0.5. The television viewing habits of a husband and wife are clearly not independent. In fact, the probability that a married woman watches the show, given that her husband does, is 0.7. Find the probability that:

a both the husband and wife watch the show
b the husband watches the show given that his wife watches it.

10 The 65 middle managers in a company are classified by age (in years) and income as follows:

Income	*Age*		
	30–39 (T)	40–49 (F)	50–69 (S)
Low (L)	13	4	1
Moderate (M)	8	10	3
High (H)	2	16	8

A middle manager is selected at random from the company. Find:

a $\Pr(L)$ **b** $\Pr(S)$ **c** $\Pr(T)$ **d** $\Pr(M)$

e $\Pr(L \cap F)$ **f** $\Pr(T \cap M)$ **g** $\Pr(L|F)$ **h** $\Pr(T|M)$

Is income independent of age? Explain your answer.

11 A consumer research organisation has studied the services provided by the 150 TV repair persons in a certain city and their findings are summarised in the following table:

	Good service (G)	*Poor service* (G')
Factory trained (F)	48	16
Not factory trained (F')	24	62

One of the TV repairers is randomly selected.

a Calculate the following probabilities:

i $\Pr(G|F)$, the probability that a factory trained repairer is one who gives good service

ii $\Pr(G \cap F)$, the probability that this person is factory trained and is giving good service

iii $\Pr(G \cup F)$, the probability that the repairer is giving good service or is factory trained or both

b Are events G and F independent?

c Are the events G and F mutually exclusive?

12 A fair coin is to be tossed three times.

a Draw a tree diagram showing all the possible outcomes and their respective probabilities.

b Find the probability that the tosses result in:

i three heads **ii** two heads and a tail in any order **iii** at most two heads

13 A spinner is divided into three regions, 1, 2 and 3, such that $\Pr(1) = 0.2$, $\Pr(2) = 0.3$ and $\Pr(3) = 0.5$.

a Draw a tree diagram showing the outcomes and their associated probabilities after two spins.

b Find the probability that:

i both spins result in a 2

ii the same number shows both times

iii the sum of the numbers showing is 4

iv the second number is larger than the first

14 Three dice are rolled. Find the probability that:

a a four is showing on each die **b** an even number is showing on each die

c at least two dice show an even number **d** all dice show the same number

15 A coin is tossed five times. Find the probability that:

a a head results from each toss
b the first three tosses are heads and the last two tails
c the result of the third toss is the same as the result of the second toss
d no two consecutive tosses have the same result

16 A bag contains six balls numbered 1, 2, 3, 4, 5, 6. A ball is chosen at random and its number recorded. A second ball is then chosen from the remaining five balls and its number recorded. Find the probability that:

a the first number is a 6
b the first number is a 6 and the second a 5
c the first number is one more than the second number
d the first number is not 5
e the second number is 5

17 A three-digit random number is chosen from the numbers 000 to 999. Find the probability that:

a the first digit is odd
b all three digits are odd
c the random number is odd

MC TEST D&D TEST

Review

Chapter summary

- Probability is a numerical measure of the chance of a particular event occurring and may be determined experimentally or by symmetry.
- Whatever method is used to determine the probability, the following rules will hold:
  - **a** $0 \leq \Pr(A) \leq 1$ for all events $A \subseteq \varepsilon$
  - **b** $\Pr(\varepsilon) = 1$
  - **c** $\Pr(\varnothing) = 0$
  - **d** $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
  - **e** $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ (the **addition rule**).
- Probabilities associated with compound events are sometimes able to be calculated more easily from a probability table.
- $\Pr(A|B)$ describes the **conditional probability** of event A occurring given that event B has already occurred, and

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

  or $\Pr(A \cap B) = \Pr(A|B)\Pr(B)$ (the **multiplication rule**).
- The probabilities associated with multi-stage experiments can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).
- If two events are independent:
  - **a** $\Pr(A|B) = \Pr(A)$

    and whether or not B has occurred has no effect on the probability of A occurring.
  - **b** $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

Multiple-choice questions

1 If the probability of Chris scoring 50 or more marks in the exam is 0.7, then the probability he scores less than 50 marks is

A 0 **B** 0.3 **C** 0.4 **D** 0.7 **E** 0.8

2 A spinner is coloured red, yellow, blue and green. When spun the probability that it lands on red is 0.1, yellow is 0.2 and blue is 0.4. What is the probability that it lands on green?

A 0.1 **B** 0.2 **C** 0.3 **D** 0.4 **E** 0.5

3 Phillip is making a sign and has cut the letters of the word THEATRETTE out of wood, and placed them in his tool box. If a letter is selected at random from the toolbox, then the probability that it is a T is

A $\frac{2}{5}$ **B** $\frac{3}{10}$ **C** $\frac{1}{5}$ **D** $\frac{1}{6}$ **E** $\frac{3}{5}$

4 Of a group of 25 people in a restaurant, three chose a vegetarian meal, five chose fish, ten chose beef and the rest chose chicken for their main course. What is the probability that a randomly chosen diner chose chicken?

A $\frac{3}{25}$ **B** $\frac{6}{25}$ **C** $\frac{7}{25}$ **D** $\frac{2}{5}$ **E** $\frac{7}{18}$

5 Suppose that a card is chosen at random from a well-shuffled deck of 52 playing cards. What is the probability that the card is either a spade or a jack?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{17}{52}$ **D** $\frac{4}{13}$ **E** $\frac{9}{26}$

6 Suppose that 57% of swimmers in a club are female (F), that 32% of the swimmers in the club swim butterfly (B), and that 11% of the swimmers in the club are females and swim butterfly. Which of the following probability tables correctly summarises this information?

A

	B	B'	
F	0.11	0.21	0.32
F'	0.46	0.22	0.68
	0.57	0.43	1

B

	B	B'	
F	0.04	0.53	0.57
F'	0.28	0.15	0.43
	0.32	0.68	1

C

	B	B'	
F	0.18	0.39	0.57
F'	0.14	0.29	0.43
	0.32	0.68	1

D

	B	B'	
F	0.11	0.32	0.43
F'	0.21	0.36	0.57
	0.32	0.68	1

E

	B	B'	
F	0.11	0.46	0.57
F'	0.21	0.22	0.43
	0.32	0.68	1

7 If for two events A and B, $\Pr(A) = \frac{3}{8}$, and $\Pr(B) = \frac{4}{7}$, and $\Pr(A \cap B) = \frac{8}{21}$, then $\Pr(A|B)$ is

A $\frac{3}{8}$ **B** $\frac{3}{14}$ **C** $\frac{63}{64}$ **D** $\frac{21}{32}$ **E** $\frac{2}{3}$

Review

The following information relates to Questions 8 and 9.

The probability of Brett going to the gym on Monday is 0.6. If he goes to the gym on Monday then the probability that he will go again on Wednesday is 0.7. If he doesn't go to the gym on Monday then the probability that he will go on Wednesday is only 0.4.

8 The probability that he goes to the gym on both Monday and Wednesday is

A 0.36 **B** 0.24 **C** 0.42 **D** 0.16 **E** 0.28

9 The probability that he goes to the gym on Wednesday is

A 0.58 **B** 0.42 **C** 0.16 **D** 0.84 **E** 0.32

10 If A and B are independent events such that $\Pr(A) = 0.35$ and $\Pr(B) = 0.46$, then $\Pr(A \cup B)$ is

A 0.810 **B** 0.649 **C** 0.161 **D** 0.110

E Cannot be determined.

Short-answer questions (technology-free)

1 Two six-sided dice are tossed. Find the probability that:

a the sum of the values of the uppermost faces is 7 **b** the sum is not 7

2 The probability that a computer chip is operational is 0.993. What is the probability that it is not operational?

3 A whole number between 1 and 300 (inclusive) is chosen at random. Find the probability that the number is:

a divisible by 3 **b** divisible by 4 **c** divisible by 3 or by 4

4 A drawer contains 30 red socks and 20 blue socks.

a If a sock is chosen at random, its colour noted, the sock replaced and a second sock withdrawn, what is the probability that both socks are red?

b If replacement doesn't take place, what is the probability that both socks are red?

5 Box A contains 5 pieces of paper numbered 1, 3, 5, 7, 9.

Box B contains 3 pieces of paper numbered 1, 4, 9.

One piece of paper is removed at random from each box. Find the probability that the two numbers obtained have a sum that is divisible by 3.

6 A three-digit number is formed by arranging digits 1, 5 and 6 in a random order.

a List the sample space.

b Find the probability of getting a number larger than 400.

c What is the probability that an even number is obtained?

7 A letter is chosen at random from the word STATISTICIAN.

a What is the probability that it is a vowel?

b What is the probability that it is a T?

8 Given $\Pr(B) = \frac{1}{3}$, $\Pr(A|B) = \frac{2}{3}$ and $\Pr(A|B') = \frac{3}{7}$, determine:

a $\Pr(A \cap B')$ **b** $\Pr(A)$ **c** $\Pr(B'|A)$

9 Of the patients reporting to a clinic with the symptoms of sore throat and fever, 25% have a sore throat, 50% have an allergy, and 10% have both.

a What is the probability that a patient selected at random has either a sore throat, an allergy or both?

b Are the events 'sore throat' and 'allergy' independent?

10 The primary cooling unit in a nuclear power plant has reliability of 0.999. There is also a back-up cooling unit to substitute for the primary unit if it fails. The reliability of the back-up unit is 0.890. Find the reliability of the cooling system of the power plant. Assume independence.

11 A group of executives is classified according to body weight and incidence of hypertension. The proportion of the various categories is as shown:

	Overweight	*Normal weight*	*Underweight*
Hypertensive	0.1	0.08	0.02
Not hypertensive	0.15	0.45	0.20

a What is the probability that a person selected at random from this group will have hypertension?

b A person selected at random from this group is found to be overweight. What is the probability that this person is also hypertensive?

12 If $\Pr(A) = 0.6$ and $\Pr(B) = 0.5$, can A and B be mutually exclusive? Why or why not?

13 Ivan and Joe are chess players. In any game the probabilities of Ivan beating Joe, Joe beating Ivan or the game resulting in a draw are 0.6, 0.1 or 0.3 respectively.
They play a series of three games. Calculate the probability that:

a they win alternate games, with Ivan winning the first game

b the three games are drawn

c exactly two of the games are drawn

d Joe does not win a game.

14 Given an experiment such that $\Pr(A) = 0.3$, $\Pr(B) = 0.6$, $\Pr(A \cap B) = 0.2$, find:

a $\Pr(A \cup B)$ **b** $\Pr(A' \cap B')$ **c** $\Pr(A|B)$ **d** $\Pr(B|A)$

15 A die with 2 red faces and 4 blue faces is thrown three times. Each face is equally likely to face upward. Find the probability of obtaining:

a 3 red faces **b** a blue on the first, a red on the second and a blue on the third

c exactly 1 red face **d** at least 2 blue faces

e 3 blue faces given at least 2 blue faces.

Review

Extended-response questions

1 Let A and B be events in an event space ε, such that $\Pr(A) = \alpha$, $\Pr(B) = \beta$, $\Pr(A \cap B) = \gamma$. Find expressions for the probabilities of the following events in terms of α, β and γ:

a $A' \cap B$ **b** $A \cap B'$ **c** $A' \cap B'$

d $A' \cup B'$ **e** $(A \cup B)'$ **f** $(A \cap B)'$

2 Two bowls each contain eight pieces of fruit. In bowl A there are five oranges and three apples; in bowl B there is one orange and seven apples.

a For each bowl, find the probability that two pieces of fruit chosen at random will both be apples, if the first piece of fruit is not replaced before the second piece of fruit is chosen.

b For each bowl, find the probability that two pieces of fruit chosen at random will both be apples, if the first piece of fruit is replaced before the second is chosen.

c One bowl is chosen at random and from it two pieces of fruit are chosen at random without replacement. If both pieces of fruit are apples, find the probability that bowl A was chosen.

d One bowl is chosen at random and from it two pieces of fruit are chosen at random, the first being replaced before the second is chosen. If both pieces of fruit are apples, find the probability that bowl A was chosen.

3 Explain, in terms of the relation between the events A and B, the implication of each of the following:

a $\Pr(A|B) = 1$ **b** $\Pr(A|B) = 0$ **c** $\Pr(A|B) = \Pr(A)$

4 Three students, Dudley, Ted and Michael, have equal claim for an award. They decide to determine the winner by each tossing a coin, and the person whose coin falls unlike the other two will be the winner. If all the coins fall alike they will toss again.

a Determine the probability that Dudley wins on the first toss.

b Given that there is a winner on the first toss, what is the probability that it is Dudley?

c Find the probability that the winner is not decided in

i the first two tosses **ii** the first n tosses

d Given that no winner is decided in the first n tosses, what is the probability that Dudley wins on the next toss?

5 To have a stage production ready for opening night there are three tasks which must be done and, as the same people are involved in each task, these must be done in sequence. The following probabilities are estimated for the duration of the activities:

Task	*6 days*	*7 days*	*8 days*
Build scenery	0.3	0.3	0.4
Paint scenery	0.6	0.3	0.1
Print programs	0.4	0.4	0.2

a What is the probability that the building and painting of the scenery will together take exactly 15 days?

b What is the probability that all three tasks will together take exactly 22 days?

c Find the probability distribution of T, the total length of time taken to complete all three tasks.

6 On average, Emma goes home for dinner once a fortnight, and Sally once a week. Their mother cooks lasagne for dinner on one day in twelve, but Emma notices that on the days she comes to dinner the probability of lasagne is $\frac{7}{8}$.

a What is the probability that either Emma or Sally comes to dinner if their diaries are independent?

b Establish inequalities for the probability that one or other or both of them comes to dinner if nothing is known about the relationship between their diaries.

c What is the probability there will be lasagne for dinner on a day when Emma does not come to dinner?

7 A confectionary machine produces jellybeans of different shapes and colours in a random way and pours them into a giant vat. The jellybeans are produced in the proportions indicated in the table. There are millions of jellybeans in the vat.

Cross-sectional shape	*Purple*	*Green*	*Red*
Circular	0.4	0.2	0.1
Oval	0.2	0.1	0

a What is the probability that a randomly chosen jellybean is purple?

b What is the probability that a randomly chosen jellybean has an oval cross-section, given that it is purple?

c What is the probability of obtaining a green jellybean, given that it has a circular cross-section.

d Three jellybeans are withdrawn from the vat. What is the probability of obtaining one of each colour. (Assume replacement.)

e Ten jellybeans are taken from the vat and put in a jar. The number of each type of jellybean is indicated in the table.

Cross-sectional shape	*Purple*	*Green*	*Red*
Circular	4	2	1
Oval	2	1	0

i Three jellybeans are removed from the jar without replacement. What is the probability that there is one of each colour?

ii If three jellybeans are taken from the jar without replacement and it is known that exactly two of them have oval cross-sections, what is the probability that all three are purple?

CHAPTER

9

Counting Methods

Objectives

- To introduce **addition** and **multiplication** principles.
- To define and apply the concept of arrangements.
- To define and apply the concept of selections.
- To define and compute ${}^nC_r$.
- To relate combinations to **Pascal's Triangle**.
- To apply counting methods to **probability**.

In Chapter 8 it was found that often, when determining the probability of an event occurring, the number of outcomes contained in both the event and the sample space of interest need to be known. To do this, the sample space was listed and the outcomes counted. Sometimes a tree diagram was used or a table constructed to make sure that all the possibilities were accounted for.

When dealing with more complicated probability problems, listing the sample space and the event becomes too difficult. There may be hundreds of outcomes for a particular experiment, and even if they were comparatively easy to list we would soon tire of the task. In this chapter we will look at ways of counting the number of outcomes for various experiments and this will enable us to deal with more complicated probability problems.

9.1 Addition and multiplication principles

Some people find the decision about what to wear when they get up in the morning to be very difficult, and the more clothes they own, the more complex the decision becomes! Let us consider the number of choices they might have by looking at some examples.

Example 1

Sandi can't decide whether to wear a windcheater or a jacket. She has four windcheaters and two jackets. How many choices does she have?

Solution

As Sandi is going to wear a windcheater *or* a jacket she has a total of six choices from among these two items.

Example 2

Sandi's next choice is whether to wear jeans or a skirt. She has three pairs of jeans and four skirts. How many choices does she have?

Solution

Once again, as Sandi will wear jeans *or* a skirt, she has a total of seven choices from these two items.

Addition rule

In general, to choose between alternatives simply add up the available number for each alternative.

Example 3

At the library Alan is having trouble deciding which book to borrow. He has a choice of three mystery novels, three biographies or two science fiction books. How many choices of book does he have?

Solution

As he is choosing between alternatives (mystery novels *or* biographies *or* science fiction) then he has a total of $3 + 3 + 2 = 8$ choices.

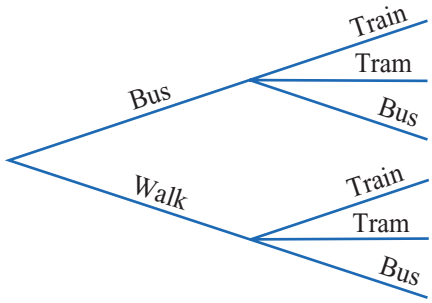
Sometimes the question arises of determining the number of possibilities when making successive choices.

Example 4

When travelling from home to school James takes a bus or walks to the main road, where he can catch a tram or a train or another bus to his destination. How many ways does James have for travelling to school?

Solution

A tree diagram may help to answer this question.



By counting the branches of the tree diagram it is found that there are six choices. This answer could be deduced by noting there are two choices for the first part of the journey and three choices for the second, and $2 \times 3 = 6$.

Multiplication rule

When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

Example 5

Consider Sandi's situation. She has six choices of windcheater or jacket, and seven choices of jeans or skirts. How many choices does she have for a complete outfit?

Solution

As Sandi will wear either a windcheater or jacket and jeans or a skirt, then we cannot consider these to be alternative choices. We could draw a tree diagram to list the possibilities, but this would be arduous. Using the multiplication rule, however, we can quickly determine the number of choices to be $6 \times 7 = 42$.

Exercise 9A

Examples 1–3

1 Find how many choices of book are possible if one book is to be selected from the following:

a eight novels, three dictionaries
b three mysteries, two dramas, seven science fiction
c twenty-two romances, fourteen mysteries, one autobiography
d ten novels, three biographies, twelve encyclopedias, four atlases

Examples 4, 5

2 Find how many different meals are possible if three courses (one entrée, one main course and one dessert) are to be selected from a menu that lists:

- **a** three entrées, four main courses, five desserts
- **b** ten entrées, ten main courses, five desserts
- **c** five entrées, seven main courses, ten desserts
- **d** eight entrées, eight main courses, eight desserts.

3 The menu in a restaurant lists four choices of entrée, eight of main course and four of dessert. Find the number of choices of meal possible:

- **a** if one of each of the three courses must be chosen
- **b** if you can choose to omit the entrée.

4 John cannot decide how to spend his evening. He can read a book, watch a video or go to the movies. If he can choose between three books, seven videos and ten movies, how many different choices does he have?

5 A student has to select a two-unit study for her course. She has to choose one unit in each semester. In semester one she has to choose one of two mathematics units, three language units and four science units. In semester two she has a choice of two history units, three geography units and two art units. How many choices does she have for the complete study?

6 Dominic is travelling from Melbourne to Brisbane and has the following choices. He can fly directly from Melbourne to Brisbane on one of three airlines, or he can fly from Melbourne to Sydney on one of four airlines and then travel from Sydney to Brisbane with one of five bus lines, or he can go on one of three bus lines directly from Melbourne to Brisbane. In how many ways could Dominic travel from Melbourne to Brisbane?

7 A particular new model of car is available in five choices of colour, three choices of transmission, four types of interior and two types of engine. Air conditioning is optional. How many different types of car are possible?

8 A company uses one letter followed by four digits for product codes. If any of the letters A–Z is allowed in the first position, and any of the digits 0–9 in the next four positions, how many different product codes are possible? (The letters and digits may be used more than once.)

9 In Victoria a licence plate generally consists of three letters followed by three numbers. If any of the letters A–Z is allowed in the first three positions, and any of the digits 0–9 in the second three positions, how many different number plates are possible? (The letters and digits may be used more than once.)

10 Morse code consists of a succession of dots and dashes. The symbols formed by the code may consist of one, two, three or four dots or dashes. How many different symbols may be represented by this code?

9.2 Arrangements

Arrangements are concerned with arranging objects in a definite order.

Example 6

How many ways are there of arranging three different books on a shelf?

Solution

Consider the bookshelf as having three possible positions in which books can be placed:

position 1	position 2	position 3

As we have three books, there are three choices of book to place in position 1. Having placed a book in position 1, there are only two choices of book for position 2, and similarly after two books are placed there is only one choice of book for position 3.

	position 1	position 2	position 3
number of choices	3	2	1

Using the multiplication principle stated in the previous section, we know that the total number of choices will be the product of these individual choices:

Number of arrangements of three books in a row is $3 \times 2 \times 1 = 6$

In general, if n objects are arranged in a row there are n choices for the first position, $n - 1$ choices for the second position, $n - 2$ choices for the third position and so on, until 1 is reached.

Thus:

The number of ways of arranging n objects in a row is

$$n \times (n-1) \times (n-2) \times (n-3) \cdots \times 2 \times 1$$

Example 7

Twelve students are to have their photos taken. How many ways are there of arranging the group if they all sit in a row?

Solution

As there are twelve students the number of arrangements is

$$12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 479\,001\,600$$

which is rather a large number of choices!

Continuing to write the expression $n(n-1)(n-2)(n-3)\cdots 2\times 1$ can be rather cumbersome, so for convenience this is written as $n!$ which is read as 'n factorial'.

The notation $n!$ (read n factorial) is an abbreviation for the product of all the integers from n down to 1:

$$n! = n\times(n-1)\times(n-2)\times(n-3)\times\cdots\times 2\times 1$$

Not all arrangements require the use of every object available, as is shown in the following example.

Example 8

A painter is to paint the five circles of the Olympic flag. He cannot remember the colours to use for any of the circles, but he knows they should all be different. He has eight colours of paint available. In how many ways can he paint the circles on the flag?

Solution

In a similar manner to the previous example we list the painter's choices:

Circle	1	2	3	4	5
Number of choices	8	7	6	5	4

Thus the total number of arrangements possible is $8\times 7\times 6\times 5\times 4 = 6720$.

Could the factorial notation be used to express the answer to Example 8? In that example the number of arrangements of eight objects in groups of five was to be counted. The answer is

$$\begin{aligned} 8\times 7\times 6\times 5\times 4 &= \frac{8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{3\times 2\times 1} \\ &= \frac{8!}{3!} \end{aligned}$$

This is the number of ways of choosing and arranging five objects from eight different ones, or the number of arrangements of eight objects taken five at a time.

In general, the number of arrangements of n objects in groups of size r is given by:

$$\frac{n!}{(n-r)!} = n\times(n-1)\times(n-2)\ldots(n-r+1)$$

Arrangements are also called permutations in mathematics, and this expression for the number of arrangements of n objects in groups of size r is often denoted by the symbol ${}^nP_r$.

Example 9

Find the number of different four-figure numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if each digit:

a can only be used once **b** can be used more than once

Solution

a As we are arranging nine objects (n) in groups of four (r):

$$\frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024$$

b There are nine choices for each of the four positions, so the total number of choices is $9 \times 9 \times 9 \times 9 = 9^4 = 6561$.

Consider the number of arrangements of n objects in a group of size n. From first principles we have found that this is equal to $n!$ Using our rule for the number of arrangements of n objects in groups of size n gives us the answer of:

$$\frac{n!}{(n-n)!} = \frac{n!}{0!}$$

Equating these expression gives an important definition for 0!:

$$0! = 1$$

If a more complicated arrangement is required in general, the restriction should be dealt with first, as shown in the following example.

Example 10

How many different even four-figure numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, if each digit may be used only once?

Solution

As the number is to be even it must end in 2, 4, 6, or 8, and so we have four choices for the last digit. Having now selected the last digit we need to choose any three digits from the remaining seven to fill up the other places. Consider each of the four positions to be filled as a 'box', and write in each box the number of choices available for that position. The total number of arrangements is then found by multiplying together the numbers in the boxes. Remember, the restriction should be dealt with first. Thus:

7	6	5		4
number of choices for the first three positions				number of choices for the last digit if the number is to be even

Multiplying gives $7 \times 6 \times 5 \times 4 = 840$.

Exercise 9B

1 Evaluate:

a $3!$ **b** $5!$ **c** $7!$ **d** $2!$ **e** $0!$ **f** $1!$

2 Evaluate:

a $\frac{5!}{3!}$ **b** $\frac{9!}{7!}$ **c** $\frac{3!}{0!}$ **d** $\frac{8!}{6!}$ **e** $\frac{5!}{0!}$ **f** $\frac{10!}{7!}$

Example 6 **3** In how many ways can five books be arranged in a row on a shelf?

Example 7 **4** In how many ways can seven students be seated on a row of seven chairs?

5 In how many ways can four pictures be hung in a row on a wall?

6 In how many ways can six cups be hung on six hooks?

Example 8 **7** In how many ways can three pictures be hung along a wall if there are ten pictures available?

8 If there are eight swimmers in the final of a 1500 m freestyle event, in how many ways can the first three places be filled?

9 Find the number of ways in which the letters of the word TROUBLE can be arranged:

a if they are all used **b** if they are used three at a time

10 Find the number of ways in which the letters of the word PANIC can be arranged:

a if they are all used **b** if they are used four at a time

Example 9 **11** Find the number of four-letter code words that can be made from the letters of the word COMPLEX:

a if no letter can be used more than once **b** if the letters can be re-used

12 Find how many code words of three or four letters can be made from the letters of the word NUMBER:

a if no letter can be used more than once **b** if the letters can be re-used

Example 10 **13** If no digit can be used more than once, find how many numbers formed from the digits 3, 4, 5, 6, 7 are:

a three-digit numbers **b** three-digit numbers and even

c greater than 700

14 If no number can be used more than once, Find how many numbers can be formed from the digits 3, 4, 5, 6, 7, 8, which are:

a two- or three-digit numbers **b** six-digit numbers and even

c greater than 7000

15 Four boys and two girls sit in a line on stools in front of a counter. Find the number of ways in which they can arrange themselves:

a if there are no restrictions **b** if the two girls wish to be at the ends

9.3 Selections

In the previous section methods for counting the number of ways in which objects can be chosen and then arranged were discussed. Sometimes the interest is solely in the number of different groups of objects that can be selected.

Example 11

Four flavours of ice cream — vanilla, chocolate, strawberry and caramel — are available at the school canteen. How many different double-cone selections are possible if two different flavours must be used?

Solution

The possibilities are:

vanilla and chocolate

vanilla and strawberry

vanilla and caramel

chocolate and strawberry

chocolate and caramel

strawberry and caramel

giving a total of six different selections.

In this example the selection 'vanilla and chocolate' is considered to be the same as 'chocolate and vanilla', and so was counted only once. Such choices without regard to order are called selections or combinations and the symbol ${}^{n}C_{r}$ is used to represent the number of ways in which groups of size r can be chosen from a total of n objects when order is unimportant.

When the total group size n is not large the combinations can be listed, but obviously a more efficient method is preferable. Consider again Example 8 concerning the colours on the Olympic flag. In this example the colours to be used are first chosen, and then they are arranged on the flag. This is shown as:

Choose the colours	Arrange them	Possible arrangements
${}^{8}C_{5}$	$\times 5!$	$\frac{8!}{3!}$

So, if ${}^{8}C_{5} \times 5! = \frac{8!}{3!}$, an expression for ${}^{8}C_{5}$ can be found by dividing both sides by 5!. That is:

$${}^{8}C_{5} = \frac{8!}{3!\,5!}$$

Note that the two figures on the bottom line (3 and 5) add to 8.

In general, the number of combinations of n objects in groups of size r is

$$ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n \times (n-1) \times (n-2) \cdots (n-r+1)}{r!} = \frac{n!}{r!(n-r)!} $$

A commonly used alternative notation for ${}^nC_r$ is $\binom{n}{r}$.

A graphics calculator can be used to determine values of ${}^nC_r$. To do this enter the value of n, then select MATH and scroll across to **PRB**. Select **3:nCr** then press ENTER. Now enter the value of r required, and again select ENTER.

Example 12

Consider again Example 11. The number of combinations of four ice creams in groups of size two is

$$ {}^4C_2 = \frac{4!}{2!2!} = \frac{4 \times 3}{2 \times 1} = 6 $$

This is the same as the answer we had before.

Once again, not all combination problems are so straightforward, as shown by the following example.

Example 13

A team of three boys and three girls is to be chosen from a group of eight boys and five girls. How many different teams are possible?

Solution

Three boys can be chosen from eight in ${}^8C_3$ ways, and three girls from five in ${}^5C_3$ ways. Thus the total number of possible teams is:

$$ {}^8C_3 \times {}^5C_3 = 56 \times 10 = 560 $$

Pascal's Triangle

Consider the possibilities when choosing from small groups of objects. In the first instance, consider when the group has only one object, A. One object can be chosen from the group in only one way: ${}^1C_1 = 1$. Next, consider when the group has two objects, A and B.

Both of them can be chosen in only 1 way:	(AB)	${}^2C_2 = 1$
One of them can be chosen in 2 ways:	(A) (B)	${}^2C_1 = 2$

In order to build up the pattern, it can be said that in both of these cases there is only one way in which to choose no objects from a group: ${}^1C_0 = 1$ and ${}^2C_0 = 1$.

Suppose that the original group has three objects, A, B, and C.

Three objects can be chosen in 1 way:	(ABC)	${}^3C_3 = 1$
Two objects can be chosen in 3 ways:	$(AB)\,(BC)\,(AC)$	${}^3C_2 = 3$
One object can be chosen in 3 ways:	$(A)\,(B)\,(C)$	${}^3C_1 = 3$
No objects can be chosen in 1 way:	$(\,)$	${}^3C_0 = 1$

Choosing from four objects we have:

$${}^4C_4 = 1 \qquad {}^4C_3 = 4 \qquad {}^4C_2 = 6 \qquad {}^4C_1 = 4 \qquad {}^4C_0 = 1$$

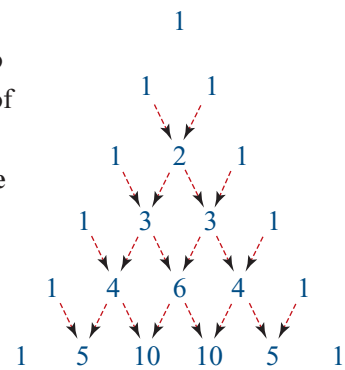
Rearranging these choices allows us to show a pattern. We will also add a top line for a group of 0 objects.

0 objects			1			
1 object		1		1		
2 objects		1	2	1		
3 objects	1	3	3	1		
4 objects	1	4	6	4	1	
5 objects	?	?	?	?	?	?

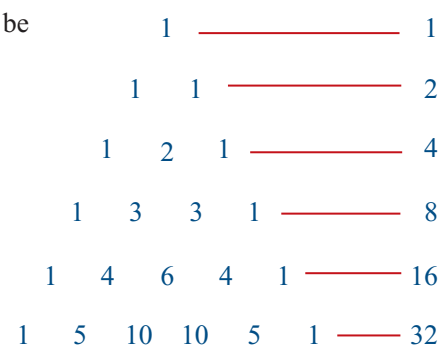
The pattern can be continued indefinitely by noting:

- The first and last number in the row are always 1.
- The number in any position in a row is the sum of the two numbers in the row above which are to the left and right of it, as shown.

This triangle of numbers is usually called **Pascal's Triangle** after the French mathematician Blaise Pascal who made ingenious use of it in his studies of probability.



An interesting application of Pascal's Triangle can be discovered by adding each row:



Is there a pattern in the sequence?

Row	1	2	3	4	5	6	n
Sum	1	2	4	8	16	32	?

It can be seen from the table that the sum of the entries in row n appears to be 2^n, and this is in fact true. That is:

$$^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

This can be used to solve some problems concerned with combinations.

Example 14

Nick is making a list for his party. He has seven friends, and he can't decide how many to invite. If he may choose any number from one to all seven to come to the party, how many possible party lists does he have? (Assume he will invite at least one person to his party.)

Solution

Nick may invite one person to the party and he has $^7C_1 = 7$ ways of doing this. If he invites two people to the party he has $^7C_2 = 21$ ways of doing this.

Continuing in this way we can see that Nick's total number of choices is:

$$^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7$$

Since we know that:

$$^7C_0 + {}^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7 = 2^7$$

then the required answer is $2^7 - {}^7C_0 = 2^7 - 1 = 128 - 1 = 127$

Skillsheet TEACHER

Exercise 9C

Example 11

1 For each of the following examples determine the number of selections possible by listing the possibilities:

a An ice cream with two different scoops is selected from a choice of vanilla, chocolate and strawberry.

b Two students from the group of Jaime, Georgia and Wey are chosen to represent the class in debating.

c Two students from the group of Thomas, William, Jackson and Phillip are chosen for the tennis team.

d Three scarves are chosen from a blue scarf, a green scarf, a red scarf and a white scarf.

Example 12

2 Evaluate:

a $^5C_3$ **b** $^5C_2$ **c** $^7C_4$ **d** $^7C_3$

Compare your answers for parts a and b, and for parts c and d.

3 Evaluate:

a ${}^{20}C_{18}$ **b** ${}^{100}C_{99}$ **c** ${}^{100}C_{2}$ **d** ${}^{250}C_{248}$

4 Evaluate:

a $\binom{6}{3}$ **b** $\binom{7}{1}$ **c** $\binom{8}{2}$ **d** $\binom{50}{48}$

5 How many netball teams of seven can be chosen from 13 players?

6 An ice cream parlour has 25 different flavours of ice cream available. How many different three-scoop ice cream sundaes are available if three different flavours are to be used and the order of the scoops does not matter?

7 How many different hands of seven cards can be dealt from a normal pack of 52 cards?

8 In Tattslotto six numbers are selected from 45. How many different possible selections are there? (Do not attempt to consider supplementary numbers.)

Example 13

9 A student has the choice of three mathematics subjects and four science subjects. In how many ways can they choose to study one mathematics and two science subjects?

10 A survey is to be conducted, and eight people are to be chosen from a group of 30.

a In how many different ways could the eight be chosen?

b If the group contains 10 men and 20 women, how many groups containing exactly two men are possible?

11 From a standard 52-card deck, how many 7-card hands have exactly 5 spades and 2 hearts?

12 In how many ways can a committee of five be selected from eight women and four men:

a without restriction? **b** if there must be three women on the committee?

13 Six females and five males are interviewed for five positions. If all are found to be acceptable for any position, in how many ways could the following combinations be selected?

a three males and two females **b** four females and one male

c five female **d** five people regardless of sex

e at least four females

14 The selectors of a sporting team need to choose 10 athletes from the 15 track and 12 field athletes who have qualified to compete.

a How many groups are possible?

b How many groups would contain track athletes only?

c How many groups would contain field athletes only?

d How many groups would contain half track and half field athletes?

15 A student representative committee of five is to be chosen from four male and six female students. How many committees could be selected which contain more female than male students?

Example 14 **16** Joanne is offered a selection of five different sweets. She can choose to pass or to select any number of them. In total how many choices does she have?

17 Eight people have auditioned for roles in a play. The director can choose none, or any number of them for his production. In how many ways can selections be made from these eight people, any number at a time?

18 How many colours can be obtained by mixing a set volume of red, blue, green, orange and white paints if any number of paints can be used at a time?

19 How many varieties of fruit salad, using at least two fruits, can be obtained from apples, oranges, pears, passionfruit, kiwi fruit and nectarines, taken any number at a time?

20 In how many ways can a group of six people be divided into:

a two equal groups?

b two unequal groups, if there must be at least one person in each group?

9.4 Applications to probability

As already established in Chapter 8, the probability of an event occurring may be determined by dividing the number of favourable outcomes by the total number of possible outcomes. Establishing the number of favourable outcomes and the total number of outcomes is often achieved by using permutations and combinations.

Example 15

Four-letter 'words' are to be made by arranging letters of the word SPECIAL. What is the probability the 'word' will start with a vowel?

Solution

Arranging the seven letters in groups of four may be done in $7 \times 6 \times 5 \times 4$ different ways, i.e. 840, and this is the number of possible outcomes. Consider the number of arrangements which start with a vowel. Since there are three vowels we have 3 choices for the first letter. Having done this, we have six letters remaining which are to be placed in the three remaining positions, and this can be done in $6 \times 5 \times 4$ different ways. Thus the number of arrangements which start with a vowel is:

$$3 \times 6 \times 5 \times 4 = 360$$

Thus, the probability of the arrangement starting with a vowel is:

$$\frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} = \frac{360}{840} = \frac{3}{7}$$

Example 16

Three students are to be chosen to represent the class in a debate. If the class consists of six boys and four girls, what is the probability that the teams will contain:

i exactly one girl? **ii** at least two girls?

Solution

Since there is a total of 10 students, the number of possible teams is ${}^{10}C_3 = 120$.

i If the team is to contain one girl then this can be done in ${}^{4}C_1 = 4$ different ways. Having placed one girl, the other two places must be filled by boys, and this can be done in ${}^{6}C_2 = 15$ different ways. Thus the total number of teams containing one girl and two boys is $4 \times 15 = 60$ and the probability that the team contains exactly one girl is $\frac{60}{120} = 0.5$.

ii If the team is to contain at least two girls then it may contain two *or* three girls. The number of teams containing:

exactly two girls is $\binom{6}{1} \times \binom{4}{2} = 36$

exactly three girls is $\binom{6}{0} \times \binom{4}{3} = 4$

Thus the total number of teams containing at least two girls is 40, and the probability of this is $\frac{40}{120} = \frac{1}{3}$.

Exercise 9D

1 A four-digit number (with no repetitions) is to be formed from the set of numbers $\{1, 2, 3, 4, 5, 6\}$. Find the probability that the number:

a is even **b** is odd

Example 15

2 Three-letter 'words' are to be made by arranging the letters of the word COMPUTER. What is the probability the 'word' will start with a vowel?

3 Three letters are chosen at random from the word HEART and arranged in a row. Find the probability that:

a the letter H is first **b** the letter H is chosen **c** both vowels are chosen

4 Three men and three women are to be randomly seated in a row. Find the probability that both the end places will be filled by women.

Example 16

5 A netball team of seven players is to be chosen from six men and seven women. Find the probability that the selected team contains more men than women.

6 Bill is making a sandwich. He may choose any combination of the following: lettuce, tomato, carrot, cheese, cucumber, beetroot, onion, ham. Find the probability that:

a the sandwich contains ham
b the sandwich contains three ingredients
c the sandwich contains at least three ingredients

7 A bag contains five white, six red and seven blue balls. If three balls are selected at random, without replacement, find the probability they are:

a all red **b** all different colours

8 A four-digit number (with no repetitions) is to be formed from the set of numbers $\{0, 1, 2, 3, 4, 5, 6, 7\}$. Find the probability that the number:

a is even **b** is odd **c** is less than 4000
d is less than 4000, given that it is greater that 3000

9 Susie chooses four pieces of bubble gum from a jar containing five red, two blue and three green pieces of bubble gum. Calculate the probability that Susie selects:

a no green bubble gum
b at least one green bubble gum
c at least one green bubble gum and at least one red bubble gum
d at least one red bubble gum given that there is at least one green bubble gum

10 A hand of five cards is dealt from a normal pack of 52 cards. Find the probability that the hand will contain:

a no aces **b** at least one ace **c** the ace of spades
d the ace of spades given that there is at least one ace

11 A committee of three is to be chosen from a group of four men and five women. Find the probability that the committee contains:

a all women **b** at least one woman
c exactly two men given that at least one man is chosen

MC TEST D&D TEST

Review

Chapter summary

- $n!$ (n factorial) $= n(n-1)(n-2)\ldots 2 \times 1$ and $0! = 1$.
- Evaluation of ${}^nC_r$ gives the number of combinations or selections of n objects in subgroups of size r. Order is not important.
- When the combination problem involves restrictions, deal with these first.
- Combinations may be used when determining probabilities. In the appropriate cases, the probability is given by dividing the number of favourable outcomes by the total number of outcomes.

Multiple-choice questions

1 For his holiday reading Geoff has selected eight detective novels, three biographies and four science fiction books, but he only has room in his case for three books. If he selects one book from each group, how many combinations of book are possible?

A 15 **B** 28 **C** 56 **D** 20 **E** 96

2 Georgia is choosing her five subjects for Year 12. She has already chosen three subjects. She will choose one of the three mathematics subjects, and either one of five languages or one of three science subjects. How many different subject combinations are possible?

A 11 **B** 15 **C** 9 **D** 24 **E** 45

3 In how many ways can 10 people be arranged in a queue at the bank?

A $10!$ **B** $\frac{10!}{2!}$ **C** $\frac{10!}{2!8!}$ **D** $\frac{10!}{8!}$ **E** ${}^{10}C_1$

4 How many different numberplates can be made using two letters followed by four digits, if neither the letters nor the digits can be repeated?

A 8 **B** 720 **C** 5690 **D** 3 276 000 **E** 6 760 000

5 ${}^{21}C_3$ is equal to:

A $21!$ **B** $\frac{21!}{3!}$ **C** $\frac{21!}{18!3!}$ **D** $\frac{21!}{18!}$ **E** $\frac{18!3!}{21!}$

6 In how many ways can a hand of six cards be dealt from a pack of 52 cards?

A $6!$ **B** ${}^{52}C_6$ **C** ${}^{46}C_6$ **D** $\frac{52!}{6!}$ **E** $52!$

7 In how many ways can three DVDs be chosen from a group of 12 DVDs?

A 12 **B** 36 **C** 220 **D** 1320 **E** 79 833 600

8 A class consists of 10 girls and 14 boys. In how many ways could a committee of two girls and two boys be chosen?

A ${}^{10}C_2 \times {}^{14}C_2$ **B** ${}^{24}C_4$ **C** $10! \times 14!$ **D** $10 \times 9 \times 14 \times 13$ **E** $\frac{10!14!}{4!}$

9 Three-letter 'words' are to be made by arranging the letters of the word METHODS. What is the probability that the word begins with a vowel?

A $\frac{1}{105}$ **B** $\frac{1}{21}$ **C** $\frac{1}{2}$ **D** $\frac{5}{26}$ **E** $\frac{2}{7}$

10 What is the probability that a team of four chosen at random from a group of eight friends, four males and four females, would consist of three women and one man?

A $\frac{1}{2}$ **B** $\frac{3}{4}$ **C** $\frac{3}{8}$ **D** $\frac{3}{64}$ **E** $\frac{8}{35}$

Short-answer questions (technology-free)

1 Evaluate:

a ${}^{1000}C_{998}$ **b** ${}^{1000\,000}C_{999\,999}$ **c** ${}^{1000\,000}C_{1}$

2 How many integers from 100 to 999, inclusive, have three different digits?

3 How many different three-digit house numbers can be constructed from six brass numerals 1, 2, 3, 4, 5, 6?

4 A supermarket sells n different brands of liquid dishwashing soap. Each brand offers four different sized bottles, small, medium, large and economy, and each is available as either lemon-scented or pine-scented. How many different types of liquid dishwashing soap bottles are available at this supermarket?

5 Of the integers from 1000 to 9999 how many have at least one digit a 5 or 7?

6 A bushwalking club has 80 members, 50 men and 30 women. A committee consisting of two men and one woman is to be selected. How many different committees are possible?

7 There are five vowels and 21 consonants in the English alphabet. How many different four-letter 'words' can be formed that contain two different vowels and two different consonants?

8 A pizza restaurant offers the following toppings: onions, green peppers, mushrooms, anchovies and pepperoni.

a How many different kinds of pizza with three different toppings can be ordered?

b How many different kinds, with any number of toppings (between none and all five) can be ordered?

9 Seven people are to be seated in a row. Calculate the number of ways in which this can be done so that two particular persons, A and B, always have exactly one of the others between them.

10 Three letters are chosen at random from the word OLYMPICS and arranged in a row. What is the probability that:

a the letter O is first? **b** the letter Y is chosen? **c** both vowels are chosen?

Extended-response questions

1 Permutations are formed using all of the digits 1, 2, 3, 4, . . . , 9 without repetition. Determine the number of possible permutations in each of the following cases:

a Even and odd digits alternate.

b The digits 1 and 2 are together but not necessarily in that order.

Review

2 There are 10 chairs in a row.
- **a** In how many ways can three people be seated?
- **b** In how many of these will the two end chairs be occupied?
- **c** In how many of these will the two end chairs be empty?

3 All possible three-digit numbers are formed from the odd digits $\{1, 3, 5, 7, 9\}$.
- **a** How many such numbers are possible if each digit is used only once?
- **b** How many of the numbers from part a are larger than 350?

4 In how many ways can a committee of four be chosen from five married couples if:
- **a** all individuals are eligible for selection?
- **b** the committee must consist of two women and two men?
- **c** a husband and wife cannot both be selected?

5 Geoff has five flat batteries, and 10 charged batteries. Unfortunately his little brother mixes them up, and he can't tell them apart. He selects four batteries at random for his calculator.
- **a** How many different combinations of the 15 batteries could Geoff select?
- **b** In how many of these are all four batteries charged?
- **c** In how many of these is at least one battery flat?

6 There are seven mints and 11 jubes in the lolly jar. Steve puts his hand in the jar and selects four lollies at random.
- **a** How many different combinations of the lollies are there?
- **b** In how many of these are there no mints?
- **c** In how many of these are there two mints and two jubes?

7 In Tattslotto, a player picks a selection of six numbers from the numbers 1 to 45. To determine the winners eight numbers are chosen at random – the first six are designated as the winning numbers, and the other two as the supplementary numbers. Prizes are as follows:

Division 1:	6 winning numbers
Division 2:	5 winning numbers and 1 supplementary
Division 3:	5 winning numbers
Division 4:	4 winning numbers
Division 5:	3 winning numbers and 1 supplementary

Find the number of combinations which satisfy each of the divisions, and hence the probabilities of winning each of the five divisions.

8 In Keno, a player selects between three and 10 numbers from 1 to 80. Each selection is called a 'spot'. If you select five numbers, you are playing a 'Spot 5' game. To determine the winners, 20 numbers are drawn randomly from the 80 numbers. If your selected numbers are among the 20, you win. The amount you win depends on the 'spot' you are playing.
Find the probability of winning a:
- **a** 'Spot 6' game
- **b** 'Spot 5' game

CHAPTER

10

Discrete Probability Distributions and Simulation

Objectives

- To demonstrate the basic ideas of discrete random variables.
- To introduce the concept of a probability distribution for a discrete random variable.
- To introduce and investigate applications of the binomial probability distribution.
- To show that simulation can be used to provide **estimates of probability** which are close to exact solutions.
- To use **simulation techniques** to provide solutions to probability problems where an exact solution is too difficult to determine.
- To use coins and dice as **simulation models**.
- To introduce and use **random number tables**.

10.1 Discrete random variables

In Chapter 8 the notion of the probability of an event occurring was explored, where an event was defined as any subset of a sample space. Sample spaces which were not sets of numbers were frequently encountered. For example, when a coin is tossed three times the sample space is:

$$\varepsilon = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

If it is only the number of heads that is of interest, however, a simpler sample space could be used whose outcomes are numbers. Let X represent the number of heads in the three tosses of the coin, then the possible values of X are 0, 1, 2, and 3. Since the actual value that X will take is the outcome of a random experiment, X is called a **random variable**. Mathematically a random variable is a function that assigns a number to each outcome in the sample space ε.

A random variable X is said to be **discrete** if it can assume only a countable number of values. For example, suppose two balls are selected at random from a jar containing several white (W) and black balls (B). A random variable X is defined as the number of white balls obtained in the sample. Thus X is a discrete random variable which may take one of the values 0, 1, or 2. The sample space of the experiment is:

$$S = \{WW, WB, BW, BB\}$$

Each outcome in the sample space corresponds to a value of X, and vice-versa.

Experimental outcome	*Value of X*
WW	$X = 2$
BW	$X = 1$
WB	$X = 1$
BB	$X = 0$

Many events can be associated with a given experiment. Some examples are:

Event	*Sample outcomes*
One white ball: $X = 1$	$\{WB, BW\}$
At least one white ball: $X \geq 1$	$\{WW, WB, BW\}$
No white balls: $X = 0$	$\{BB\}$

A **probability distribution** can be thought of as the theoretical description of a random experiment. Consider tossing a die 600 times – we *might* obtain the following results:

x	1	2	3	4	5	6
Frequency	98	104	93	97	108	100
Experimental probability	$\frac{98}{600}$	$\frac{104}{600}$	$\frac{93}{600}$	$\frac{97}{600}$	$\frac{108}{600}$	$\frac{100}{600}$

Theoretically, the frequencies will be equal, regardless of the number of trials (provided they are sufficiently large). This can most easily be expressed by giving probabilities – for instance, let X be 'the outcome of tossing a die'.

x	1	2	3	4	5	6
Theoretical frequency	100	100	100	100	100	100
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The table gives the **probability distribution** of X.

The probability distribution of X, $p(x) = \Pr(X = x)$ is a function that assigns probabilities to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take. For *any* discrete probability function the following must be true:

1 The minimum possible value of $p(x)$ is zero, and the maximum possible value of $p(x)$ is 1. That is:

$$0 \leq p(x) \leq 1$$

for every value x that X can take.

2 All values of $p(x)$ in every probability distribution must sum to exactly 1.

To determine the probability that X lies in an interval, we add together the probabilities that X takes all values included in that interval, as shown in the following example.

Example 1

Consider the function:

x	1	2	3	4	5
$\Pr(X = x)$	$2c$	$3c$	$4c$	$5c$	$6c$

a For what value of c is this a probability distribution? **b** Find $\Pr(3 \leq X \leq 5)$.

Solution

a To be a probability distribution we require

$$\begin{aligned} 2c + 3c + 4c + 5c + 6c &= 1 \\ 20c &= 1 \\ c &= \frac{1}{20} \end{aligned}$$

b $$\begin{aligned} \Pr(3 \leq X \leq 5) &= \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5) \\ &= \frac{4}{20} + \frac{5}{20} + \frac{6}{20} \\ &= \frac{15}{20} = \frac{3}{4} \end{aligned}$$

Example 2

The table shows a probability distribution with random variable X.

x	1	2	3	4	5	6
$\Pr(X = x)$	0.2	0.2	0.07	0.17	0.13	0.23

Give the following probabilities:

a $\Pr(X > 4)$ **b** $\Pr(2 < X < 5)$ **c** $\Pr(X \geq 5 | X \geq 3)$

Solution

a $\Pr(X > 4) = \Pr(X = 5) + \Pr(X = 6) = 0.13 + 0.23 = 0.36$

b $\Pr(2 < X < 5) = \Pr(X = 3) + \Pr(X = 4) = 0.07 + 0.17 = 0.24$

c $\Pr(X \geq 5 | X \geq 3) = \dfrac{\Pr(X \geq 5)}{\Pr(X \geq 3)}$ (as $X \geq 5$ and $X \geq 3$ implies $X \geq 5$)

$$= \frac{\Pr(X = 5) + \Pr(X = 6)}{\Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5) + \Pr(X = 6)}$$

$$= \frac{0.13 + 0.23}{0.07 + 0.17 + 0.13 + 0.23} = \frac{0.36}{0.6}$$

$$= \frac{3}{5}$$

Example 3

The following distribution table gives the probabilities for the number of people on a carnival ride at a paticular time of day.

No. of people (t)	0	1	2	3	4	5
$\Pr(T = t)$	0.05	0.2	0.3	0.2	0.1	0.15

Find:

a $\Pr(T > 4)$ **b** $\Pr(1 < T < 5)$ **c** $\Pr(T < 3 | T < 4)$

Solution

a $\Pr(T > 4) = \Pr(T = 5) = 0.15$

b $\Pr(1 < T < 5) = \Pr(T = 2) + \Pr(T = 3) + \Pr(T = 4) = 0.6$

c $\Pr(T < 3 | T < 4) = \dfrac{\Pr(T < 3)}{\Pr(T < 4)}$

$$= \frac{0.55}{0.75}$$

$$= \frac{11}{15}$$

Exercise 10A

1 A random variable X can take the values $x = 1, 2, 3, 4$. Indicate whether or not each of the following is a probability function for such a variable, and if not, give reasons:

a $p(1) = 0.05$ $p(2) = 0.35$ $p(3) = 0.55$ $p(4) = 0.15$

b $p(1) = 0.125$ $p(2) = 0.5$ $p(3) = 0.25$ $p(4) = 0.0625$

c $p(1) = 13\%$ $p(2) = 69\%$ $p(3) = 1\%$ $p(4) = 17\%$
d $p(1) = 51$ $p(2) = 12$ $p(3) = 34$ $p(4) = 3$
e $p(1) = 0.66$ $p(2) = 0.32$ $p(3) = -0.19$ $p(4) = 0.21$

2 For each of the following write a probability statement in terms of the discrete random variable X showing the probability that:

a X is equal to 2 **b** X is greater than 2 **c** X is at least 2
d X is less than 2 **e** X is 2 or more
f X is more than 2 **g** X is no more than 2
h X is greater than or equal to 2 **i** X is less than or equal to 2
j X is no less than 2 **k** X is greater than 2 and less than 5

3 A random variable X can take the values 0, 1, 2, 3, 4, 5. List the set of values that X can take for each of the following probability statements:

a $\Pr(X = 2)$ **b** $\Pr(X > 2)$ **c** $\Pr(X \geq 2)$
d $\Pr(X < 2)$ **e** $\Pr(X \leq 2)$ **f** $\Pr(2 \leq X \leq 5)$
g $\Pr(2 < X \leq 5)$ **h** $\Pr(2 \leq X < 5)$ **i** $\Pr(2 < X < 5)$

Example 1

4 Consider the following function:

x	1	2	3	4	5
$\Pr(X = x)$	k	$2k$	$3k$	$4k$	$5k$

a For what value of k is this a probability distribution?
b Find $\Pr(2 \leq X \leq 4)$.

5 The number of 'no-shows' on a scheduled airline flight has the following probability distribution:

r	0	1	2	3	4	5	6	7
$p(r)$	0.09	0.22	0.26	0.21	0.13	0.06	0.02	0.01

Find the probability that:

a more than four people do not show up for the flight
b at least two people do not show up for the flight.

6 Suppose Y is a random variable with the distribution given in the table.

y	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\Pr(Y = y)$	0.08	0.13	0.09	0.19	0.20	0.03	0.10	0.18

Find:

a $\Pr(Y \leq 0.50)$ **b** $\Pr(Y > 0.50)$ **c** $\Pr(0.30 \leq Y \leq 0.80)$

Example 2

7 The table shows a probability distribution with random variable X.

x	1	2	3	4	5	6
$\Pr(X = x)$	0.1	0.13	0.17	0.27	0.20	0.13

Give the following probabilities:

a $\Pr(X > 3)$ **b** $\Pr(3 < X < 6)$ **c** $\Pr(X \geq 4 | X \geq 2)$

8 Suppose that a fair coin is tossed three times.

a List the eight equally likely outcomes.

b If X represents the number of heads shown, determine $\Pr(X = 2)$.

c Find the probability distribution of the random variable X.

d Find $\Pr(X \leq 2)$.

e Find $\Pr(X \leq 1 | X \leq 2)$.

9 When a pair of dice is rolled, 36 equally likely outcomes are possible. Let Y denote the sum of the dice.

a What are the possible values of the random variable Y?

b Find $\Pr(Y = 7)$.

c Determine the probability distribution of the random variable Y.

10 When a pair of dice is rolled, 36 equally likely outcomes are possible. Let X denote the larger of the values showing on the dice. If both dice come up the same, then X denotes the common value.

a What are the possible values of the random variable X?

b Find $\Pr(X = 4)$.

c Determine the probability distribution of the random variable X.

11 A dart is thrown at a circular board with a radius of 10 cm. The board has three rings: a bullseye of radius 3 cm, a second ring with an outer radius of 7 cm, and a third ring with outer radius 10 cm. Assume that the probability of the dart hitting a region R is given by

$$\Pr(R) = \frac{\text{area of } R}{\text{area of dartboard}}$$

a Find the probability of scoring a bullseye.

b Find the probability of hitting the middle ring.

c Find the probability of hitting the outer ring.

12 Suppose that a fair coin is tossed three times. You lose \$3.00 if three heads appear and \$2.00 if two heads appear. You win \$1.00 if one head appears and \$3.00 if no heads appear; \$$Y$ is the amount you win or lose.

a Find the probability distribution of the random variable Y.

b Find $\Pr(Y \leq 1)$.

10.2 Sampling without replacement

Consider the sort of probability distribution that arises from the most common sampling situation.

A jar contains three mints and four toffees, and Bob selects two (without looking). If the random variable of interest is the number of mints he selects, then this can take values of 0, 1 or 2. Suppose this was done experimentally many times (say 50). The following results may be obtained:

Number of mints	0	1	2
Number of times observed	18	24	8

Let X be the number of mints Bob selects. From the table, the probabilities can be estimated for each outcome as:

$$\Pr(X = 0) \approx \frac{18}{50} = 0.36$$
$$\Pr(X = 1) \approx \frac{24}{50} = 0.48$$
$$\Pr(X = 2) \approx \frac{8}{50} = 0.16$$

Obviously, it is not desirable that an experiment needs to be carried out every time a situation like this arises and it is not necessary. It is possible to work out the theoretical probability for each value of the random variable by using the knowledge of combinations from Chapter 9.

Consider the situation in which the sample of two sweets contains no mints. Then it must contain two toffees. Thus Bob has selected no mints from the three available, and two toffees from the four available, which gives the number of favourable outcomes as:

$$\binom{3}{0}\binom{4}{2}$$

Now the number of possible choices Bob has of choosing two sweets from seven is $\binom{7}{2}$ and thus the probability of Bob selecting no mints is:

$$\Pr(X = 0) = \frac{\binom{3}{0}\binom{4}{2}}{\binom{7}{2}}$$
$$= \frac{6}{21}$$
$$= \frac{2}{7}$$

Similarly, we can determine:

$$\Pr(X=1)=\frac{\binom{3}{1}\binom{4}{1}}{\binom{7}{2}} \quad \text{and} \quad \Pr(X=2)=\frac{\binom{3}{2}\binom{4}{0}}{\binom{7}{2}}$$

$$=\frac{12}{21} \qquad\qquad =\frac{3}{21}$$

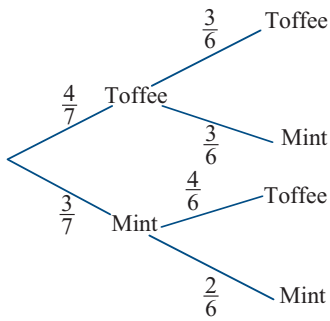
$$=\frac{4}{7} \qquad\qquad =\frac{1}{7}$$

Thus the probability distribution for X is:

x	0	1	2
$\Pr(X=x)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

Note that the probabilities in the table add up to 1. If they did not add to 1 it would be known that an error had been made.

This problem can also easily be completed with a tree diagram.



Therefore $\Pr(X=0)=\frac{4}{7}\times\frac{3}{6}=\frac{2}{7}$

$$\Pr(X=1)=\frac{4}{7}\times\frac{3}{6}+\frac{3}{7}\times\frac{4}{6}=\frac{2}{7}+\frac{2}{7}=\frac{4}{7}$$

and $\Pr(X=2)=\frac{3}{7}\times\frac{2}{6}=\frac{1}{7}$

This can be considered as a sequence of two trials in which the second is dependent on the first.

Instead of evaluating the probabilities for all the values of X and listing them in a table, the probability distribution could be given as a rule and, providing the values of X are specified for which the rule is appropriate, the same information as before is available. In this case the rule is:

$$\Pr(X=x)=\frac{\binom{3}{x}\binom{4}{2-x}}{\binom{7}{2}},\ x=0,1,2$$

This example is an application of a distribution which is commonly called the **hypergeometric distribution**.

Example 4

Marine biologists are studying a group of dolphins which live in a small bay. They know there are 12 dolphins in the group, four of which have been caught, tagged and released to mix back into the population. If the researchers return the following week and catch another group of three dolphins, what is the probability that two of these will already be tagged?

Solution

Let X equal the number of tagged dolphins in the second sample.

We wish to know the probability of selecting two of the four tagged dolphins, and one of the eight non tagged dolphins, when a sample of size 3 is selected from a population of size 12. That is:

$$\Pr(X = 2) = \frac{\binom{4}{2}\binom{8}{1}}{\binom{12}{3}}$$
$$= \frac{12}{55}$$

Exercise 10B

Example 4

1 A company employs 30 salespersons, 12 of whom are men and 18 are women. Five salespersons are to be selected at random to attend an important conference. What is the probability of selecting two men and three women?

2 An electrical component is packaged in boxes of 20. A technician randomly selects three from each box for testing. If there are no faulty components, the whole box is passed. If there are any faulty components, the box is sent back for further inspection. If a box is known to contain four faulty components, what is the probability it will pass?

3 A pond contains seven gold and eight black fish. If three fish are caught at random in a net, find the probability that at least one of them is black.

4 A researcher has caught, tagged and released 10 birds of a particular species into the forest. If there are known to be 25 of this species of bird in the area, what is the probability that another sample of five birds will contain three tagged ones?

5 A tennis instructor has 10 new and 10 used tennis balls. If he selects six balls at random to use in a class, what is the probability that there will be at least two new balls?

6 A jury of six persons was selected from a group of 18 potential jurors, of whom eight were female and 10 male. The jury was supposedly selected at random, but it contained only one female. Do you have any reason to doubt the randomness of the selection? Explain your reasons.

10.3 Sampling with replacement: the binomial distribution

Suppose a fair six-sided die is rolled four times and a random variable X is defined as the number of 3s observed. An approximate probability distribution for this random variable could be found by repeatedly rolling the die four times, and observing the outcomes. On one occasion the four rolls of the die was repeated 100 times and the following results noted:

x	0	1	2	3	4
No. of times observed	50	44	5	1	0

From this table the probabilities for each outcome could be estimated:

$$\Pr(X = 0) \approx \frac{50}{100} = 0.50$$
$$\Pr(X = 1) \approx \frac{40}{100} = 0.44$$
$$\Pr(X = 2) \approx \frac{5}{100} = 0.05$$
$$\Pr(X = 3) \approx \frac{1}{100} = 0.01$$
$$\Pr(X = 4) \approx \frac{0}{100} = 0$$

The theoretical probability distribution can be determined in the following way.

One possible outcome of the experiment is *TTNN*, where *T* represents a 3 and *N* represents not a 3.

The probability of this particular outcome (that is, in this order) is

$$\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$$

How many different arrangements of *T, T, N*, and *N* are there? Listing we find that there are six: *TTNN, TNTN, TNNT, NTTN, NTNT, NNTT*. The number of arrangements could be found without listing them by recognising that this is equal to $\binom{4}{2}$, the number of ways of placing the two *T*s in the four available places.

Thus, the probability of obtaining exactly two 3s when a fair die is tossed four times is:

$$\Pr(X = 2) = \binom{4}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$$
$$= \frac{25}{216}$$

Continuing in this way the entire probability distribution can be defined as given in the table. (Note that the probabilities shown do not add to exactly 1 owing to rounding errors.)

x	0	1	2	3	4
$\Pr(X = x)$	0.4822	0.3858	0.1157	0.0154	0.0008

It would be convenient to be able to use a formula to summarise the probability distribution. In this case it is:

$$\Pr(X = x) = \binom{4}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{n-x} \quad x = 0, 1, 2, 3, 4$$

This is an example of the **binomial probability distribution**, which has arisen from a binomial experiment. A **binomial experiment** is one that possesses the following properties:

- The experiment consists of a number, n, of identical trials.
- Each trial results in one of two outcomes, which are usually designated as either a **success**, S, or a **failure**, F.
- The probability of success on a single trial, p say, is constant for all trials (and thus the probability of failure on a single trial is $(1 - p)$).
- The trials are independent (so that the outcome on any trial is not affected by the outcome of any previous trial).

The random variable of interest, X, is the number of successes in n trials of a binomial experiment. Thus, X has a binomial distribution and the rule is:

$$\Pr(X = x) = \binom{n}{x}(p)^x(1-p)^{n-x} \quad x = 0, 1, \ldots, n$$

where $$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

Example 5

Rainfall records for the city of Melbourne indicate that, on average, the probability of rain falling on any one day in November is 0.4. Assuming that the occurrence of rain on any day is independent of whether or not rain falls on any other day, find the probability that rain will fall on any three days of a chosen week.

Solution

Since there are only two possible outcomes on each day (rain or no rain), the probability of rain on any day is constant (0.4) regardless of previous outcomes. The situation described is a binomial experiment. In this example occurrence of rain is considered as a success, and so define X as the number of days on which it rains in a given week. Thus X is a binomial random variable with $p = 0.4$ and $n = 7$.

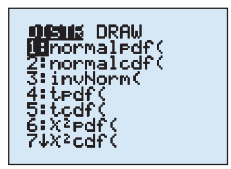
$$\Pr(X = x) = \binom{7}{x}(0.4)^x(0.6)^{7-x}, \quad x = 0, 1, \ldots, 7$$

and
$$\Pr(X = 3) = \binom{7}{3}(0.4)^3(0.6)^{7-3}$$
$$= \frac{7!}{3!4!} \times 0.064 \times 0.13$$
$$= 0.290\,304 \qquad \text{(values held in calculator)}$$

Using a graphics calculator

A graphics calculator can be used to evaluate probabilities for many distributions, including the binomial distribution.

To access the probability distributions stored in the calculator, select 2ND DISTR, which will give the following screen.



Scrolling down, two references to the binomial distribution can be found:

- **0:binompdf(** is the command used to determine probabilities for the binomial distribution of the form $\Pr(X = x)$. Here **pdf** refers to **probability distribution function**.
- **A:binomcdf(** is the command used to determine probabilities for the binomial distribution of the form $\Pr(X \leq x)$. Here **cdf** refers to **cumulative distribution function**.

Examples of how and when to use each of these functions are given in Example 6.

CAS APPENDIX

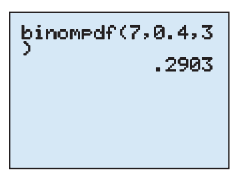
Example 6

For the situation described in Example 5, use the graphics calculator to find the probability that:

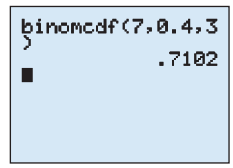
a rain will fall on any three days of a chosen week
b rain will fall on no more than three days of a chosen week
c rain will fall on at least three days of a chosen week.

Solution

a As shown in Example 5, $\Pr(X = 3)$ is required, where $n = 7$ and $p = 0.4$.
Select binompdf(as described earlier, and then enter the values of n, p, and x in that order as shown on the screen. Press ENTER to find the required value.
Once again it is found that $\Pr(X = 3) = 0.2903$.



b Here $\Pr(X \leq 3)$ is required. Select binomcdf(as described earlier, and again enter the values of n, p, and x in that order as shown on the screen. Press ENTER to find the required value.



Here $\Pr(X \leq 3) = 0.7102$

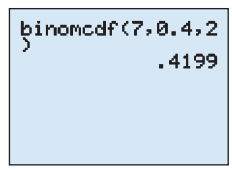
c This time $\Pr(X \geq 3)$ is required and

$$\Pr(X \geq 3) = \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7)$$

The graphics calculator does not have a built-in function to calculate $\Pr(X \geq x)$, but binomcdf(can be used to evaluate this probability by recognising that:

$$\begin{aligned}\Pr(X \geq 3) &= 1 - (\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)) \\ &= 1 - \Pr(X \leq 2)\end{aligned}$$

Select binomcdf(as described earlier, and again enter the values of n, p, and x in that order as shown on the screen. Press ENTER to find the required value.



Here $\Pr(X \leq 2) = 0.4199$
and hence $\Pr(X \geq 3) = 1 - 0.4199 = 0.5801$

The graphics calculator can also be used to display the binomial probability distribution graphically. This is shown in Example 7.

Example 7

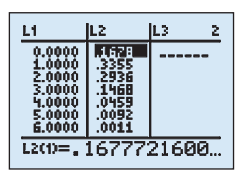
Use the graphics calculator to plot the probability distribution function

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n$$

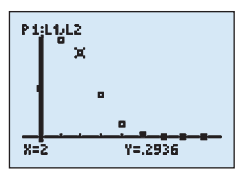
for $n = 8$ and $p = 0.2$.

Solution

Enter the numbers 0, 1, . . ., 8 into L1 (these are the values that the random variable X can take here). Take the cursor to the top of the list L2 and press ENTER. The probabilities associated with each value of X can be stored in L2 using the command binompdf(8, 0.2) → L2.



To plot the graph select 2ND STAT PLOT and choose a scatterplot with L1 as the Xlist and L2 as the YList (this is the default). Selecting **9:ZoomStat** from the ZOOM menu gives the following plot. Use TRACE to read the values from the screen.



Example 8

The probability of winning a prize in a game of chance is 0.25. What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.9?

Solution

Since the probability of winning each game is the same each time the game is played, this is an example of a binomial distribution, with the probability of success $p = 0.25$. We are being asked to find the value of n such that:

$$\Pr(X \geq 2) > 0.9$$

or, equivalently, $\Pr(X < 2) < 0.1$

$$\Pr(X < 2) = \Pr(X = 0) + \Pr(X = 1)$$

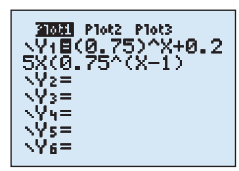
$$= \binom{n}{0} 0.25^0(0.75)^n + \binom{n}{1} 0.25^1(0.75)^{n-1}$$

$$= (0.75)^n + n0.25^1(0.75)^{n-1} \quad \text{since} \quad \binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{1} = n$$

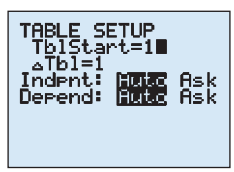
It is required to find the value of n such that:

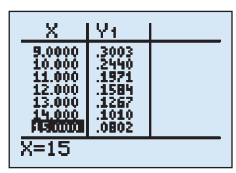
$$(0.75)^n + 0.25n(0.75)^{n-1} < 0.1$$

This is not an inequality that can be solved algebraically; however, the graphics calculator can be used to solve this equation. Select Y= and enter the equation as shown.



Now select 2ND TBLSET and set up as shown. Finally select 2ND TABLE and scroll down until the value of the equation is less than 0.1.





Thus, this game must be played at least 15 times to ensure that the probability of winning at least twice is more than 0.9.

Example 9

The probability of an archer obtaining a maximum score from a shot is 0.4. Find the probability that out of five shots the archer obtains the maximum score:

a three times

b three times, given that it is known that she obtains the maximum score at least once.

Solution

a Let X be the number of maximum scores from 5 shots.

$$\begin{aligned}\Pr(X = 3) &= {}^5C_3(0.4)^3(0.6)^2 \\ &= 10 \times 0.064 \times 0.36 \\ &= 0.2304 \quad \left(= \frac{144}{625}\right)\end{aligned}$$

b
$$\begin{aligned}\Pr(X = 3 | X > 0) &= \frac{\Pr(X = 3)}{\Pr(X > 0)} \\ &= \frac{0.2304}{1 - \Pr(X = 0)} \\ &= \frac{0.2304}{1 - 0.6^5} \\ &= \frac{0.2304}{0.92224} \\ &= 0.2498 \quad \text{(correct to 4 decimal places)}\end{aligned}$$

Exercise 10C

Example 5

1 For the binomial distribution $\Pr(X = x) = \binom{6}{x}(0.3)^x(0.7)^{6-x}$, $x = 0, 1, \ldots, 6$, find:

a $\Pr(X = 3)$ **b** $\Pr(X = 4)$

Example 6

2 For the binomial distribution $\Pr(X = x) = \binom{10}{x}(0.1)^x(0.9)^{10-x}$, $x = 0, 1, \ldots, 10$, find:

a $\Pr(X = 2)$ **b** $\Pr(X \leq 2)$

3 A fair die is rolled 60 times. Use your graphics calculator to find the probability of observing:

a exactly ten 6s **b** fewer than ten 6s **c** at least ten 6s

4 Rainfall records for the city of Melbourne indicate that, on average, the probability of rain falling on any one day in November is 0.35. Assuming that the occurrence of rain on any day is independent of whether or not rain falls on any other day, find the probability that:

a rain will fall on the first three days of a given week, but not on the other four

b rain will fall on exactly three days of a given week, but not on the other four

c rain will fall on at least three days of a given week.

5 A die is rolled seven times and the number of 2s that occur in the seven rolls is noted. Find the probability that:

a the first roll is a 2 and the rest are not

b exactly one of the seven rolls results in a 2.

6 If the probability of a female child being born is 0.5, use your graphics calculator to find the probability that, if 100 babies are born on a certain day, more than 60 of them will be female.

7 A breakfast cereal manufacturer places a coupon in every tenth packet of cereal entitling the buyer to a free packet of cereal. Over a period of two months a family purchases five packets of cereal.

a Find the probability distribution of the number of coupons in the five packets.

b What is the most probable number of coupons in the five packets?

8 If the probability of a female child being born is 0.48, find the probability that a family with exactly three children has at least one child of each sex.

9 An insurance company examines its records and notes that 30% of accident claims are made by drivers aged under 21. If there are 100 accident claims in the next 12 months, use your graphics calculator to determine the probability that 40 or more of them are made by drivers aged under 21.

10 A restaurant is able to seat 80 customers inside, and many more at outside tables. Generally, 80% of their customers prefer to sit inside. If 100 customers arrive one day, use your graphics calculator to determine the probability that the restaurant will seat inside all those who make this request.

11 A supermarket has four checkouts. A customer in a hurry decides to leave without making a purchase if all the checkouts are busy. At that time of day the probability of each checkout being free is 0.25. Assuming that whether or not a checkout is busy is independent of any other checkout, calculate the probability that the customer will make a purchase.

12 An aircraft has four engines. The probability that any one of them will fail on a flight is 0.003. Assuming the four engines operate independently, find the probability that on a particular flight:

a no engine failure occurs **b** not more than one engine failure occurs

c all four engines fail

13 A market researcher wishes to determine if the public has a preference for one of two brands of cheese, brand A or brand B. In order to do this, 15 people are asked to choose which cheese they prefer. If there is actually no difference in preference:

a What is the probability that 10 or more people would state a preference for brand A?

b What is the probability that 10 or more people would state a preference for brand A or brand B?

14 It has been discovered that 4% of the batteries produced at a certain factory are defective. A sample of 10 is drawn randomly from each hour's production and the number of defective batteries is noted. In what percentage of these hourly samples would there be a least two defective batteries? Explain what doubts you might have if a particular sample contained six defective batteries.

15 An examination consists of 10 multiple-choice questions. Each question has four possible answers. At least five correct answers are required to pass the examination.

a Suppose a student guesses the answer to each question. What is the probability the student will make:

i at least three correct guesses? **ii** at least four correct guesses?

iii at least five correct guesses?

b How many correct answers do you think are necessary to decide that the student is not guessing each answer? Explain your reasons.

16 An examination consists of 20 multiple-choice questions. Each question has four possible answers. At least 10 correct answers are required to pass the examination. Suppose the student guesses the answer to each question. Use your graphics calculator to determine the probability that the student passes.

Example 7 **17** Plot the probability distribution function $\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$, for $n = 10$ and $p = 0.3$

18 Plot the probability distribution function $\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$, for $n = 15$ and $p = 0.6$.

Example 8 **19** What is the least number of times a fair coin should be tossed in order to ensure that:

a the probability of observing at least one head is more than 0.95?

b the probability of observing more than one head is more than 0.95?

20 What is the least number of times a fair die should be rolled in order to ensure that:

a the probability of observing at least one 6 is more than 0.9?

b the probability of observing more than one 6 is more than 0.9?

21 Geoff has determined that his probability of hitting an ace when serving at tennis is 0.1. What is the least number of balls he must serve to ensure that:

a the probability of hitting at least one ace is more than 0.8?

b probability of hitting more than one ace is more than 0.8?

22 The probability of winning in a game of chance is known to be 0.05. What is the least number of times Phillip should play the game in order to ensure that:

a the probability that he wins at least once is more than 0.90?

b the probability that he wins at least once is more than 0.95?

Example 9 **23** The probability of a shooter obtaining a maximum score from a shot is 0.7. Find the probability that out of five shots the shooter obtains the maximum score:

a three times

b three times, given that it is known that he obtains the maximum score at least once.

24 Each week a security firm transports a large sum of money between two places. The day on which the journey is made is varied at random and, in any week, each of the five days from Monday to Friday is equally likely to be chosen. (In the following, give answers correct to 4 decimal places.)
Calculate the probability that in a period of 10 weeks Friday will be chosen:

a two times **b** at least two times
c exactly three times, given it is chosen at least two times

10.4 Solving probability problems using simulation

Simulation is a very powerful and widely used procedure which enables us to find approximate answers to difficult probability questions. It is a technique which **imitates** the operation of the real-world system being investigated. Some problems are not able to be solved directly and simulation allows a solution to be obtained where otherwise none would be possible. In this section some specific probability problems are looked at which may be solved by using simulation, a valuable and legitimate tool for the statistician.

Example 10

What is the probability that a family of five children will include at least four girls?

Solution

This problem could be simulated by tossing a coin five times, once for each child, using a model based on the following **assumptions**:

- There is a probability of 0.5 of each child being female.
- The sex of each child is independent of the sex of the other children. That is, the probability of a female child is always 0.5.

Since the probability of a female child is 0.5, then tossing a fair coin is a suitable **simulation model**. Let a head represent a female child and a tail a male child. A **trial** consists of tossing the coin five times to represent one complete family of five children and the **result** of the trial is the number of female children obtained in the trial. To estimate the required probability several trials need to be conducted. How many trials are needed to estimate the probability? As we have already noted in Section 8.2, the more repetitions of an experiment the better the estimate of the probability. Initially about 50 trials could be considered.

An example of the results that might be obtained from 10 trials is given in the table on the next page:

Trial number	*Simulation results*	*Number of heads*
1	*THHTT*	2
2	*HHHTH*	4
3	*HHHTH*	4
4	*HTTTH*	2
5	*HTHHH*	4
6	*HTTTH*	2
7	*TTHHH*	3
8	*HTHHT*	3
9	*TTTHH*	2
10	*HHTTT*	2

Continuing in this way, the following results were obtained for 50 trials:

Number of heads	*Number of times obtained*
0	1
1	8
2	17
3	13
4	10
5	1

The results in the table can be used to estimate the required probability. Since at least four heads were obtained in 11 trials, estimate the probability of at least four female children as $\frac{11}{50}$ or 0.22. Of course, since this probability has been estimated experimentally, repeating the simulations would give a slightly different result, but we would expect to obtain approximately this value most of the time.

Example 10 can be recognised as a situation involving a binomial random variable, with $n = 5$ and $p = 0.5$. Thus the exact answer to the question 'What is the probability that a family of five children will include at least four girls?' is:

$$\begin{aligned}\Pr(X \geq 4) &= \binom{5}{4}(0.5)^4(0.5)^1 + \binom{5}{5}(0.5)^5(0.5)^0 \\ &= 0.1875\end{aligned}$$

This is reasonably close to the answer obtained from the simulation.

In Example 10 simulation was used to provide an estimate of the value of a particular probability. Simulation is also widely used to estimate the values of other quantities which are of interest in a probability problem. We may wish to know the average result, the largest result, the number of trials required to achieve a certain result, and so on. An example of this type of problem is given in Example 11.

Example 11

A pizza store is giving away Batman souvenirs with each pizza bought. There are six different souvenirs available, and a fan decides to continue buying the pizzas until all six are obtained. How many pizzas will need to be bought, on average, to obtain the complete set of souvenirs?

Solution

As there are more than two outcomes of interest, a coin is not a suitable simulation model, but a fair six-sided die could be used. Each of the six different souvenirs is represented by one of the six sides of the die. Rolling the die and observing the outcome is equivalent to buying a pizza and noting which souvenir was obtained. This simulation model is based on the following assumptions:

- The six souvenirs all occur with equal frequency.
- The souvenir obtained with one pizza is independent of the souvenirs obtained with the other pizzas.

A trial would consist of rolling the die until all of the six numbers 1, 2, 3, 4, 5 and 6 have been observed and the result of the trial is the number of rolls necessary to do this. The results of one trial are shown:

5 2 5 2 2 2 3 3 1 2 6 3 5 4

In this instance, 14 pizzas were bought before the whole set was obtained. Of course, we would not expect to buy 14 pizzas every time – this is just the result from one trial. To estimate the required probability we would need to conduct several trials. The following is an example of the results that might be obtained from 50 trials. In each case the number listed represents the number of pizzas that were bought to obtain a complete set of Batman souvenirs:

14	8	12	11	16	8	8	11	15	26	14	20	11	13	35
23	19	14	10	10	20	9	10	14	29	13	7	15	15	22
9	10	14	16	14	17	12	10	24	13	19	27	31	11	9
16	21	22	8	9										

To estimate the number of pizzas that need to be bought, the average of the numbers obtained in these simulations is calculated. Thus we estimate that, in order to collect the complete set of souvenirs, it would be necessary to buy approximately

$$\frac{14 + 8 + 12 + 11 + 16 \cdots + 16 + 21 + 22 + 8 + 9}{50} \approx 15 \text{ pizzas}$$

Since Example 11 is not a situation with which we are normally familiar, it is difficult to obtain an exact solution to the problem. Thus simulation has enabled us to find an answer to a question that is far too difficult for us to solve theoretically.

In practice there are situations where coins and dice may not be used. Other methods of simulation need to be adopted to deal with a wide range of situations. Suppose we wished to determine how many pizzas would need to be bought, on average, to obtain a complete set of eight souvenirs. This time we need to generate random numbers from 1 to 8 and a six-sided die would no longer be appropriate, but there are other methods that could be used. We could construct a spinner with eight equal sections marked from 1 to 8, or we could mark eight balls from 1 to 8 and draw them (with replacement) from a bowl, or one of a number of other methods. Often, when we wish to simulate and coins and dice are not appropriate we use random number tables, which will be discussed in the next section.

Exercise 10D

Examples 10+11

1 A teacher gives her class a test consisting of 10 'true or false' questions. Use simulation to estimate the probability of a student who guesses the answer to every question, and gets at least seven correct. Use your knowledge of the binomial distribution to find an exact answer to this question.

Example 11

2 Because of overpopulation, some countries are trying to limit their birth rate. One country decides to implement a 'one son' policy, in which a family is allowed to continue having children until their first son is born. Use simulation to answer the following questions:

a What is the ratio of girls to boys in this country?

b What is the average family size in this country?

10.5 Random number tables

To extend the number of applications which can be simulated, a more flexible method of generating data is needed. In practice this is often done by using random number tables, which can be adapted to almost any situation. Random number tables consist of the digits 0 to 9 and are constructed so that each of the digits is equally likely to appear in any position in the table. You could generate your own random number tables by drawing balls numbered 0 to 9 from a bag, replacing each ball every time so that the probability is unchanged for each draw.

An examination of a set of random number tables shows that they consist of a list of digits with no apparent pattern or order. They are usually in blocks of five digits with spaces between every fifth row to allow ease of movement around the table.

To use the tables select a starting point at random, by dropping a pencil on the tables for instance, and then proceed around the table in a specific direction. This direction can be vertical, horizontal, diagonal or whatever is chosen, but the method of movement must be consistent during the particular simulation session.

Random number table			
86541	76173	75204	86985
55785	37652	39743	48699
72924	25688	76717	90494
60165	34668	04915	22256
88362	73974	88715	61973
99543	77560	63191	93767
11455	50193	78250	57919
75632	33716	23548	28717
99772	35699	69760	08925
55166	35468	44094	97745
70041	92573	39582	94721
09793	27734	02039	47940
93208	51461	42913	01229
87861	95053	99631	06851
88758	65504	82859	99350
38007	13045	04337	13596
58598	50064	56579	22246
21903	13871	29732	74382
90925	65050	01864	35758
04573	65678	50035	45409
21509	03139	51765	64811
29568	19637	93406	36455
24803	37259	06578	79348
36874	38883	56760	51548
65540	71436	95999	92932
82937	64298	18134	09227
31586	70322	16634	61797
94708	72900	64798	11745
57624	48870	81394	54295
69132	44509	75684	52312
60284	06394	43778	17119
66040	21411	75357	66688
11450	19652	23549	05419
18271	85381	89117	59026
49909	02934	18217	54863
27084	94761	02202	92919
35938	46791	17387	79738
12694	35629	49770	02507
31796	28529	80589	33885
30994	81918	24311	92036

Example 12

There are five movie star cards available in a certain brand of bubble gum. Sally only wants the one with her favourite star on it and she decides to continue buying packets of bubble gum until she gets the one she wants. How many packets of bubble gum will she need to buy, on average, to do this?

Solution

Before deciding on a simulation model we need to clearly state our assumptions:

- The five cards all occur with equal frequency.
- The card obtained with one bubble gum packet is independent of each of the cards in the other packets.

First consider the possible outcomes. There are five cards which could generate five equally likely outcomes, each one represented by a different card. There are 10 different digits in the random number tables allowing the designation of two digits for each outcome. Let us use 0 and 1 to represent obtaining the card we want (a success), and 2, 3, 4, 5, 6, 7, 8 and 9 to represent obtaining any one of the other four cards (a failure).

$$\underbrace{0\ \ 1}_{\text{success}} \qquad \underbrace{2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9}_{\text{failure}}$$

Suppose the section of the random number tables in use looks like this:

5 5 7 8 5 3 7 6 5 2 3 9 7 4 3 4 8 6 9 9

7 2 9 2 4 2 5 6 8 8 7 6 7 1 7 9 0 4 9 4

6 0 1 6 5 3 4 6 6 8 0 4 9 1 5 2 2 2 5 6

8 8 3 6 2 7 3 9 7 4 8 8 7 1 5 6 1 9 7 3

Proceeding through the tables keeping the same pattern we record:

Trial 1: 5 6 8 8 7 6 7 1

As soon as a 0 or 1 is reached the trial is complete and the total number of packets bought is recorded, which in this case is eight. We then continue with the same pattern from the next digit for Trial 2 and so on.

Trial 2:	7	9	0						Number of packets = 3
Trial 3:	4	9	4	6	0				Number of packets = 5
Trial 4:	1								Number of packets = 1
Trial 5:	6	5	3	4	6	6	8	0	Number of packets = 8

This process could continue until we had the results from about 50 trials.

A possible set of results might look like this:

8	3	5	1	8	3	4	2	5	2
4	7	2	9	2	3	6	4	4	6
4	5	12	10	1	2	12	2	3	1
5	5	6	1	2	3	4	2	1	6
4	1	2	5	2	1	4	3	4	16

From these simulation results it can be estimated that, on average, Sally would need to buy about 4.3 packets of bubble gum in order to obtain the card she wants.

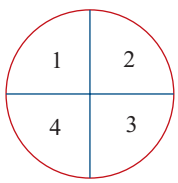
Random number tables may be used in many situations. If we wish to consider a problem involving a probability of success of 0.4, we can use the digits 0,1, 2 and 3 to indicate success, and 4, 5, 6, 7, 8 and 9 to indicate failure. If the probability is 0.36 we can use pairs of digits from the tables, where 00−35 indicates success, and 36−99 failure, and so on.

Exercise 10E

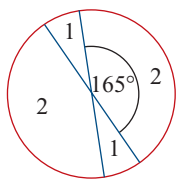
Example 12

1 How would you use random number tables to simulate the outcomes of each spinner?

a



b



2 Use simulation to estimate the number of pizzas we would need to buy if the number of Batman souvenirs described in Example 11 was extended to 10.

3 Use the information contained in Example 12 and simulation to estimate the number of bubble gum packets Sally would need to buy if:

a she wishes to collect two cards in the set (that is, two different cards)

b she wishes to have two copies of her favourite card (that is, two of the same card).

4 A teacher gives the class a test consisting of 10 multiple-choice questions, each with five alternatives. Use simulation to estimate the probability of a student who guesses the answer to every question getting at least seven correct. Use your knowledge of the binomial distribution to find an exact answer to this question.

5 An infectious disease has a one-day infection period and after that the person is immune. Six people live on an otherwise deserted island. One person catches the disease and randomly visits one other person for help during the infectious period. The second person is infected and visits another person at random during the next day (their infection period). The process continues, with one visit per day, until an infectious person visits an immune person and the disease dies out.

a Use simulation to estimate the average number of people who will be infected before the disease dies out.

b Use random number tables to extend this problem to different size populations.

MC TEST

D&D TEST

Chapter summary

- A **discrete** random variable X is one which can assume only a countable number of values. Often these values are whole numbers, but not necessarily.
- The **probability distribution** of X, $p(x) = \Pr(X = x)$ is a function that assigns probabilities to each value of X. It can be represented by a formula, a table or a graph, and must give a probability $p(x)$ for every value that X can take.
  For **any** discrete probability function the following must be true:
  - **a** The minimum value of $p(x)$ is zero, and the maximum value is 1, for every value of X. That is, $0 \leq p(x) \leq 1$ for all x.
  - **b** The sum of all values of $p(x)$ must be exactly 1.
- The **binomial distribution** arises when counting the number of successes in a sample chosen from an infinite population, or from a finite population with replacement. In either case, the probability, p, of a success on a single trial remains constant for all trials. If the experiment consists of a number, n, of identical trials, and the random variable of interest, X, is the number of successes in n trials, then:

$$\Pr(X = x) = \binom{n}{x} (p)^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n$$

  where $\binom{n}{x} = \dfrac{n!}{x!(n-x)!}$
- Simulation is a simple and legitimate method for finding solutions to problems when an exact solution is difficult, or impossible, to find.
- In order to use simulation to solve a problem a clear statement of the problem and the underlying assumptions must be made.
- A model must be selected to generate outcomes for a simulation. Possible choices for physical simulation models are coins, dice and spinners. Random number tables, calculators and computers may also be used.
- Each trial should be defined and repeated several times (at least 50).
- The results from all the trials should be recorded and summarised appropriately to provide an answer to a problem.

Multiple-choice questions

1 Consider the following table which represents the probability distribution of the variable X.

x	0	1	2	3	4
$\Pr(X = x)$	k	$2k$	$3k$	$2k$	k

For the table to represent a probability function, the value of k is

A $\frac{1}{10}$ **B** $\frac{1}{9}$ **C** $\frac{1}{5}$ **D** $\frac{1}{7}$ **E** $\frac{1}{8}$

Review

2 Suppose that the random variable X has the probability distribution given in the table:

x	1	2	3	4	5	6
$\Pr(X = x)$	0.05	0.23	0.18	0.33	0.14	0.10

Then $\Pr(X \geq 5)$ is equal to

A 0.24 **B** 0.10 **C** 0.90 **D** 0.76 **E** 0.14

3 Suppose that there are two apples and three oranges in a bag. A piece of fruit is drawn from the bag. If the fruit is an apple, it is not replaced and a second piece of fruit is drawn and the process is repeated until an orange is chosen. If X is the number of pieces of fruit drawn *before* an orange is chosen then the possible values for X are

A $\{0\}$ **B** $\{0, 1\}$ **C** $\{0, 1, 2\}$ **D** $\{0, 1, 2, 3\}$ **E** $\{1, 2, 3\}$

4 Which one of the following random variables has a binomial distribution?

A the number of tails observed when a fair coin is tossed 10 times
B the number of times a player rolls a die before a 6 is observed
C the number of SMS messages a students sends in a day
D the number of people at the AFL Grand Final
E the number of accidents which occur per day at a busy intersection

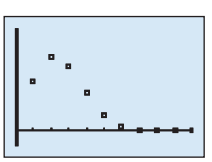
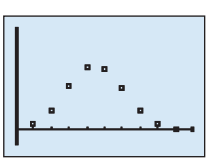
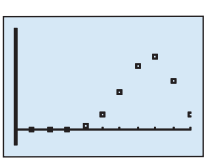
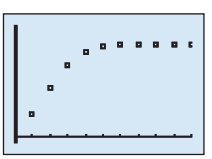
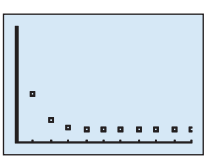
5 Suppose that X is the number of male children born into a family. If the distribution of X is binomial, with probability of success of 0.48, the probability that a family with six children will have exactly three male children is

A 0.48×3 **B** $(0.48)^3$ **C** $(0.48)^6$ **D** ${}^6C_3(0.48)^3$ **E** ${}^6C_3(0.48)^3(0.52)^3$

6 The probability that a student will be left-handed is known to be 0.23. If nine students are selected at random for the softball team then the probability that at least one of these students is left-handed is given by

A $(0.23)^9$ **B** ${}^9C_1(0.23)^1(0.77)^8$ **C** $1 - {}^9C_0(0.23)^0(0.77)^9$
D $1 - {}^9C_0(0.23)^0(0.77)^9 - {}^9C_1(0.23)^1(0.77)^8$ **E** $(0.23)^9 + {}^9C_1(0.23)^1(0.77)^8$

7 Which one of the following graphs best represents the shape of a binomial probability distribution of the random variable X with 10 independent trials and probability of success 0.2?

A 
B 
C 
D 
E 

8 If the probability that a mathematics student in a certain state is male is 0.56, and if 60 students are chosen at random from that state, then the probability that at least 30 of those chosen are male is closest to

A 0.066 **B** 0.210 **C** 0.790 **D** 0.857 **E** 0.143

Questions 9 and 10 refer to the following information:

Tom is choosing lucky numbers from a box. The probability of winning a prize with any one of the lucky numbers is 0.1, and whether or not a prize is won on a single draw is independent of any draw.

9 Suppose Tom draws 10 lucky numbers. The probability he wins three or four times is

A 0.0574 **B** 0.0686 **C** 0.0112 **D** 0.0702 **E** 0.9984

10 Suppose Tom plays a sequence of n games. If the probability of winning at least one prize is more than 0.90, then the smallest value n can take is closest to

A 1 **B** 2 **C** 15 **D** 21 **E** 22

Short-answer questions (technology-free)

1 For the probability distribution

x	0	1	2	3	4
$\Pr(X = x)$	0.12	0.25	0.43	0.12	0.08

calculate:

a $\Pr(X \leq 3)$ **b** $\Pr(X \geq 2)$ **c** $\Pr(1 \leq X \leq 3)$

2 A box contains 100 cards. Twenty-five cards are numbered 1, 28 are numbered 2, 30 are numbered 3 and 17 are numbered 4. One card will be drawn from the box and its number X observed. Give the probability distribution of X.

3 From the six marbles numbered as shown, two marbles will be drawn without replacement.

① ① ① ① ② ②

Let X denote the sum of the numbers on the selected marbles. List the possible values of X and determine the probability distribution.

4 Two of the integers $\{1, 2, 3, 6, 7, 9\}$ are chosen at random. (An integer can be chosen twice.) Let X denote the sum of the two integers.

a List all choices and the corresponding values of X.

b List the distinct values of X.

c Obtain the probability distribution of X.

5 For a binomial distribution with $n = 4$ and $p = 0.25$, find the probability of:

a three or more successes **b** at most three successes **c** two or more failures

Review

6 Twenty-five per cent of trees in a forest have severe leaf damage from air pollution. If three trees are selected at random, find the probability that:

a two of the selected trees have severe leaf damage

b at least one has severe leaf damage.

7 In a large batch of eggs, one in three is found to be bad. What is the probability that of four eggs there will be:

a no bad egg? **b** exactly one bad egg? **c** more than one bad egg?

8 In a particular village the probability of rain falling on any given day is $\frac{1}{4}$.
Calculate the probability that in a particular week rain will fall on:

a exactly three days **b** less than three days **c** four or more days

9 Previous experience indicates that, of the students entering a particular diploma course, $p\%$ will successfully complete it. One year, 15 students commence the course. Calculate, in terms of p, the probability that:

a all 15 students successfully complete the course

b only one student fails

c no more than two students fail.

10 The probability of winning a particular game is $\frac{3}{5}$. (Assume all games are independent.)

a Find the probability of winning at least one game when the game is played three times.

b Given that, when the game is played m times, the probability of winning exactly two games is three times the probability of winning exactly one game, find the value of m.

Extended-response questions

1 For a particular random experiment $\Pr(A|B) = 0.6$, $\Pr(A|B') = 0.1$ and $\Pr(B) = 0.4$.
The random variable X takes the value 4 if both A and B occur, 3 if only A occurs, 2 if only B occurs, and 1 if neither A nor B occur.

a Specify the probability distribution of X. **b** Find $\Pr(X \geq 2)$.

2 The number of times a paper boy hits the front step of a particular house in a street in a randomly selected week is given by the random variable X, which can take values 0, 1, 2, 3, 4, 5, 6, 7. The probability distribution for X is given in the table.

x	0	1	2	3	4	5	6	7
$\Pr(X = x)$	0	k	0.1	0.2	0.2	0.3	0.1	0

a **i** Find the value of k.

ii Find the probability that he hits the front step more than three times.

iii Find the probability that he hits the front step more than four times, given that he hits the front step more than three times.

b It is found that there are 10 houses on the round for which the paper boy's accuracy is given exactly by the distribution above. Therefore the probability of hitting the front step of any one of these 10 houses three or four times a week is 0.4.

i Find the probability (correct to 4 decimal places) that out of the 10 houses he hits the front step of exactly four *particular* houses three or four times a week.

ii Find the probability (correct to 4 decimal places) that out of 10 houses he hits the front step any four houses three or four times a week.

3 A bag contains three blue cards and two white cards that are identical in all respects except colour. Two cards are drawn at random and without replacement from the bag.

a Find the probability that the two cards are of different colour.

If the cards are of a different colour, two fair coins are tossed and the number of heads recorded. If the cards are of the same colour, the two fair coins are each tossed twice and the number of heads recorded.

Let X be the number of heads recorded.

b Find:

i $\Pr(X = 0)$ **ii** $\Pr(X = 2)$

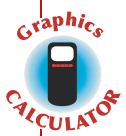
The events A and B are defined as follows:

A occurs if the two cards drawn are of the same colour, B occurs if $X \geq 2$.

c Find:

i $\Pr(A \cup B)$ **ii** $\Pr(B|A)$

4 An examination consists of 20 multiple-choice questions. Each question has five possible answers. At least 10 correct answers are required to pass the examination. Suppose the student guesses the answer to each question.



a Use your graphics calculator to determine the probability that the student passes.

b Given that the student has passed, what is the probability that they scored at least 80% on the test?

5 Jolanta is playing a game of chance. She is told that the probability of winning at least once in every five games is 0.99968. Assuming that the probability of winning each game is constant, what is her probability of winning in any one game?

6 A five-letter 'word' is formed by selecting letters from the word BINOMIAL. Each letter is replaced after selection so that it may be chosen more than once. Find the probability that the 'word' contains at least one vowel.

7 Suppose that a telephone salesperson has a probability of 0.05 of making a sale on any phone call.

a What is the probability that they will make at least one sale in the next 10 calls?

b How many calls should the salesperson make in order to ensure that the probability of making at least one sale is more than 90%?

Review

8 Suppose that, in flight, aeroplane engines fail with probability q, independently of each other, and a plane will complete the flight successfully if at least half of its engines do not fail.

a Find, in terms of q, the probability that a two-engine plane completes the flight successfully.

b Find, in terms of q, the probability that a four-engine plane completes the flight successfully.

c For what values of q is a two-engine plane to be preferred to a four-engine one?

9 Use simulation to estimate the probability that in a group of 10 people at least two of them will have their birthday in the same month. (Assume that each month is equally likely for any person.)

10 In general 45% of people have type O blood. Assuming that donors arrive independently and randomly at the blood bank, use simulation to answer the following questions.

a If 10 donors came in one day, what is the probability of at least four having type O blood?

b On a certain day, the blood bank needs four donors with type O blood. How many donors, on average, should they have to see in order to obtain exactly four with type O blood?

11 Sixteen players are entered in a tennis tournament. Use simulation to estimate how many matches a player will play, on average:

a if the player has a 50% chance of winning each match

b if the player has a 70% chance of winning each match.

12 Consider a finals series of games in which the top four teams play off as follows:

Game 1:	Team A vs Team B
Game 2:	Team C vs Team D
Game 3:	Winner of game 2 plays loser of game 1
Game 4:	Winner of game 3 plays winner of game 1

The winner of game 4 is then the winner of the series.

a Assuming all four teams are equally likely to win any game, use simulation to model the series.

b Use the results of the simulation to estimate the probability that each of the four teams wins the series.

CHAPTER

11

Revision

Revision of Chapters 8–10

11.1 Multiple-choice questions

Questions marked with a † are based on Chapter 10.

1 Mary and Ann try to guess the month in which the other was born. The probability that both guess correctly is

A $\frac{1}{2}$ **B** $\frac{1}{6}$ **C** $\frac{1}{4}$ **D** $\frac{1}{24}$ **E** $\frac{1}{144}$

2 Bag A contains 2 white and 3 black balls. Bag B contains 3 white and 2 black balls. If one ball is drawn from each bag the probability that they are of different colours is

A $\frac{6}{25}$ **B** $\frac{10}{25}$ **C** $\frac{13}{25}$ **D** $\frac{21}{25}$ **E** $\frac{24}{25}$

3 Two dice are thrown. The probability of getting a sum that is greater than or equal to 12 is

A 0 **B** $\frac{1}{6}$ **C** $\frac{1}{12}$ **D** $\frac{1}{18}$ **E** $\frac{1}{36}$

4 A group consists of four boys and three girls. If two of them are chosen at random (without replacement), the probability that a boy and a girl are chosen is

A $\frac{2}{7}$ **B** $\frac{4}{7}$ **C** $\frac{12}{49}$ **D** $\frac{24}{49}$ **E** $\frac{27}{49}$

5 If X and Y are mutually exclusive events such that $\Pr(X) = \Pr(Y')$, then $\Pr(X \cup Y)$ is

A $\Pr(X) \times \Pr(Y)$ **B** $\Pr(X)$ **C** $\Pr(Y)$ **D** 0 **E** 1

†6 In 1974, England won the toss in 250 of the 500 Tests played. The probability that England wins the toss exactly 250 times in the next 500 Tests is

A 1 **B** $\left(\frac{1}{2}\right)^{250}$ **C** $\left(\frac{1}{2}\right)^{500}$

D $\binom{500}{250}\left(\frac{1}{2}\right)^{250}$ **E** $\binom{500}{250}\left(\frac{1}{2}\right)^{500}$

Revision

7 If six fair dice are rolled, the probability of getting at least one 4 is

A $\frac{4}{6}$ **B** $\left(\frac{5}{6}\right)^6$ **C** $1-\left(\frac{5}{6}\right)^6$ **D** $\frac{1}{6}$ **E** $\frac{1}{3}$

8 If a card is randomly drawn from a well-shuffled bridge card deck (52 cards), the probability of getting a heart or a jack is

A $\frac{1}{52}$ **B** $\frac{5}{13}$ **C** $\frac{4}{13}$ **D** $\frac{7}{52}$ **E** $\frac{1}{26}$

9 A bag contains k red marbles and 1 white marble. Two marbles are drawn without replacement. The probability that both are red is

A $\frac{k}{(k+1)^2}$ **B** $\frac{k-1}{k+1}$ **C** $\frac{k}{k+1}$ **D** $\frac{2k}{k+1}$ **E** $\frac{2}{k+1}$

10 Two cards drawn at random from a pack.

i The first card is replaced and the pack shuffled before the second is drawn.

ii There is no such replacement.

The ratio of the probabilities that both are aces is

A 8:3 **B** 5:3 **C** 4:3 **D** 17:13 **E** 52:51

11 The probability of Bill hitting the bullseye with a single shot is $\frac{1}{2}$. The probability that Charles does the same is $\frac{1}{4}$. Bill has 2 shots and Charles has 4. The ratio of the probability of each player hitting the bullseye at least once is

A 64:27 **B** 2:1 **C** 32:27 **D** 192:175 **E** 64:85

12 The number of arrangements which can be made using all the letters of the word RAPIDS, if the vowels are together, is

A 30 **B** 60 **C** 120 **D** 240 **E** 720

13 The number of ways in which n books can be chosen from $m+n$ different books is

A $\frac{(m+n)!}{n!}$ **B** $(m+n)!-m!$ **C** $(m+n)!-n!$

D $\frac{(m+n)!}{m!}$ **E** $\frac{(m+n)!}{m!n!}$

14 The number of different teams of seven which can be selected from a squad of 12 players is

A 792 **B** 84 **C** 5040 **D** 120 **E** 396

15 The number of four-letter code words which can be made using the letters P, Q, R, S if repetitions are allowed is

A 16 **B** 24 **C** 64 **D** 128 **E** 256

16 Six cards labelled 1, 2, 3, 4, 5 and 6 are put into a box. Three cards are then drawn from the box (without replacement). The probability that the three cards are all labelled with odd numbers is

A $\frac{1}{2}$ **B** $\frac{1}{4}$ **C** $\frac{1}{8}$ **D** $\frac{1}{12}$ **E** $\frac{1}{20}$

17 A football ground has six entry gates (I, J, K, L, M and N) and 10 exit gates (Q, R, S, T, U, V, W, X, Y and Z). The number of ways two people can enter by the same entry gate and leave by a different exit gate is

A $6 \times 10 + 1 \times 9$ **B** $6 \times 10 \times 9$ **C** $6 \times 10 + 6 \times 9$
D $6 + 10 + 9$ **E** $6 \times 10 \times 2$

18 If $\Pr(A \cap B) = \frac{1}{5}$ and $\Pr(B) = \frac{1}{2}$ and $\Pr(B|A) = \frac{1}{3}$, then

A $\Pr(A|B) = \frac{3}{5}$ and $\Pr(A) = \frac{1}{5}$ **B** $\Pr(A|B) = \frac{1}{3}$ and $\Pr(A) = \frac{2}{5}$
C $\Pr(A|B) = \frac{2}{5}$ and $\Pr(A) = \frac{3}{5}$ **D** $\Pr(A|B) = \frac{2}{3}$ and $\Pr(A) = \frac{3}{5}$
E $\Pr(A|B) = \frac{2}{5}$ and $\Pr(A) = \frac{2}{5}$

19 A pair of unbiased dice is thrown. The probability of the sum of the numbers uppermost being 10 is

A $\frac{1}{36}$ **B** $\frac{2}{36}$ **C** $\frac{3}{36}$ **D** $\frac{10}{36}$ **E** $\frac{18}{36}$

20 If the letters of the word HEADS are arranged in random order, then the probability that the letters are in alphabetical order is

A $\frac{1}{120}$ **B** $\frac{1}{60}$ **C** $\frac{1}{24}$ **D** $\frac{1}{5}$ **E** $\frac{1}{4}$

21 A box contains twelve red balls and four green balls. A ball is selected at random from the box and not replaced and then a second ball is drawn. The probability that the two balls are both green is equal to

A $\frac{1}{4}$ **B** $\frac{1}{16}$ **C** $\frac{3}{64}$ **D** $\frac{1}{8}$ **E** $\frac{1}{20}$

†**22** A hockey player attempts to score a goal in a practice session. The probability of each shot scoring is $\frac{1}{8}$, independently of the outcome of any other shot. The probability that the player hits four goals out of six shots is

A $\left(\frac{1}{8}\right)^4\left(\frac{7}{8}\right)^2$ **B** $15\left(\frac{7}{8}\right)^4\left(\frac{1}{8}\right)^2$ **C** $20\left(\frac{7}{8}\right)^4\left(\frac{1}{8}\right)^2$
D $\left(\frac{7}{8}\right)^4\left(\frac{1}{8}\right)^2$ **E** $15\left(\frac{1}{8}\right)^4\left(\frac{7}{8}\right)^2$

†**23** In an experiment the constant probability that a trial will be successful is p. Three trials are to be made. The probability that not more than one will be successful is

A $(1-p)^3$ **B** $3p(1-p)^2$ **C** $(1-p)^2(1+2p)$
D $p^2(3-2p)$ **E** p^3

Revision

†**24** It is found that an archer hits a target with a probability of 0.8. The probability that he hits the target at least once from 10 shots is

A $1 - {}^{10}C_1(0.8)(0.2)^9$ **B** ${}^{10}C_1(0.8)^9(0.2)^1$ **C** ${}^{10}C_1(0.8)(0.2)^9$
D $1 - (0.2)^{10}$ **E** $1 - (0.8)^{10}$

†**25** The probability that Trevor hits the bullseye on a dartboard is 0.15. The least number of darts that Trevor should throw to ensure that the probability he throws at least one bullseye is more than 0.9 is

A 2 **B** 6 **C** 14 **D** 15 **E** 29

11.2 Extended-response questions

1 The heights (measured to the nearest centimetre) of 56 plants grown under experimental conditions are given in the following table.

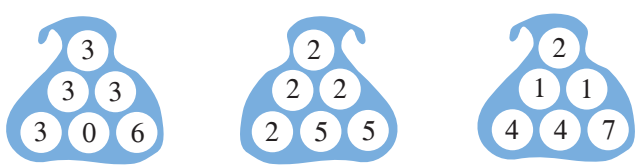
Note: The table gives the number of plants for which the measured height lies in a given interval.

Interval	(0, 10]	(10, 20]	(20, 30]	(30, 40]	(40, 50]	(50, 60]	(60, 70]
Number of plants	1	2	4	6	13	22	8

a If one plant is selected at random, find the probability that its height is:
- **i** greater than 50 cm
- **ii** either greater than 50 cm or less than or equal to 30 cm
- **iii** greater than 40 cm, given that it is greater than 30 cm.

b The probability that a randomly selected plant produces a flower is $\frac{6}{7}$ and the probability that it will suffer from a disease is $\frac{1}{4}$. Assume independence.
- **i** Find the probability that a single plant selected at random produces a flower but does not suffer from disease.
- **ii** Find the probability that a single plant selected at random produces a flower, does not suffer from disease and has a height greater than 50 cm.

2 Three people, *A*, *B* and *C*, are each given a bag containing six numbered balls as shown in the diagram below.



A *B* *C*

a Calculate the probability that *B* draws a higher number than *C*.

b Calculate the probability that *A* draws a higher number than both *B* and *C*.

3 Eight cards bearing the letters P, A, R, A, L, L, E, L are placed in a box. Three cards are drawn out at random without replacement. Calculate the probability that:

a the second card bears the letter L
b the three cards bear the letters A, L, E in that order
c the three cards bear the letters A, L, E in any order
d the first two cards bear different letters.

4 Bella Scotsimal, a fortune teller, claims to be able to predict the sex of unborn children. In fact, on each occasion she is consulted the probability that she makes a correct prediction is 0.6, independent of any other prediction.
One afternoon Bella is consulted by 10 women. Find, correct to 2 significant figures, the probability that:

a her first 8 predictions are correct and her last two are wrong
b she makes exactly 8 correct predictions
c she makes at least 8 correct predictions
d she makes exactly 8 correct predictions, given that she makes at least 8.

5 Of the employees in a large factory, $\frac{1}{8}$ travel to work by bus, $\frac{3}{8}$ by train, and the remainder by car. Those travelling by bus have a probability of $\frac{1}{3}$ of being late, those by train will be late with probability $\frac{1}{5}$, and those by car will be late with probability $\frac{3}{4}$.

a Draw and complete a tree diagram and calculate the probability that an employee chosen at random will be late.
b If an employee is late, calculate the probability that he travelled by car.

6 For the purpose of choosing a team for a quiz, a class is split into three groups. Group A contains two boys and three girls. Group B contains one boy and three girls, and group C contains two boys and one girl. An unbiased die is thrown. If a 1, 2 or 3 appears, a random choice will be made from group A. If a 4 or 5 appears, a random choice will be made from group C. If a 6 appears, a random choice will be made from group B.

a Calculate the probability that a boy will be chosen when the first choice is made.
b Given that a girl is chosen when the first choice is made, calculate the probability that she will be chosen from:
i group A **ii** group B

7 Suppose that in a game of chance bag A contains balls numbered 1 and 2, bag B contains balls numbered 3, 4 and 5, and bag C contains balls numbered 3 and 5.

Revision

a David chooses one bag at random and then draws one ball randomly from the bag.

i Find the probability that the ball drawn is a 4.

ii Find the probability that the ball drawn is a 3.

b After David has drawn a ball in the way described in part a, he puts it back into the original bag. Next, Sally chooses one bag at random and then draws one ball randomly from the bag.

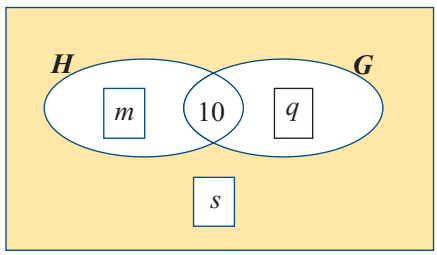
Find the probability that:

i the balls drawn by David and Sally are both numbered 4

ii the balls drawn by David and Sally are both numbered 3 and from the same bag.

8 Of the 100 students at a particular level in a school, 40 take History, 45 take Geography and 10 take both subjects.

a A Venn diagram interpreting this situation is as shown:



m is the number of students who take History only.
q is the number of students who take Geography only.
s is the number of students who take neither subject.

i Find the values of m, q and s.

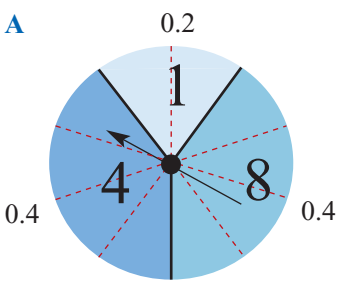
ii Find the number of students who take either History or Geography but not both.

b If a student is chosen at random, find the probability that he or she takes History but not Geography.

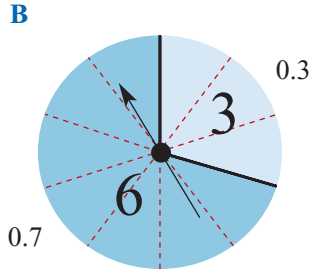
c If a student is chosen randomly from the students who do not take History, find the probability that he or she takes Geography.

9 Players *A, B* and *C* each have a disc and a spinning arrow which when spun comes to rest with its head pointing towards one sector. The probability of the arrowhead stopping at any sector and the score recorded by the player is shown in the diagram. For example, *A* scores 1, 4 or 8 with probabilities 0.2, 0.4 or 0.4 respectively.

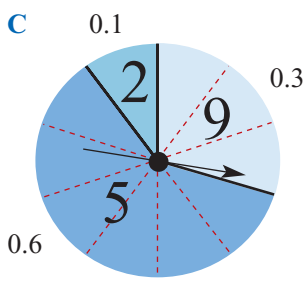
A



B



C



If *A, B* and *C* all spin their arrows simultaneously, find the probability (assuming independence) that:

a the total of the scores is 23 **b** *B* scores more than *C* **c** *C* scores more than *A*

10 The journey from town A to town D passes through towns B and C. There are three roads from A to B, four from B to C and five from C to D.

a Find the number of different routes that can be taken from A to D.

There are road works on one of the roads between A and B, on two roads between B and C and on three roads between C and D.

b Find the number of different routes that can be taken from A to D without encountering any road works.

c Calculate the probability that a traveller from A to D, picking a route at random, encounters road works at each stage of their journey.

11 A, B and C fire one shot each at a target.

The probability that A will hit the target is $\frac{1}{5}$, the probability that B will hit the target is $\frac{1}{4}$ and the probability that C will hit the target is $\frac{1}{3}$.

If they fire together, calculate the probability that:

a all three shots hit the target

b only C's shot hits the target

c at least one shot hits the target

d given that only one shot hits the target, it is the shot by C.

12 In a game at a fair, darts are thrown at a board. The outer square is of side 100 cm and each of the nine inner squares is of side 20 cm.

4	9	2
3	5	7
8	1	6

a Calculate, in cm^2, the area of:

i the larger outer square

ii one of the inner squares

iii the shaded region.

The number shown in each region is the score obtained by a dart hitting that region. A dart hitting the shaded region scores 0. Assume that all darts hit the board and that each point on the board is equally likely to be hit.

b Find the probability that with one dart the score will be

i 7 **ii** at least 7 **iii** 0

c Each turn consists of throwing three darts. Calculate the probability of scoring:

i a total of 18 with the first two darts **ii** a total of 24 with the three darts

13 Eighteen numbers are arranged in three groups of six as follows:

Group A: 0, 6, 6, 6, 6, 10
Group B: 1, 1, 1, 7, 7, 11
Group C: 3, 3, 3, 3, 8, 8

One number is drawn at random from each group. Given that a, b and c denote the number drawn from groups A, B and C respectively, calculate the probability that:

a $c < b$ **b** a is greater than both b and c **c** $c > a + b$

14 Phillip has noted that when he is fishing the probability that he will catch a trout on any day is 0.2, independent of whether or not he catches a trout on any other day. He stops fishing for the day if he catches a trout.

a Suppose that Phillip goes fishing five days in a row. What is the probability that he will catch:

i no trout?

ii exactly two trout?

iii at least one trout?

b What is the least number of days on which he should fish to ensure that:

i the probability of catching at least one trout is more than 0.9?

ii the probability of catching more than one trout is more than 0.9?

15 Suppose that the probability of a team scoring a goal from a corner in a hockey game is 0.3.

a Suppose that the team has 10 corners. What is the probability, correct to 4 decimal places, that the team will score:

i one goal?

ii at least two goals?

iii no more than two goals?

b What is the least number of corners which would be required to ensure that:

i the probability of scoring at least one goal is more than 0.95?

ii the probability of scoring more than one goal is more than 0.95?

CHAPTER

12

Exponential Functions and Logarithms

Objectives

- To define and understand **exponential functions**.
- To **sketch graphs** of the various types of exponential functions.
- To understand the rules for manipulating **exponential and logarithmic expressions**.
- To solve **exponential equations**.
- To evaluate **logarithmic expressions**.
- To solve equations using **logarithmic methods**.
- To sketch graphs of functions of the form $y = \log_a x$ and simple transformations of this.
- To understand and use a range of **exponential models**.
- To **sketch graphs** of exponential functions.
- To **apply** exponential functions to solving problems.

The function $f(x) = ka^x$, where k is a non zero constant and the base a is a positive real number other than 1, is called an **exponential function** (or index function).

Consider the following example of an exponential function. Assume that a particular biological organism reproduces by dividing every minute. The following table shows the population, P, after n one-minute intervals (assuming that all organisms are still alive).

n	0	1	2	3	4	5	6	n
P	1	2	4	8	16	32	64	2^n

Thus P defines a function which has the rule $P = 2^n$, an exponential (or index) function.

12.1 Graphs of exponential functions

Two types of graphs will be examined.

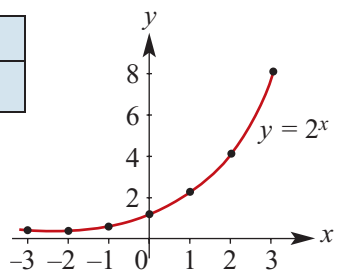
1 Graphs of $y = a^x$, $a > 1$

Example 1

Plot the graph of $y = 2^x$, and examine the table of values for $-3 \leq x \leq 3$. A graphics calculator can be used.

Solution

x	-3	-2	-1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8



It can be observed that, as negative values of increasing magnitude are considered, the graph approaches the x-axis from above. The x-axis is said to be an **asymptote**. As the magnitude of negative x-values becomes larger, 2^x takes values closer and closer to zero, but never reaches zero, i.e. the graph gets closer and closer to the x-axis. This is written as $x \to -\infty, y \to 0$ from the positive side, or as $x \to -\infty, y \to 0^+$.

The y-axis intercept for the graph is (0, 1). The range of the function is R^+.

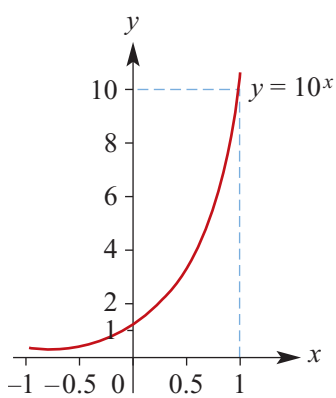
Example 2

Plot the graph of $y = 10^x$ and examine the table of values for $-1 \leq x \leq 1$. A graphics calculator can be used.

Solution

x	-1	-0.5	0	0.5	1
$y = 10^x$	0.1	0.316	1	3.16	10

The x-axis is an asymptote, and the y-axis intercept is (0, 1). The y-values increase as the x-values increase. This rate of increase for $y = 10^x$ is greater than that for $y = 2^x$ for a given value of x.



It is worth noting at this stage that, for a and b positive numbers greater than 1, there is a positive number k such that $a^k = b$. This can be seen from the graphs of $y = 2^x$ and $y = 10^x$ above. Using a graphics calculator to solve $2^k = 10$ graphically gives $k = 3.321928\ldots$ Hence $10^x = (2^{3.321928\ldots})^x = 2^{3.321928\ldots x}$. This means that the graph of $y = 10^x$ can be obtained from the graph of $y = 2^x$ by a dilation of factor $\frac{1}{k} = \frac{1}{3.321928\ldots}$ from the y-axis.

This shows that all graphs of the form $y = a^x$, where $a > 1$, are related to each other by dilations from the y-axis.

This will be discussed again later in the chapter.

2 Graphs of $y = a^x$, $0 < a < 1$

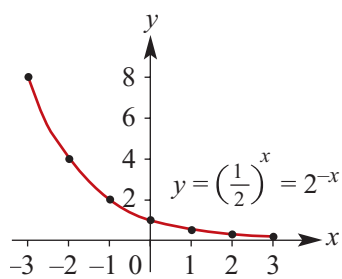
Example 3

Plot the graph of $y = \left(\frac{1}{2}\right)^x$ and examine the table of values for $-3 \leq x \leq 3$. A graphics calculator can be used.

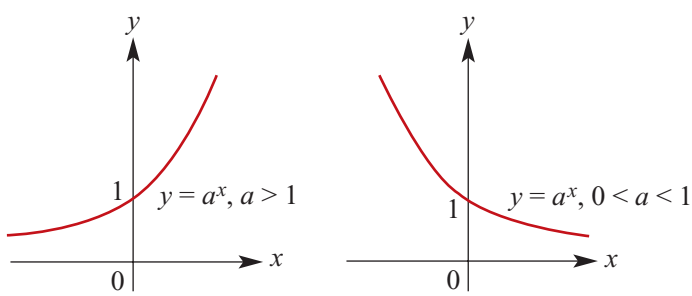
Solution

x	-3	-2	-1	0	1	2	3
$y = \left(\frac{1}{2}\right)^x = 2^{-x}$	8	4	2	1	0.5	0.25	0.125

The x-axis is an asymptote, and the y-axis intercept is (0, 1). For this graph the y-values decrease as the x-values increase. This is written as $x \to \infty$, $y \to 0$ from the positive side, or as $x \to \infty$, $y \to 0^+$. The range of the function is R^+.



In general:



For $0 < a < 1$, $y = a^x$ is equivalent to $y = b^{-x}$, where $b = \frac{1}{a}$.

The graph of $y = a^{-x}$ is obtained from the graph of $y = a^x$ by a reflection in the y-axis. Thus, for example, the graph of $y = \left(\frac{1}{2}\right)^x$ is obtained from the graph of $y = 2^x$ by a reflection in the y-axis, and vice versa.

If $a = 1$, the resulting graph is a horizontal line with equation $y = 1$.

Using a graphics calculator

Example 4

Plot the graph of $y = 2^x$ on a graphics calculator and hence find:

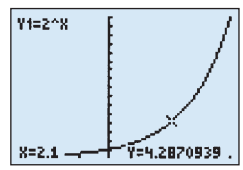
a the value of y when $x = 2.1$, correct to 3 decimal places

b the value of x when $y = 9$.

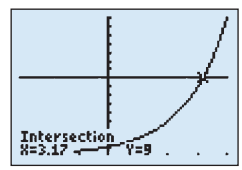
Solution

Plot the graph for $-3 \leq x \leq 4$.

a Select **1:value** from the **CALC** menu and enter $x = 2.1$.
$y = 4.287$ when $x = 2.1$ (Answer correct to 3 decimal places.)



b Enter Y2 = 9 in the **Y=** window and then select **5:intersect** from the **CALC** menu.



When $y = 9$, $x = 3.170$, correct to 3 decimal places.

Transformations of exponential functions

Example 5

Sketch the graphs of each of the following functions. Give equations of asymptotes and the y-axis intercepts, and state the range of each of the functions. (x-axis intercepts need not be given.)

a $f: R \to R,\ f(x) = 2^x + 3$

b $f: R \to R,\ f(x) = 2 \times 3^x + 1$

c $f: R \to R,\ f(x) = -3^x + 2$

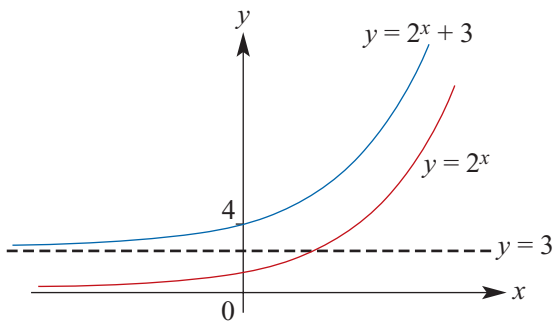
Solution

a For the function $f: R \to R,\ f(x) = 2^x + 3$, the corresponding graph is obtained by transforming the graph of $y = 2^x$ by a translation of 3 units in the positive direction of the y-axis.

The asymptote of $y = 2^x$, with equation $y = 0$, is transformed to the asymptote with equation $y = 3$ for the graph of $f(x) = 2^x + 3$. The asymptotic behaviour can be described as $x \to -\infty$, $y \to 3$ from the positive side, or as $x \to -\infty$, $y \to 3^+$.

As $f(0) = 2^0 + 3 = 4$, the y-axis intercept is 4.

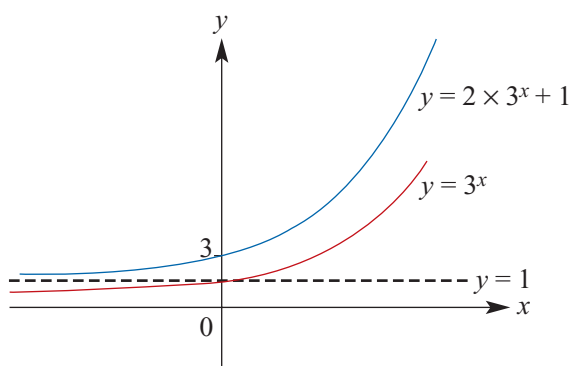
The range of the function $f: R \to R$, $f(x) = 2^x + 3$ is $(3, \infty)$.



b For the function $f: R \to R$, $f(x) = 2 \times 3^x + 1$, the corresponding graph is obtained by transforming the graph of $y = 3^x$ by a dilation of factor 2 from the x-axis, followed by a translation of 1 unit in the positive direction of the y-axis.

The asymptote of $y = 3^x$, with equation $y = 0$, is transformed to the asymptote $y = 1$ for the graph of $f(x) = 2 \times 3^x + 1$.

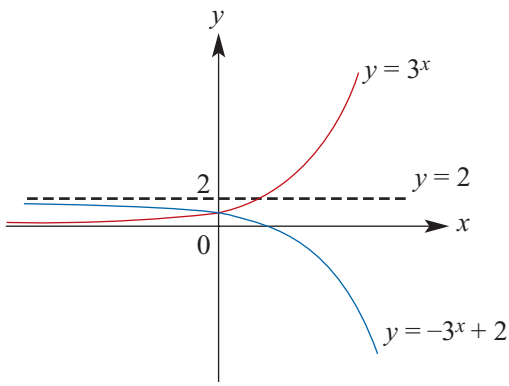
The asymptotic behaviour is described as $x \to -\infty$, $y \to 1$ from the positive side, or as $x \to -\infty$, $y \to 1^+$. The y-axis intercept is given by $f(0) = 2 \times 3^0 + 1 = 3$. The range of the function $f: R \to R$, $f(x) = 2 \times 3^x + 1$ is $(1, \infty)$.



c For the function $f: R \to R$, $f(x) = -3^x + 2$, the corresponding graph is obtained by transforming the graph of $y = 3^x$ by reflection in the x-axis followed by a translation of 2 units in the positive direction of the y-axis.

The asymptote of $y = 3^x$, with equation $y = 0$, is transformed to the asymptote $y = 2$ for the graph of $f(x) = -3^x + 2$.

The asymptotic behaviour is described by as $x \to -\infty$, $y \to 2$ from the negative side, or as $x \to -\infty$, $y \to 2^-$. The y-axis intercept is given by $f(0) = -3^0 + 2 = 1$. The range of the function $f: R \to R$, $f(x) = -3^x + 2$ is $(-\infty, 2)$.



Example 6

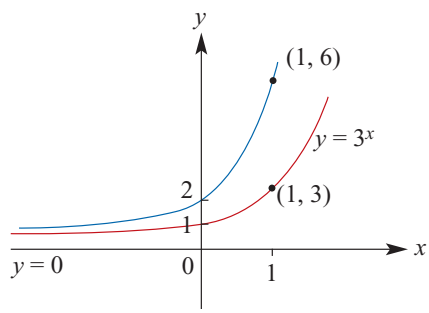
Sketch the graphs of each of the following:

a $y = 2 \times 3^x$ **b** $y = 3^{2x}$ **c** $y = 3^{\frac{x}{2}}$ **d** $y = -3^{2x} + 4$

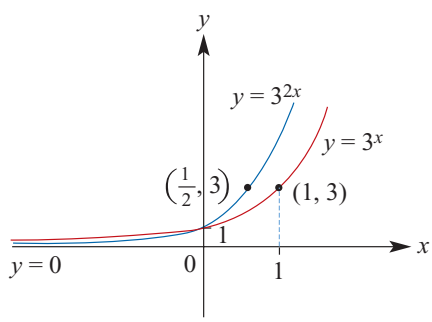
Solution

a The graph of $y = 2 \times 3^x$ is obtained from the graph of $y = 3^x$ by a dilation of factor 2 from the x-axis.

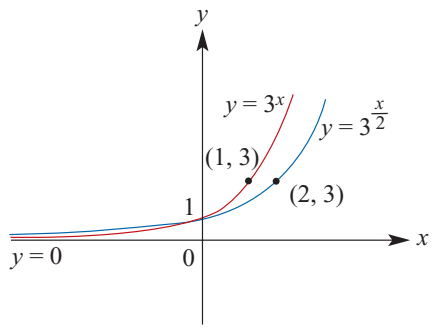
The horizontal asymptote for both graphs has equation $y = 0$.



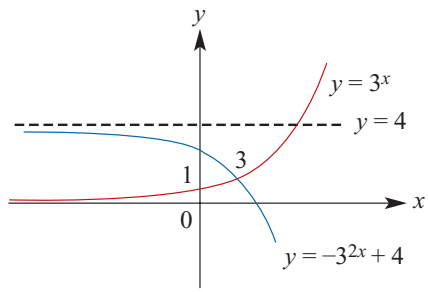
b The graph of $y = 3^{2x}$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ from the y-axis.
(In the notation introduced in Chapter 5, write it as $(x, y) \to \left(\frac{1}{2}x, y\right)$.
Then describe the transformation as $x' = \frac{1}{2}x$ and $y' = y$, and hence $x = 2x'$ and $y = y'$.
The graph of $y = 3^x$ is mapped to the graph of $y' = 3^{2x'}$)
The horizontal asymptote for both graphs has equation $y = 0$.



c The graph of $y = 3^{\frac{x}{2}}$ is obtained from the graph of $y = 3^x$ by a dilation of factor 2 from the y-axis. (In the notation introduced in Chapter 5, write it as $(x, y) \to (2x, y)$.) Then describe the transformation as $x' = 2x$ and $y' = y$, and hence $x = \dfrac{x'}{2}$ and $y = y'$. The graph of $y = 3^x$ is mapped to the graph of $y' = 3^{\frac{x'}{2}}$).



d The graph of $y = -3^{2x} + 4$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ from the y-axis, followed by a reflection in the x-axis then by a translation of 4 units in the positive direction of the y-axis.



Exercise 12A

1 Using a graphics calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 1.8^x$ **b** $y = 2.4^x$ **c** $y = 0.9^x$ **d** $y = 0.5^x$

2 Using a graphics calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 2 \times 3^x$ **b** $y = 5 \times 3^x$ **c** $y = -2 \times 3^x$ **d** $y = -5 \times 3^x$

Example 4 **3** Using a graphics calculator plot the graph of $y = 2^x$ for $x \in [-4, 4]$ and hence find the solution of the equation $2^x = 14$.

4 Using a graphics calculator plot the graph of $y = 10^x$ for $x \in [-0.4, 0.8]$ and hence find the solution of the equation $10^x = 6$.

Example 5 **5** Sketch the graphs of each of the following functions. Give equations of asymptotes and y-axis intercepts, and state the range of each of the functions. (x-axis intercepts need not be given.)

a $f: R \to R,\ f(x) = 3 \times 2^x + 2$ **b** $f: R \to R,\ f(x) = 3 \times 2^x - 3$

c $f: R \to R,\ f(x) = -3^x - 2$ **d** $f: R \to R,\ f(x) = -2 \times 3^x + 2$

e $f: R \to R,\ f(x) = \left(\dfrac{1}{2}\right)^x + 2$ **f** $f: R \to R,\ f(x) = -2 \times 3^x - 2$

Example 6 **6** Sketch the graphs of each of the following:

a $y = 2 \times 5^x$ **b** $y = 3^{3x}$ **c** $y = 5^{\frac{x}{2}}$ **d** $y = -3^{2x} + 2$

12.2 Reviewing rules for exponents (indices)

We shall review the rules for manipulating the exponential expression a^x, where a is a real number called the **base**, and x is a real number called the **exponent** or **index**. Other words which are synonyms for *index* are **power** and **logarithm**.

Multiplication: $a^m \times a^n$

If m and n are positive integers

then $a^m = a \times a \times a \cdots \times a$

$\leftarrow m \text{ terms} \rightarrow$

and $a^n = a \times a \cdots \times a$

$\leftarrow n \text{ terms} \rightarrow$

$\therefore \quad a^m \times a^n = (a \times a \times a \cdots \times a) \times (a \times a \cdots \times a)$

$\leftarrow m \text{ terms} \rightarrow \quad \leftarrow n \text{ terms} \rightarrow$

$= (a \times a \times a \cdots \times a)$

$\leftarrow m + n \text{ terms} \rightarrow$

$= a^{m+n}$

Rule 1

To multiply two numbers in exponent form with the **same base**, **add** the exponents.

$$a^m \times a^n = a^{m+n}$$

CAS APPENDIX

Example 7

Simplify each of the following:

a $2^3 \times 2^{12}$ **b** $x^2y^3 \times x^4y$ **c** $2^x \times 2^{x+2}$ **d** $3a^2b^3 \times 4a^3b^2$

Solution

a $2^3 \times 2^{12} = 2^{3+12}$
$= 2^{15}$

b $x^2y^3 \times x^4y = x^2 \times x^4 \times y^3 \times y$
$= x^6y^4$

c $2^x \times 2^{x+2} = 2^{x+x+2}$
$= 2^{2x+2}$

d $3a^2b^3 \times 4a^3b^2 = 3 \times 4a^2 \times a^3 \times b^3 \times b^2$
$= 12a^5b^5$

Division: $a^m \div a^n$

If m and n are positive integers and $m > n$

$\leftarrow m \text{ terms} \rightarrow$

then $a^m \div a^n = \dfrac{a \times a \times a \ldots a}{a \times a \times a \ldots a}$

$\leftarrow n \text{ terms} \rightarrow$

$= a \times a \times a \ldots a$ (by cancelling)

$\leftarrow (m - n) \text{ terms} \rightarrow$

$= a^{m-n}$

Rule 2

To divide two numbers in exponent form with the **same base**, **subtract** the exponents.

$$a^m \div a^n = a^{m-n}$$

Define $a^0 = 1$ for $a \neq 0$ and $a^{-n} = \dfrac{1}{a^n}$ for $a \neq 0$.

Note: **Rule 1** and **Rule 2** also hold for negative indices m, n for $a \neq 0$.

For example:

$$2^4 \times 2^{-2} = \frac{2^4}{2^2} = \frac{2 \times 2 \times 2 \times 2}{2 \times 2} = 2^2 \quad \text{(i.e. } 2^{4+(-2)}\text{)}$$

$$2^{-4} \div 2^2 = \frac{1}{2^4} \times \frac{1}{2^2} = \frac{1}{2^4 \times 2^2} = 2^{-6} \quad \text{(i.e. } 2^{-4-2}\text{)}$$

$$2^3 \div 2^3 = 2^3 \times \frac{1}{2^3} = 1 = 2^0 \quad \text{(i.e. } 2^{3-3}\text{)}$$

Example 8

Simplify each of the following:

a $\dfrac{x^4y^3}{x^2y^2}$ **b** $\dfrac{b^{4x} \times b^{x+1}}{b^{2x}}$ **c** $\dfrac{16a^5b \times 4a^4b^3}{8ab}$

Solution

a $\dfrac{x^4y^3}{x^2y^2} = x^{4-2}y^{3-2}$

$= x^2y$

b $\dfrac{b^{4x} \times b^{x+1}}{b^{2x}} = b^{4x+x+1-2x}$

$= b^{3x+1}$

c $\dfrac{16a^5b \times 4a^4b^3}{8ab} = \dfrac{16 \times 4}{8} \times a^{5+4-1} \times b^{1+3-1}$

$= 8a^8b^3$

Raising the power: $(a^m)^n$

Consider the following:

$$(2^3)^2 = 2^3 \times 2^3 = 2^{3+3} = 2^6 = 2^{3 \times 2}$$
$$(4^3)^4 = 4^3 \times 4^3 \times 4^3 \times 4^3 = 4^{3+3+3+3} = 4^{12} = 4^{3 \times 4}$$
$$(a^2)^5 = a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{2+2+2+2+2} = a^{10} = a^{2 \times 5}$$

In general $(a^m)^n = a^{m \times n}$.

Rule 3

To raise the power of a to another power, **multiply** the indices.

$$(a^m)^n = a^{m \times n}$$

This rule holds for all integers m and n.

Products and quotients

Consider $(ab)^n$ and $\left(\frac{a}{b}\right)^n$.

$$\begin{aligned}(ab)^n &= (ab) \times (ab) \times \cdots \times (ab)\\ &\quad \leftarrow \quad n \text{ terms} \quad \rightarrow\\ &= (a \times a \times \cdots a) \times (b \times b \times \cdots b)\\ &\quad \leftarrow n \text{ terms} \rightarrow \quad \leftarrow n \text{ terms} \rightarrow\\ &= a^n b^n\end{aligned}$$

Rule 4

$$(ab)^n = a^n b^n$$

$$\begin{aligned}\left(\frac{a}{b}\right)^n &= \frac{a}{b} \times \frac{a}{b} \times \cdots \frac{a}{b}\\ &= \frac{a^n}{b^n}\end{aligned}$$

Rule 5

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Example 9

Simplify the following, expressing the answers in positive exponent form:

a 8^{-2} **b** $\left(\frac{1}{2}\right)^{-4}$ **c** $\frac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}}$ **d** $\frac{3^{2n} \times 6^n}{8^n \times 3^n}$

Solution

a $8^{-2} = \frac{1}{8^2}$

$= \frac{1}{(2^3)^2}$

$= \frac{1}{2^6}$

b $\left(\frac{1}{2}\right)^{-4} = \frac{1}{2^{-4}}$

$= 2^4$

c $\frac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}} = \frac{3^{-3} \times 2^4 \times 3^4 \times 3^{-3} \times 2^{-3} \times 2^{-3}}{3^{-4} \times 3^{-4} \times 2^{-2}}$

$= \frac{3^{-2} \times 2^{-2}}{3^{-8} \times 2^{-2}}$

$= 3^6$

d $\frac{3^{2n} \times 6^n}{8^n \times 3^n} = \frac{(3^n \times 3^n) \times (3^n \times 2^n)}{2^{3n} \times 3^n} = \frac{3^n \times 3^n}{2^{2n}}$

$= \left(\frac{3}{2}\right)^{2n}$

Exercise 12B

1 Use the stated rule for each of the following to give an equivalent expression in simplest form:

a $x^2 \times x^3$ **b** $2 \times x^3 \times x^4 \times 4$ Rule 1

c $\dfrac{x^5}{x^3}$ **d** $\dfrac{4x^6}{2x^3}$ Rule 2

e $(a^3)^2$ **f** $(2^3)^2$ Rule 3

g $(xy)^2$ **h** $(x^2y^3)^2$ Rule 4 (also use Rule 3 for h)

i $\left(\dfrac{x}{y}\right)^3$ **j** $\left(\dfrac{x^3}{y^2}\right)^2$ Rule 5 (also use Rule 3 for j)

2 Simplify the following:

Example 7

a $x^3 \times x^4 \times x^2$ **b** $2^4 \times 4^3 \times 8^2$ **c** $3^4 \times 9^2 \times 27^3$

d $(q^2p)^3 \times (qp^3)^2$ **e** $a^2b^{-3} \times (a^3b^2)^3$ **f** $(2x^3)^2 \times (4x^4)^3$

g $m^3p^2 \times (m^2n^3)^4 \times (p^{-2})^2$ **h** $2^3a^3b^2 \times (2a^{-1}b^2)^{-2}$

3 Simplify the following:

Example 8

a $\dfrac{x^3y^5}{xy^2}$ **b** $\dfrac{16a^5b \times 4a^4b^3}{8ab}$

c $\dfrac{(-2xy)^2 \times 2(x^2y)^3}{8(xy)^3}$ **d** $\dfrac{(-3x^2y^3)^2}{(2xy)^3} \times \dfrac{4x^4y^3}{(xy)^3}$

Example 9

4 Simplify each of the following, expressing your answer in positive exponent form:

a $m^3n^2p^{-2} \times (mn^2p)^{-3}$ **b** $\dfrac{x^3yz^{-2} \times 2(x^3y^{-2}z)^2}{xyz^{-1}}$ **c** $\dfrac{a^2b \times (ab^{-2})^{-3}}{(a^{-2}b^{-1})^{-2}}$

d $\dfrac{a^2b^3c^{-4}}{a^{-1}b^2c^{-3}}$ **e** $\dfrac{a^{2n-1} \times b^3 \times c^{1-n}}{a^{n-3} \times b^{2-n} \times c^{2-2n}}$

5 Simplify each of the following:

a $3^{4n} \times 9^{2n} \times 27^{3n}$ **b** $\dfrac{2^n \times 8^{n+1}}{32^n}$ **c** $\dfrac{3^{n-1} \times 9^{2n-3}}{6^2 \times 3^{n+2}}$

d $\dfrac{2^{2n} \times 9^{2n-1}}{6^{n-1}}$ **e** $\dfrac{25^{2n} \times 5^{n-1}}{5^{2n+1}}$ **f** $\dfrac{6^{x-3} \times 4^x}{3^{x+1}}$

g $\dfrac{6^{2n} \times 9^3}{27^n \times 8^n \times 16^n}$ **h** $\dfrac{3^{n-2} \times 9^{n+1}}{27^{n-1}}$ **i** $\dfrac{8 \times 2^5 \times 3^7}{9 \times 2^7 \times 81}$

6 Simplify and evaluate:

a $\dfrac{(8^3)^4}{(2^{12})^2}$ **b** $\dfrac{(125)^3}{(25)^2}$ **c** $\dfrac{(81)^4 \div 27^3}{9^2}$

12.3 Rational exponents

$a^{\frac{1}{n}}$, where n is a natural number, is defined to be the n^{th} root of a, and denoted $\sqrt[n]{a}$. If $a \geq 0$ then $a^{\frac{1}{n}}$ is defined for all $n \in N$. If $a < 0$ then $\sqrt[n]{a}$ is only defined for n odd. (Remember that only real numbers are being considered.)

$$a^{\frac{1}{n}} = \sqrt[n]{a}, \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

Using this notation for square roots:

$$\sqrt{a} = \sqrt[2]{a} = a^{\frac{1}{2}}$$

Further, the expression a^x can be defined for rational exponents, i.e. when $x = \frac{m}{n}$, where m and n are integers, by defining

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

Example 10

Evaluate:

a $9^{-\frac{1}{2}}$ **b** $16^{\frac{5}{2}}$ **c** $64^{-\frac{2}{3}}$

Solution

a $9^{-\frac{1}{2}} = \frac{1}{9^{\frac{1}{2}}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

b $16^{\frac{5}{2}} = \left(16^{\frac{1}{2}}\right)^5 = \left(\sqrt[2]{16}\right)^5 = 4^5 = 1024$

c $64^{-\frac{2}{3}} = \frac{1}{64^{\frac{2}{3}}} = \frac{1}{\left(64^{\frac{1}{3}}\right)^2} = \frac{1}{\left(\sqrt[3]{64}\right)^2} = \frac{1}{4^2} = \frac{1}{16}$

Earlier we stated the rules:

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(a^m)^n = a^{m \times n}$$

where m and n are integers.

Note: These rules are applicable for all rational exponents.

$$a^{\frac{m}{q}} \times a^{\frac{n}{p}} = a^{\frac{m}{q}+\frac{n}{p}}$$
$$a^{\frac{m}{q}} \div a^{\frac{n}{p}} = a^{\frac{m}{q}-\frac{n}{p}}$$
$$\left(a^{\frac{m}{q}}\right)^{\frac{n}{p}} = a^{\frac{m}{q}\times\frac{n}{p}}$$

Example 11

Simplify:

a $\dfrac{3^{\frac{1}{4}} \times \sqrt{6} \times \sqrt[4]{2}}{16^{\frac{3}{4}}}$ **b** $(x^{-2}y)^{\frac{1}{2}} \times \left(\dfrac{x}{y^{-3}}\right)^4$

Solution

a $$\frac{3^{\frac{1}{4}} \times \sqrt{6} \times \sqrt[4]{2}}{16^{\frac{3}{4}}} = \frac{3^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{\left(16^{\frac{1}{4}}\right)^3}$$
$$= \frac{3^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{2^3} = \frac{3^{\frac{3}{4}} \times 2^{\frac{3}{4}}}{2^3}$$
$$= \frac{3^{\frac{3}{4}}}{2^{\frac{12}{4}-\frac{3}{4}}} = \frac{3^{\frac{3}{4}}}{2^{\frac{9}{4}}}$$

b $$(x^{-2}y)^{\frac{1}{2}} \times \left(\frac{x}{y^{-3}}\right)^4 = x^{-1}y^{\frac{1}{2}} \times \frac{x^4}{y^{-12}}$$
$$= x^3 \times y^{\frac{25}{2}}$$

Exercise 12C

Example 10

1 Evaluate each of the following:

a $125^{\frac{2}{3}}$ **b** $243^{\frac{3}{5}}$ **c** $81^{-\frac{1}{2}}$ **d** $64^{\frac{2}{3}}$

e $\left(\dfrac{1}{8}\right)^{\frac{1}{3}}$ **f** $32^{-\frac{2}{5}}$ **g** $125^{-\frac{2}{3}}$ **h** $32^{\frac{4}{5}}$

i $1000^{-\frac{4}{3}}$ **j** $10\,000^{\frac{3}{4}}$ **k** $81^{\frac{3}{4}}$ **l** $\left(\dfrac{27}{125}\right)^{\frac{1}{3}}$

Example 11

2 Simplify:

a $\sqrt[3]{a^2b} \div \sqrt{ab^3}$ **b** $(a^{-2}b)^3 \times \left(\dfrac{1}{b^{-3}}\right)^{\frac{1}{2}}$ **c** $\dfrac{45^{\frac{1}{3}}}{9^{\frac{3}{4}} \times 15^{\frac{3}{2}}}$

d $2^{\frac{3}{2}} \times 4^{-\frac{1}{4}} \times 16^{-\frac{3}{4}}$ **e** $\left(\dfrac{x^3y^{-2}}{3^{-3}y^{-3}}\right)^{-2} \div \left(\dfrac{3^{-3}x^{-2}y}{x^4y^{-2}}\right)^2$ **f** $\left(\sqrt[5]{a^2}\right)^{\frac{3}{2}} \times \left(\sqrt[3]{a^5}\right)^{\frac{1}{5}}$

3 Simplify each of the following:

a $(2x-1)\sqrt{2x-1}$ **b** $(x-1)^2\sqrt{x-1}$ **c** $(x^2+1)\sqrt{x^2+1}$

d $(x-1)\sqrt[3]{x-1}$ **e** $\dfrac{1}{\sqrt{x-1}} + \sqrt{x-1}$ **f** $(5x^2+1)\sqrt[3]{5x^2+1}$

12.4 Solving exponential equations and inequations

Method 1

Express both sides of the equation as exponents to the same base and then equate the exponents since, if $a^x = a^y$ then $x = y$.

Example 12

Find the value of x for which:

a $4^x = 256$ **b** $3^{x-1} = 81$ **c** $5^{2x-4} = 25^{-x+2}$

Solution

a
$$4^x = 256$$
$$4^x = 4^4$$
$$\therefore x = 4$$

b
$$3^{x-1} = 81$$
$$3^{x-1} = 3^4$$
$$x - 1 = 4$$
$$\therefore x = 5$$

c
$$5^{2x-4} = 25^{-x+2}$$
$$= (5^2)^{-x+2}$$
$$= 5^{-2x+4}$$
$$\therefore 2x - 4 = -2x + 4$$
$$4x = 8$$
$$x = 2$$

Example 13

Solve $9^x = 12 \times 3^x - 27$.

Solution

$$(3^x)^2 = 12 \times 3^x - 27$$

Let $y = 3^x$

Then $y^2 = 12y - 27$

$$y^2 - 12y + 27 = 0$$
$$(y-3)(y-9) = 0$$

Therefore $y - 3 = 0$ or $y - 9 = 0$

$y = 3$ or $y = 9$

Hence $3^x = 3^1$ or $3^x = 3^2$

and $x = 1$ or $x = 2$

Method 2

Using a graphics calculator

Example 14

Solve $5^x = 10$ correct to 2 decimal places.

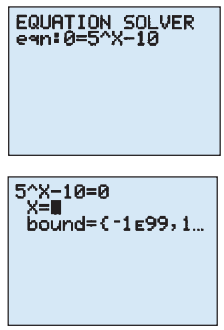
Solution

In Example 4 a graphical approach to a solution of an equation of this type was presented. An alternative is the following.

Choose **0:Solver** . . . from the **MATH** menu and complete as shown in the first window.

The second window appears as shown.

(Press ALPHA and ENTER to obtain the result.)



$$x = 1.4306765580\ldots$$

$x = 1.43$ (correct to 2 decimal places)

Solution of inequalities

The property that $a^x > a^y \Leftrightarrow x > y$, where $a \in (1, \infty)$, and $a^x > a^y \Leftrightarrow x < y$ when $a \in (0, 1)$ is used.

Example 15

Solve for x in each of the following:

a $16^x > 2$ **b** $2^{-3x+1} < \frac{1}{16}$

Solution

a
$$\begin{aligned} & 2^{4x} > 2^1 \\ \Leftrightarrow\ & 4x > 1 \\ \Leftrightarrow\ & x > \frac{1}{4} \end{aligned}$$

b
$$\begin{aligned} & 2^{-3x+1} < 2^{-4} \\ \Leftrightarrow\ & -3x + 1 < -4 \\ \Leftrightarrow\ & -3x < -5 \\ \Leftrightarrow\ & x > \frac{5}{3} \end{aligned}$$

Note: Use of a graphics calculator helps to 'visualise' the inequality.
Sketch the graph of $y = 16^x$ and find x by the use of **intersect** to solve $16^x > 2$.

Exercise 12D

Example 12

1 Solve for x in each of the following:

a $3^x = 27$ **b** $4^x = 64$ **c** $49^x = 7$

d $16^x = 8$ **e** $125^x = 5$ **f** $5^x = 625$

g $16^x = 256$ **h** $4^{-x} = \frac{1}{64}$ **i** $5^{-x} = \frac{1}{125}$

2 Solve for n in each of the following:

a $5^n \times 25^{2n-1} = 125$ **b** $3^{2n-4} = 1$ **c** $3^{2n-1} = \frac{1}{81}$

d $\frac{3^{n-2}}{9^{1-n}} = 1$ **e** $3^{3n} \times 9^{-2n+1} = 27$ **f** $2^{-3n} \times 4^{2n-2} = 16$

g $2^{n-6} = 8^{2-n}$ **h** $9^{3n+3} = 27^{n-2}$ **i** $4^{n+1} = 8^{n-2}$

j $32^{2n+1} = 8^{4n-1}$ **k** $25^{n+1} = 5 \times 390\,625$

l $125^{4-n} = 5^{6-2n}$ **m** $4^{2-n} = \dfrac{1}{2048}$

3 Solve the following exponential equations:

a $2^{x-1} \times 4^{2x+1} = 32$ **b** $3^{2x-1} \times 9^{x} = 243$ **c** $(27 \times 3^{x})^2 = 27^{x} \times 3^{\frac{1}{2}}$

Example 13 **4** Solve for x:

a $4(2^{2x}) = 8(2^{x}) - 4$ **b** $8(2^{2x}) - 10(2^{x}) + 2 = 0$

c $3 \times 2^{2x} - 18(2^{x}) + 24 = 0$ **d** $9^{x} - 4(3^{x}) + 3 = 0$

Example 14 **5** Use a graphics calculator to solve each of the following, correct to 2 decimal places:

a $2^{x} = 5$ **b** $4^{x} = 6$ **c** $10^{x} = 18$ **d** $10^{x} = 56$

Example 15 **6** Solve for x in each of the following:

a $7^{x} > 49$ **b** $8^{x} > 2$ **c** $25^{x} \leq 5$ **d** $3^{x+1} < 81$

e $9^{2x+1} < 243$ **f** $4^{2x+1} > 64$ **g** $3^{2x-2} \leq 81$

12.5 Logarithms

Consider the statement $2^3 = 8$.

This may be written in an alternative form:

$$\log_2 8 = 3$$

which is read as 'the logarithm of 8 to the base 2 is equal to 3'.

In general, if $a \in R^+ \setminus \{1\}$ and $x \in R$ then the statements $a^x = n$ and $\log_a n = x$ are equivalent.

$$a^x = y \text{ is equivalent to } \log_a y = x$$

Further examples:

- $3^2 = 9$ is equivalent to $\log_3 9 = 2$
- $10^4 = 10\,000$ is equivalent to $\log_{10} 10\,000 = 4$
- $a^0 = 1$ is equivalent to $\log_a 1 = 0$

Example 16

Without the aid of a calculator evaluate the following:

a $\log_2 32$ **b** $\log_3 81$

Solution

a Let $\log_2 32 = x$

$2^x = 32$

$2^x = 2^5$

Therefore $x = 5$, giving $\log_2 32 = 5$.

b Let $\log_3 81 = x$

$3^x = 81$

$3^x = 3^4$

Therefore $x = 4$, giving $\log_3 81 = 4$.

Laws of logarithms

The index rules are used to establish other rules for computations with logarithms.

1 Let $a^x = m$ and $a^y = n$, where m, n and a are positive real numbers.

$$\therefore \quad mn = a^x \times a^y$$
$$= a^{x+y}$$

$\log_a mn = x + y$ and, since $x = \log_a m$ and $y = \log_a n$, it follows that:

$$\log_a mn = \log_a m + \log_a n \qquad \textbf{Rule 1}$$

For example:

$$\log_{10} 200 + \log_{10} 5 = \log_{10} (200 \times 5)$$
$$= \log_{10} 1000$$
$$= 3$$

2 $\dfrac{m}{n} = \dfrac{a^x}{a^y} = a^{x-y}$

$\therefore \log_a \left(\dfrac{m}{n}\right) = x - y$ and so:

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n \qquad \textbf{Rule 2}$$

For example:

$$\log_2 32 - \log_2 8 = \log_2 \left(\frac{32}{8}\right)$$
$$= \log_2 4$$
$$= 2$$

3 If $m = 1$, $\log_a \left(\dfrac{1}{n}\right) = \log_a 1 - \log_a n$

$$= -\log_a n$$

Therefore

$$\log_a \left(\frac{1}{n}\right) = -\log_a n \qquad \textbf{Rule 3}$$

4
$$m^p = (a^x)^p$$
$$= a^{xp}$$
$$\therefore \log_a(m^p) = xp$$

and so

$$\log_a(m^p) = p \log_a m \qquad \textbf{Rule 4}$$

For example:

$$3 \log_3 4 = \log_3(4^3) = \log_3 64$$

Example 17

Without using a calculator simplify the following:

$$2\log_{10} 3 + \log_{10} 16 - 2\log_{10}\left(\frac{6}{5}\right)$$

Solution

$$\begin{aligned}
2\log_{10} 3 + \log_{10} 16 - 2\log_{10}\left(\frac{6}{5}\right) &= \log_{10} 3^2 + \log_{10} 16 - \log_{10}\left(\frac{6}{5}\right)^2 \\
&= \log_{10} 9 + \log_{10} 16 - \log_{10}\left(\frac{36}{25}\right) \\
&= \log_{10}\left(9 \times 16 \times \frac{25}{36}\right) \\
&= \log_{10} 100 \\
&= 2
\end{aligned}$$

Example 18

Solve each of the following equations for x:

a $\log_5 x = 3$ **b** $\log_5 (2x + 1) = 2$

c $\log_2 (2x + 1) - \log_2 (x - 1) = 4$ **d** $\log_3 (x - 1) + \log_3 (x + 1) = 1$

Solution

a $\log_5 x = 3 \Leftrightarrow x = 5^3 = 125$

b $\log_5 (2x + 1) = 2 \Leftrightarrow 2x + 1 = 5^2$

$\therefore \quad 2x + 1 = 25$

$2x = 24$

$x = 12$

c $\log_2 (2x + 1) - \log_2 (x - 1) = 4$

Then $\log_2\left(\frac{2x+1}{x-1}\right) = 4$

$\Leftrightarrow \quad \frac{2x+1}{x-1} = 2^4$

$2x + 1 = 16(x - 1)$

Then $17 = 14x$

Therefore $\frac{17}{14} = x$

d $\log_3 (x - 1) + \log_3 (x + 1) = 1$

Therefore $\log_3 [(x - 1)(x + 1)] = 1$

which implies $x^2 - 1 = 3$ and $x = \pm 2$

But the expression is not defined for $x = -2$, therefore $x = 2$.

Skillsheet TEACHER

Exercise 12E

1 Use the stated rule for each of the following to give an equivalent expression in simplest form:

a $\log_2 10 + \log_2 a$	**b** $\log_{10} 5 + \log_{10} 2$	Rule 1
c $\log_2 9 - \log_2 4$	**d** $\log_2 10 - \log_2 5$	Rule 2
e $\log_5\left(\frac{1}{6}\right)$	**f** $\log_5\left(\frac{1}{25}\right)$	Rule 3
g $\log_2(a^3)$	**h** $\log_2(8^3)$	Rule 4

Example 16

2 Without using a calculator evaluate each of the following:

a $\log_3 27$	**b** $\log_5 625$	**c** $\log_2\left(\frac{1}{128}\right)$
d $\log_4\left(\frac{1}{64}\right)$	**e** $\log_x x^4$	**f** $\log_2 0.125$
g $\log_{10} 10\,000$	**h** $\log_{10} 0.000\,001$	**i** $-3\log_5 125$
j $-4\log_{16} 2$	**k** $2\log_3 9$	**l** $-4\log_{16} 4$

Example 17

3 Without using a calculator simplify each of the following:

a $\frac{1}{2}\log_{10} 16 + 2\log_{10} 5$	**b** $\log_2 16 + \log_2 8$
c $\log_2 128 + \log_3 45 - \log_3 5$	**d** $\log_4 32 - \log_9 27$
e $\log_b b^3 - \log_b \sqrt{b}$	**f** $2\log_x a + \log_x a^3$
g $x\log_2 8 + \log_2 8^{1-x}$	**h** $\frac{3}{2}\log_a a - \log_a \sqrt{a}$

Example 18

4 Solve for x:

a $\log_3 9 = x$	**b** $\log_3 x = 3$
c $\log_5 x = -3$	**d** $\log_{10} x = \log_{10} 4 + \log_{10} 2$
e $\log_{10} 2 + \log_{10} 5 + \log_{10} x - \log_{10} 3 = 2$	**f** $\log_{10} x = \frac{1}{2}\log_{10} 36 - 2\log_{10} 3$
g $\log_x 64 = 2$	**h** $\log_5 (2x - 3) = 3$
i $\log_3 (x + 2) - \log_3 2 = 1$	**j** $\log_x 0.01 = -2$

5 Solve each of the following for x.

a $\log_x\left(\frac{1}{25}\right) = -2$	**b** $\log_4 (2x - 1) = 3$
c $\log_4 (x + 2) - \log_4 6 = 1$	**d** $\log_4 (3x + 4) + \log_4 16 = 5$
e $\log_3 (x^2 - 3x - 1) = 0$	**f** $\log_3 (x^2 - 3x + 1) = 0$

6 If $\log_{10} x = a$ and $\log_{10} y = c$, express $\log_{10}\left(\frac{100x^3 y^{-\frac{1}{2}}}{y^2}\right)$ in terms of a and c.

7 Prove that $\log_{10}\left(\frac{ab^2}{c}\right) + \log_{10}\left(\frac{c^2}{ab}\right) - \log_{10}(bc) = 0$.

8 If $\log_a\left(\frac{11}{3}\right) + \log_a\left(\frac{490}{297}\right) - 2\log_a\left(\frac{7}{9}\right) = \log_a(k)$, find k.

9 Solve each of the following equations for x:

a $\log_{10}(x^2 - 2x + 8) = 2\log_{10} x$ **b** $\log_{10}(5x) - \log_{10}(3 - 2x) = 1$

c $3\log_{10}(x - 1) = \log_{10} 8$ **d** $\log_{10}(20x) - \log_{10}(x - 8) = 2$

e $2\log_{10} 5 + \log_{10}(x + 1) = 1 + \log_{10}(2x + 7)$

f $1 + 2\log_{10}(x + 1) = \log_{10}(2x + 1) + \log_{10}(5x + 8)$

12.6 Using logarithms to solve exponential equations and inequations

In Section 12.4 two methods were shown for solving exponential equations.

If $a \in R^+ \setminus \{1\}$ and $x \in R$ then the statements $a^x = n$ and $\log_a n = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations.

Method 3 – Using logarithms

Example 19

Solve for x if $2^x = 11$.

Solution

Take either the $\log_{10}$ or $\log_e$ (since these are the only logarithmic functions available on your calculator) of both sides of the equation:

Then $\log_{10} 2^x = \log_{10} 11$

i.e. $x\log_{10} 2 = \log_{10} 11$

Therefore $x = \dfrac{\log_{10} 11}{\log_{10} 2}$

Use your calculator to evaluate both the $\log_{10} 11$ and $\log_{10} 2$.

Then $x = \dfrac{1.041}{0.301}$ (to 3 decimal places)

≈ 3.46 (to 2 decimal places)

Example 20

Solve $3^{2x-1} = 28$.

Solution

$$\log_{10} 3^{2x-1} = \log_{10} 28$$
$$(2x-1)\log_{10} 3 = \log_{10} 28$$
$$2x - 1 = \frac{\log_{10} 28}{\log_{10} 3}$$
$$= \frac{1.4472}{0.4771}$$
$$3.0331$$
$$\therefore 2x = 4.0331$$
$$x = 2.017 \qquad \text{(to 3 decimal places)}$$

Example 21

Solve $\{x: 0.7^x \geq 0.3\}$.

Solution

Taking $\log_{10}$ of both sides:

$$\log_{10} 0.7^x \geq \log_{10} 0.3$$
$$x \log_{10} 0.7 \geq \log_{10} 0.3$$
$$x \leq \frac{\log_{10} 0.3}{\log_{10} 0.7}$$

Note change of direction of inequality as $\log_{10} 0.7 < 0$.

$$x \leq \frac{-0.5229}{-0.1549}$$
$$x \leq 3.376$$

Example 22

Sketch the graph of $f(x) = 2 \times 10^x - 4$, giving the equation of the asymptote and the axes intercepts.

Solution

$f(0) = 2 \times 10^0 - 4 = 2 - 4 = -2$

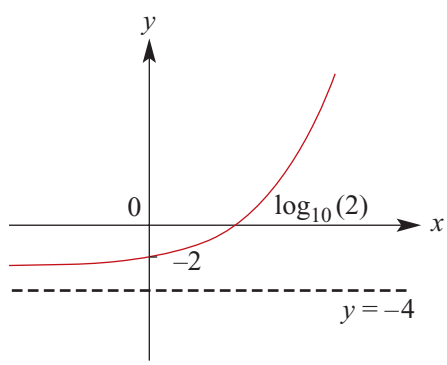
As $x \to -\infty, y \to -4$ from the positive side, or as $x \to -\infty, y \to -4^+$.

The equation of the horizontal asymptote is $y = -4$.

For the x-axis intercept consider $2 \times 10^x - 4 = 0$.

Hence $2 \times 10^x = 4$ and thus $10^x = 2$.

Therefore $x = \log_{10} 2 = 0.3010$ (correct to 4 decimal places)



Exercise 12F

1 Solve each of the following equations correct to 2 decimal places:

Example 20

a $2^x = 7$ **b** $2^x = 0.4$ **c** $3^x = 14$
d $4^x = 3$ **e** $2^{-x} = 6$ **f** $0.3^x = 2$
g $5^x = 3^{x-2}$ **h** $8^x = 2005^{x+1}$ **i** $3^{x-1} = 10$
j $0.2^{x+1} = 0.6$

Example 21

2 Solve for x. Give values correct to 2 decimal places if necessary:

a $2^x > 8$ **b** $3^x < 5$ **c** $0.3^x > 4$
d $3^{x-1} \leq 7$ **e** $0.4^x \leq 0.3$

Example 22

3 For each of the following sketch the graph of $y = f(x)$, giving the equation of the asymptote and the axes intercepts:

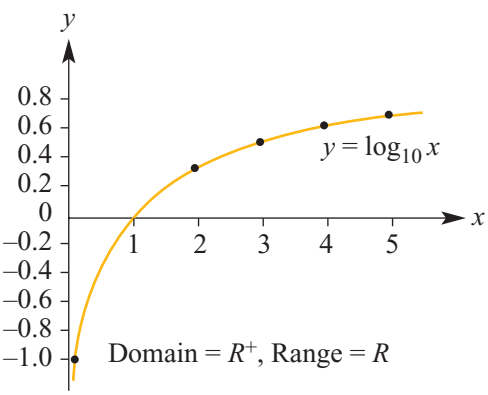
a $f(x) = 2^x - 4$ **b** $f(x) = 2 \times 3^x - 6$ **c** $f(x) = 3 \times 10^x - 5$
d $f(x) = -2 \times 10^x + 4$ **e** $f(x) = -3 \times 2^x + 6$ **f** $f(x) = 5 \times 2^x - 6$

4 In the initial period of its life a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$, where d is the diameter of the tree, in cm, t years after the beginning of this period. The diameter is 52 cm after 1 year and 80 cm after 3 years.
Calculate the values of the constants d_0 and m.

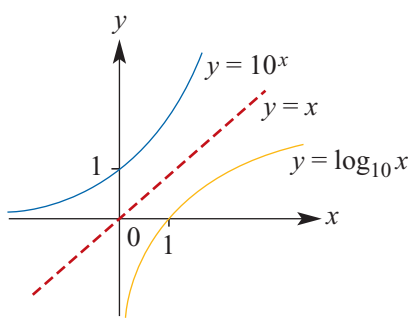
12.7 Graph of $y = \log_a x$, where $a > 1$

A table of values for $y = \log_{10} x$ is given below (the values are correct to 2 decimal places). Use your calculator to check these values.

x	0.1	1	2	3	4	5
$y = \log_{10} x$	−1	0	0.30	0.48	0.60	0.70



Domain = R^+, Range = R



The graph of $y = \log_{10} x$ is the reflection in the line $y = x$ of the graph of $y = 10^x$.

Note that $\log_{10} 1 = 0$ as $10^0 = 1$ and as $x \to 0^+, y \to -\infty$.
The inverse of a one-to-one function was introduced in Section 5.7.
The function $f^{-1}: (0, \infty) \to R, f^{-1}(x) = \log_{10} x$
is the inverse function of $f: R \to R, \quad f(x) = 10^x$

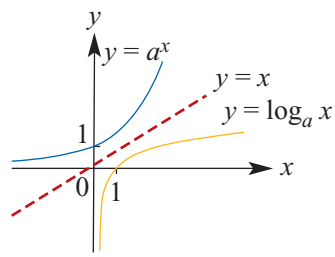
In general, $y = \log_a x$ is the rule for the inverse of the function with rule $y = a^x$, where $a > 0$ and $x > 0$.

Note that $\log_a (a^x) = x$ for all x and $a^{\log_a x} = x$ for positive values of x, as they are inverse functions.

For $a > 1$, the graphs of logarithm functions all have this same shape.

Properties of $y = a^x, a > 1$
- domain $= R$
- range $= R^+$
- $a^0 = 1$
- $x \to -\infty, y \to 0^+$

Properties of $y = \log_a x, a > 1$
- domain $= R^+$
- range $= R$
- $\log_a 1 = 0$
- $x \to 0^+, y \to -\infty$



Example 23

Find the inverse of:

a $y = 10^{2x}$ **b** $y = \log_{10} (2x)$

Solution

a $y = 10^{2x}$

Interchanging x and y gives

$x = 10^{2y}$

Therefore $2y = \log_{10} x$

and $y = \frac{1}{2} \log_{10} x$

b $y = \log_{10} (2x)$

Interchanging x and y gives

$x = \log_{10} (2y)$

Therfore $10^x = 2y$

and $y = \frac{1}{2} \times 10^x$

Example 24

Find the inverse of each of the following:

a $f(x) = 2^x + 3$ **b** $f(x) = \log_2 (x - 2)$ **c** $f(x) = 5 \times 2^x + 3$

Solution

a Let $y = 2^x + 3$

Interchanging x and y gives

$x = 2^y + 3$

$\therefore \quad x - 3 = 2^y$

and $y = \log_2 (x - 3)$

$\therefore \quad f^{-1}(x) = \log_2 (x - 3)$

domain of $f^{-1} = (3, \infty)$

b Let $y = \log_2 (x - 2)$

Interchanging x and y gives

$x = \log_2 (y - 2)$

$\therefore \quad 2^x = y - 2$

and $y = 2^x + 2$

$\therefore \quad f^{-1}(x) = 2^x + 2$

domain of $f^{-1} = R$

c Let $y = 5 \times 2^x + 3$

Interchanging x and y gives

$$x = 5 \times 2^y + 3$$

$$\therefore \quad \frac{x-3}{5} = 2^y$$

and $$y = \log_2\left(\frac{x-3}{5}\right)$$

$$\therefore \quad f^{-1}(x) = \log_2\left(\frac{x-3}{5}\right)$$

domain of $f^{-1} = (3, \infty)$

Transformations can be applied to the graphs of logarithm functions. This is shown in the following example.

Example 25

Sketch the graphs of each of the following. Give the maximal domain, the equation of the asymptote and the axes intercepts.

a $f(x) = \log_2(x-3)$ **b** $f(x) = \log_2(x+2)$ **c** $f(x) = \log_2(3x)$

Solution

a $f(x) = \log_2(x-3)$

For the function to be defined

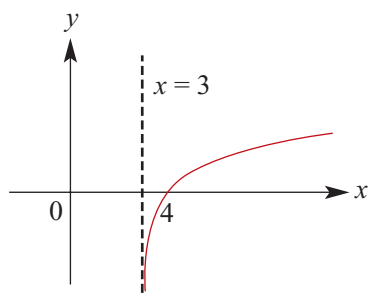
$x - 3 > 0$, i.e. $x > 3$

The maximal domain is $(3, \infty)$

x-axis intercept: $\log_2(x-3) = 0$

implies $x - 3 = 2^0$, i.e. $x = 4$

Asymptote: $x \to 3^+, y \to -\infty$



b $f(x) = \log_2(x+2)$

For the function to be defined

$x + 2 > 0$, i.e. $x > -2$

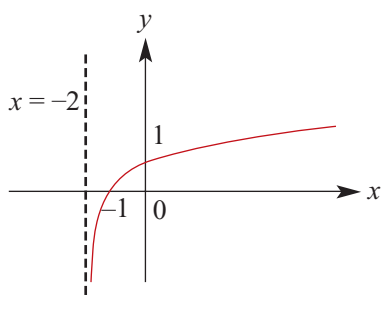
The maximal domain is $(-2, \infty)$

$f(0) = \log_2(2) = 1$

x-axis intercept: $\log_2(x+2) = 0$

implies $x + 2 = 2^0$, i.e. $x = -1$

Asymptote: $x \to -2^+, y \to -\infty$



c $f(x) = \log_2 3x$

For the function to be defined

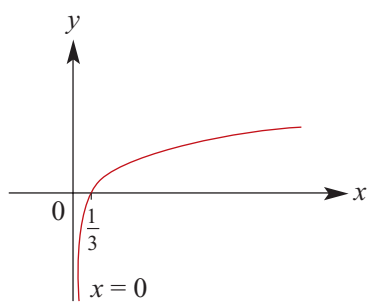
$3x > 0$, i.e. $x > 0$

The maximal domain is $(0, \infty)$

For the x-axis intercept $\log_2(3x) = 0$

implies $3x = 2^0$, i.e. $x = \frac{1}{3}$

Asymptote: $x \to 0^+, y \to -\infty$



Exercise 12G

1 Sketch the graph of each of the following and state the domain and range for each:

a $y = \log_{10}(2x)$ **b** $y = 2\log_{10} x$ **c** $y = \log_{10}\left(\frac{1}{2}x\right)$

d $y = 2\log_{10}(3x)$ **e** $y = -\log_{10} x$ **f** $y = \log_{10}(-x)$

Example 23 **2** Determine the inverse of each of the following:

a $y = 10^{0.5x}$ **b** $y = 3\log_{10} x$ **c** $y = 10^{3x}$

d $y = 2\log_{10}(3x)$

Example 24 **3** Find the rule for the inverse function of each of the following:

a $f(x) = 3^x + 2$ **b** $f(x) = \log_2(x - 3)$ **c** $f(x) = 4 \times 3^x + 2$

d $f(x) = 5^x - 2$ **e** $f(x) = \log_2(3x)$ **f** $f(x) = \log_2\left(\frac{x}{3}\right)$

g $f(x) = \log_2(x + 3)$ **h** $f(x) = 5 \times 3^x - 2$

Example 25 **4** Sketch the graphs of each of the following. Give the maximal domain, the equation of the asymptote and the axes intercepts:

a $f(x) = \log_2(x - 4)$ **b** $f(x) = \log_2(x + 3)$ **c** $f(x) = \log_2(2x)$

d $f(x) = \log_2(x + 2)$ **e** $f(x) = \log_2\left(\frac{x}{3}\right)$ **f** $f(x) = \log_2(-2x)$

5 Use a graphics calculator to solve each of the following equations correct to 2 decimal places:

a $2^{-x} = x$ **b** $\log_{10}(x) + x = 0$

6 Use a graphics calculator to plot the graphs of $y = \log_{10}(x^2)$ and $y = 2\log_{10} x$ for $x \in [-10, 10], x \neq 0$.

7 On the same set of axes plot the graph of $y = \log_{10}(\sqrt{x})$ and $y = \frac{1}{2}\log_{10} x$ for $x \in (0, 10]$.

8 Use a graphics calculator to plot the graphs of $y = \log_{10}(2x) + \log_{10}(3x)$ and $y = \log_{10}(6x^2)$

9 Find a and k such that the graph of $y = a10^{kx}$ passes through the points (2, 6) and (5, 20).

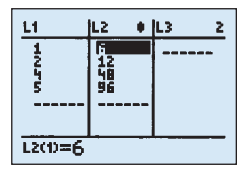
12.8 Exponential models and applications

Fitting data

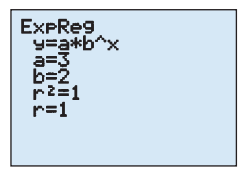
Using a graphics calculator

It is known that the points with the coordinates (1, 6), (2, 12), (4, 48), (5, 96) lie on a curve of the form $y = a \times b^x$.

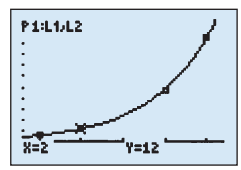
The values of a and b can be found by selecting **0:ExpReg** from the **CALC** submenu of **STAT**. Enter the points **L1** and **L2** as shown.



- In the Home screen enter **ExpReg L1**, **L2**, **Y1**, and press ENTER. The result is as shown.



- Activate **Plot** with the default settings and choose **9:ZoomStat** from the **CALC** menu. The result is as shown.The curve has the equation $y = 3 \times 2^x$.



There are many practical situations in which the relationship between variables is exponential.

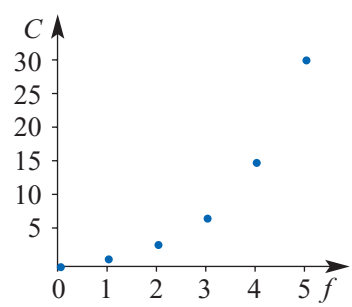
Example 26

Take a rectangular piece of paper approximately 30 cm × 6 cm. Fold the paper in half, successively, until you have folded it five times. Tabulate the times folded, f, and the number of creases in the paper, C.

Solution

Times folded, f	0	1	2	3	4	5
Creases, C	0	1	3	7	15	31

The rule connecting C and f is $C = 2^f - 1$, $f \in N \cup \{0\}$.



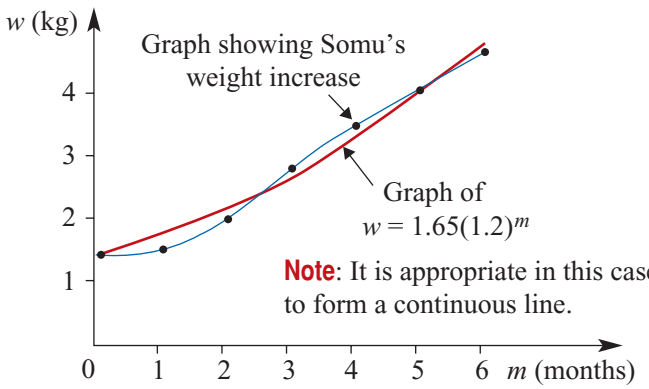
Example 27

The table below shows the increase in weight of Somu, an orang-utan born at the Eastern Plains Zoo. Draw a graph to show Somu's weight increase for the first six months.

Months, m	0	1	2	3	4	5	6
Weight, w kg	1.65	1.7	2.2	3.0	3.7	4.2	4.8

Solution

Plotting these values:



We shall now plot, on the same set of axes, the graph of the exponential function $w = 1.65(1.2)^m$, $0 \leq m \leq 6$.

Table of values

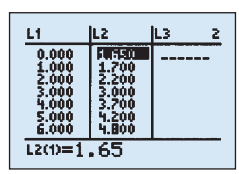
m	0	1	2	3	4	5	6
w	1.65	1.98	2.38	2.85	3.42	4.1	4.93

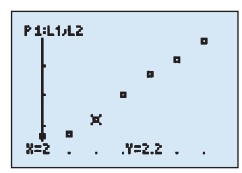
It can be seen from the graphs that the exponential model $w = 1.65(1.2)^m$ approximates to the actual weight gain and would be a useful model to predict weight gains for any future orang-utan births at the zoo. This model describes a growth rate for the first 6 months of 20% per month.

This problem can also be attempted with a graphics calculator.

Using a graphics calculator

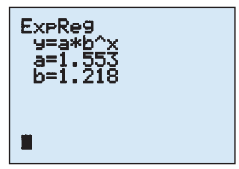
Enter the data in **L1** and **L2** as shown and display the data in a scatter plot.



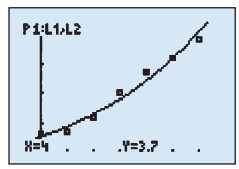


Now choose **0:ExpReg** from the **CALC** submenu of **STAT**.

In the home screen complete as **ExpReg L1, L2, Y1**. The result in the Home screen is as shown.



Press ZOOM and select **9:ZoomStat** to achieve a suitable window. The result is as shown. TRACE has been activated.



Example 28

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 11% per annum while that of the red kangaroos decreases at 5% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Solution

Let P = population of grey kangaroos at the start.

$\therefore$ number of grey kangaroos after n years $= P(1.11)^n$,

and number of red kangaroos after n years $= 10P(0.95)^n$.

When the proportions are reversed

$$P(1.11)^n = 10 \times [10P(0.95)^n]$$
$$(1.11)^n = 100(0.95)^n$$

Taking $\log_{10}$ of both sides

$$\log_{10}(1.11)^n = \log_{10} 100(0.95)^n$$
$$n \log_{10} 1.11 = \log_{10} 100 + n \log_{10} 0.95$$
$$n \times 0.04532 = 2 + n(-0.0223)$$
$$n = \frac{2}{0.0676}$$
$$= 29.6$$

i.e. the proportions of kangaroo populations will be reversed by the 30th year.

Exponential growth

The above two examples are examples of exponential change. In the following, A is a variable that is subject to exponential change.

Let A be the quantity at time t. Then $A = A_0 a^t$, where A_0 is a positive constant and a is a real number.

If $a > 1$ the model represents **growth**. If $a < 1$ the model represents **decay**.

Physical situations in which this is applicable include:

- the growth of cells
- population growth
- continuously compounded interest
- radioactive decay
- cooling of materials.

Consider a sum of money, \$10 000, invested at a rate of 5% per annum but compounded continually. That is it is compounded at every instant.

If there are n compound periods in a year, the interest paid per period is $\frac{5}{n}\%$. Therefore at the end of the year the amount of the investment, A, is

$$A = 10\,000\left(1 + \frac{5}{100n}\right)^n = 10\,000\left(1 + \frac{1}{20n}\right)^n$$

Enter the function Y1 = (1 + 1/(20X))^X. Look at the table of values with an increment of one. It is found that the value approaches 1.05127 (correct to 5 decimal places) for n large. Hence, for continuous compounding it can be written that $A = 10\,000 \times (1.05126\ldots)^x$, where x is the number of years of the investment.

Exercise 12H

Example 27

1 Find an exponential model of the form $y = ab^x$ to fit the following data.

x	0	2	4	5	10
y	1.5	0.5	0.17	0.09	0.006

2 Find an exponential model of the form $p = ab^t$ to fit the following data

t	0	2	4	6	8
p	2.5	4.56	8.3	15.12	27.56

3 A sheet of paper 0.2 mm thick is cut in half and one piece is stacked on top of the other.

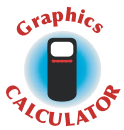
a If this process is repeated complete the following table:

Cuts, n	Sheets	Total thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	⋮
⋮	⋮	⋮
10		

b Write down a formula which shows the relationship between T and n.

c Draw a graph of T against n for $n \leq 10$.

d What would be the total thickness, T, after 30 cuts?



4 The populations (in millions), p and q, of two neighbouring American states, P and Q, over a period of 50 years from 1950 are modelled by functions $p = 1.2 \times 2^{0.08t}$ and $q = 1.7 \times 2^{0.04t}$, where t is the number of years since 1950.

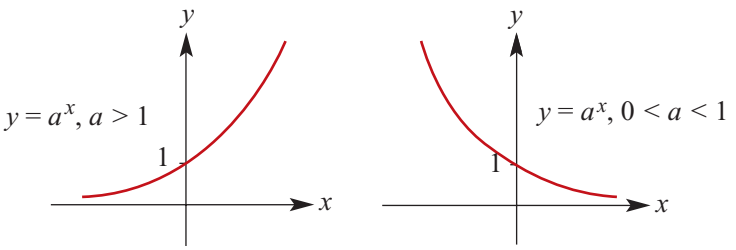
a Plot the graphs of both functions using a graphics calculator.

b Find when the population of state P is:

i equal to the population of state Q **ii** twice the population of state Q.

Chapter summary

- Groups of exponential functions
  Examples:



- To **multiply** two numbers in exponent form with the same base, **add** the exponents:
  $a^m \times a^n = a^{m+n}$
- To **divide** two numbers in exponent form with the same base, **subtract** the exponents:
  $a^m \div a^n = a^{m-n}$
- To raise the power of a to another power, **multiply** the exponents:
  $(a^m)^n = a^{m \times n}$
- If $a^x = a^y$ then $x = y$.
- For rational exponents: $a^{\frac{1}{n}} = \sqrt[n]{a}$
  and $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$
- For $a \in R^+ \setminus \{1\}$ and $x \in R$: $a^x = y$ is equivalent to $\log_a y = x$
- **Laws of logarithms**
  1 $\log_a (mn) = \log_a m + \log_a n$ **2** $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$
  3 $\log_a \left(\frac{1}{n}\right) = -\log_a n$ **4** $\log_a (m^p) = p \log_a m$
- Exponential equations can be solved by taking logarithms of both sides.
  e.g. If $2^x = 11$
  then $\log_{10} 2^x = \log_{10} 11$ and hence x can be solved.

Multiple-choice questions

1 $8x^3 \div 4x^{-3} =$
A 2 **B** $2x^0$ **C** $2x^6$ **D** $2x^{-1}$ **E** $\frac{2}{x^9}$

2 The expression $\frac{a^2b}{(2ab^2)^3} \div \frac{ab}{16a^0} =$
A $\frac{2}{a^2b^6}$ **B** $\frac{2a^2}{b^6}$ **C** $2a^2b^6$ **D** $\frac{2}{ab^6}$ **E** $\frac{1}{128ab^5}$

3 The function $f: R \to R, f(x) = 3 \times 2^x - 1$ has range
A R **B** $R \setminus \{-1\}$ **C** $(-1, \infty)$ **D** $(1, \infty)$ **E** $[1, \infty)$

Review

4 The function $f: R^+ \to R$, where $f(x) = \log_2 3x$, has an inverse function f^{-1}. The rule for f^{-1} is given by

A $f^{-1}(x) = 2^x$ **B** $f^{-1}(x) = 3^x$ **C** $f^{-1}(x) = \frac{1}{3}2^x$

D $f^{-1}(x) = 2^{\frac{x}{3}}$ **E** $f^{-1}(x) = \log_2\left(\frac{1}{3}x\right)$

5 If $\log_{10}(x-2) - 3\log_{10} 2x = 1 - \log_{10} y$, then y is equal to

A $\frac{80x^3}{x-2}$ **B** $1 + \frac{8x^3}{x-2}$ **C** $\frac{60x}{x-2}$

D $1 + \frac{6x}{x-2}$ **E** $1 - \frac{x-2}{8x^3}$

6 The solution of the equation $5 \times 2^{5x} = 10$ is x equals

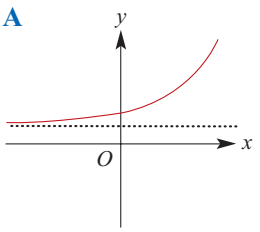
A $\frac{1}{2}$ **B** $\frac{1}{5}$ **C** $\frac{1}{5}\log_2 10$ **D** $\frac{1}{2}\log_2 5$ **E** $\frac{1}{5}2^5$

7 The equation of the asymptote of $y = 3\log_2(5x) + 2$ is

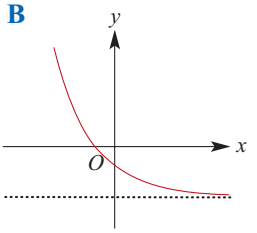
A $x = 0$ **B** $x = 2$ **C** $x = 3$ **D** $x = 5$ **E** $y = 2$

8 Which of the following graphs could be the graph of the function $f(x) = 2^{ax} + b$, where a and b are positive?

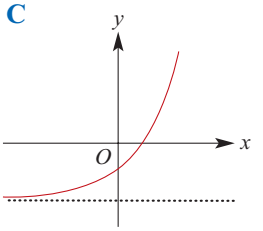
A



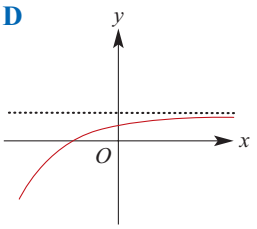
B



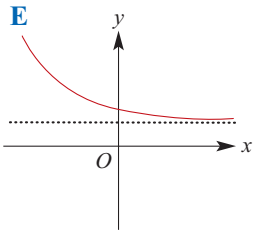
C



D



E



9 Which one of the following functions has a graph with a vertical asymptote with equation $x = b$?

A $y = \log_2(x-b)$ **B** $y = \frac{1}{x+b}$ **C** $y = \frac{1}{x+b} - b$

D $y = 2^x + b$ **E** $y = 2^{x-b}$

10 The expression $\frac{2mh}{(3mh^2)^3} \div \frac{mh}{81m^2}$ is equal to

A $\frac{6}{mh^6}$ **B** $\frac{6m^2}{h^6}$ **C** $6m^2h^6$ **D** $\frac{6}{m^2h^6}$ **E** $\frac{1}{128mh^5}$

Short-answer questions (technology-free)

1 Simplify each of the following, expressing your answer with positive index:

a $\dfrac{a^6}{a^2}$ **b** $\dfrac{b^8}{b^{10}}$ **c** $\dfrac{m^3n^4}{m^5n^6}$ **d** $\dfrac{a^3b^2}{(ab^2)^4}$

e $\dfrac{6a^8}{4a^2}$ **f** $\dfrac{10a^7}{6a^9}$ **g** $\dfrac{8(a^3)^2}{(2a)^3}$ **h** $\dfrac{m^{-1}n^2}{(mn^{-2})^3}$

i $(p^{-1}q^{-2})^2$ **j** $\dfrac{(2a^{-4})^3}{5a^{-1}}$ **k** $\dfrac{6a^{-1}}{3a^{-2}}$ **l** $\dfrac{a^4+a^8}{a^2}$

2 Use logarithms to solve each of the following equations:

a $2^x = 7$ **b** $2^{2x} = 7$ **c** $10^x = 2$ **d** $10^x = 3.6$

e $10^x = 110$ **f** $10^x = 1010$ **g** $2^{5x} = 100$ **h** $2^x = 0.1$

3 Evaluate each of the following:

a $\log_2 64$ **b** $\log_{10} 10^7$ **c** $\log_a a^2$ **d** $\log_4 1$

e $\log_3 27$ **f** $\log_2\left(\dfrac{1}{4}\right)$ **g** $\log_{10} 0.001$ **h** $\log_2 16$

4 Express each of the following as single logarithms:

a $\log_{10} 2 + \log_{10} 3$ **b** $\log_{10} 4 + 2\log_{10} 3 - \log_{10} 6$

c $2\log_{10} a - \log_{10} b$ **d** $2\log_{10} a - 3 - \log_{10} 25$

e $\log_{10} x + \log_{10} y - \log_{10} x$ **f** $2\log_{10} a + 3\log_{10} b - \log_{10} c$

5 Solve each of the following for x:

a $3^x(3^x - 27) = 0$ **b** $(2^x - 8)(2^x - 1) = 0$

c $2^{2x} - 2^{x+1} = 0$ **d** $2^{2x} - 12 \times 2^x + 32 = 0$

6 Sketch the graph of:

a $y = 2.2^x$ **b** $y = -3.2^x$ **c** $y = 5.2^{-x}$

d $y = 2^{-x} + 1$ **e** $y = 2^x - 1$ **f** $y = 2^x + 2$

7 Solve the equation $\log_{10} x + \log_{10} 2x - \log_{10}(x+1) = 0$

8 Given $3^x = 4^y = 12^z$, show that $z = \dfrac{xy}{x+y}$.

9 Evaluate $2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150$.

10 **a** Given that $\log_p 7 + \log_p k = 0$, find k.

b Given that $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$, find q.

11 Solve:

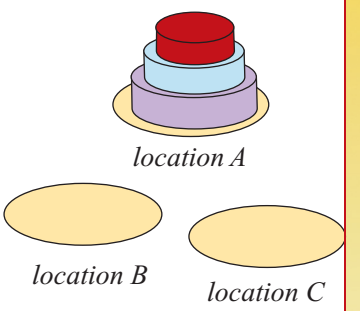
a $2 \times 4^{a+1} = 16^{2a}$ (for a) **b** $\log_2 y^2 = 4 + \log_2(y+5)$ (for y)

Review

Extended-response questions

1 This problem is based on the so-called 'Tower of Hanoi' puzzle. Given a number of different sized discs, the problem is to move a pile of discs (where n, the number of discs, is ≥ 1) to a second location (if starting at A then either to B or C) according to the following rules:

- Only one disc can be moved at a time.
- A total of only three locations can be used to 'rest' discs.
- A larger sized disc cannot be placed on top of a smaller disc.
- The task must be completed in the smallest possible number of moves.



a Using two coins complete the puzzle. Repeat first with three coins and then four coins and thus complete the table.

Number of discs, n	1	2	3	4
Minimum number of moves, M	1			

b Work out the formula which shows the relationship between M and n. Use your formula to extend the table of values for $n = 5$, 6 and 7.

c Plot the graph of M against n.

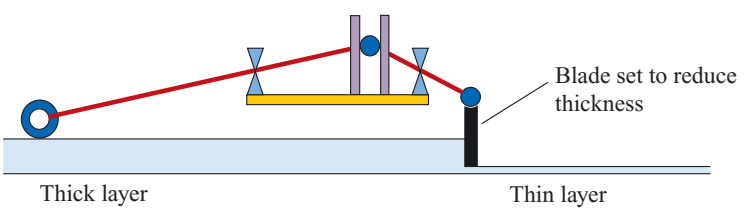
d Investigate, for both $n = 3$ and 4, to find whether there is a pattern for the number of times each particular disc is moved.

2 To control an advanced electronic machine, 2187 different switch positions are required. There are two kinds of switches available:

- Switch 1: These can be set in 9 different positions.
- Switch 2: These can be set to 3 different positions.

If n of switch type 1 and $n + 1$ of switch type 2 are used, calculate the value of n to give the required number of switch positions.

3 Research is being carried out to investigate the durability of paints of different thicknesses. The automatic machine shown in the diagram is proposed for producing a coat of paint of a particular thickness.



The paint is spread over a plate and a blade sweeps over the plate reducing the thickness of the paint. The process involves the blade moving at three different speeds.

a Operating at the initial setting the blade reduces the paint thickness to one-eighth of the original thickness. This happens n times. What fraction of the paint thickness remains? Express this as a power of $\frac{1}{2}$.

b The blade is then reset so that it removes three-quarters of the remaining paint. This happens $(n - 1)$ times. At the end of this second stage express the remaining thickness as a power of $\frac{1}{2}$.

c The third phase of the process involves the blade being reset to remove half of the remaining paint. This happens $(n - 3)$ times. At what value of n would the machine have to be set to reduce a film of paint 8192 units thick to 1 unit thick?

4 A hermit has little opportunity to replenish supplies of tea and so, to eke out supplies for as long as possible, he dries out the tea leaves after use and then stores the dried tea in an airtight box. He estimates that after each re-use of the leaves the amount of tannin in the used tea will be half the previous amount. He also estimates that the amount of caffeine in the used tea will be one-quarter of the previous amount.

The information on the label of the tea packet states that the tea contains 729 mg of caffeine and 128 mg of tannin.

a Write down expressions for the amount of caffeine when the tea leaves are re-used for the first, second, third and nth times.

b Do the same for the amount of tannin remaining.

c Find the number of times he can re-use the tea leaves if a 'tea' containing more than three times as much tannin as caffeine is undrinkable.

5 A new type of red synthetic carpet was produced in two batches. The first batch had a brightness of 15 units and the second batch 20 units. After a period of time it was discovered that the first batch was losing its brightness at the rate of 5% per year while the second lost its brightness at the rate of 6% per year.

a Write down expressions for the brightness of each batch after n years.

b A person bought some carpet from the first batch when it was a year old and some new carpet from the second batch. How long would it be before the brightness of the two carpets was the same?

6 The value of shares in Company X increased linearly over a two-year period according to the model $x = 0.8 + 0.17t$, where t is the number of months from the beginning of January 1997 and $\$x$ is the value of the shares at time t.

The value of shares in Company Y increased over the same period of time according to the model $y = 10^{0.03t}$, where $\$y$ is the value of these shares at time t months.

The value of shares in Company Z increased over the same period according to the model $z = 1.7 \log_{10}(5(x + 1))$, where $\$z$ is the value of the shares at time t months.

Use a graphic calculator to sketch the graphs of each of the functions on the one screen.

a Find the values of the shares in each of the three companies at the end of June 1997.

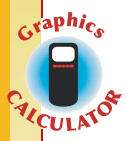
b Find the values of the shares in the three companies at the end of September 1998.

Review

c During which months were shares in Company X more valuable than shares in Company Y?

d For how long and during which months were the shares in Company X the most valuable?

7 In 2000 in a game park in Africa it was estimated that there were approximately 700 wildebeest and that their population was increasing at 3% per year. At the same time, in the park there were approximately 1850 zebras and their population was decreasing at the rate of 4% per year. Use a graphics calculator to plot the graphs of both functions.



a After how many years was the number of wildebeest greater than the number of zebras?

b It is also estimated that there were 1000 antelope and their numbers were increasing by 50 per year. After how many years was the number of antelope greater than the number of zebras?

8 Students conducting a science experiment on cooling rates measure the temperature of a beaker of liquid over a period of time. The following measurements were taken.

Time (min)	3	6	9	12	15	18	21
Temperature (°C)	71.5	59	49	45.5	34	28	23.5

a Find an exponential model to fit the data collected.

b Use this model to estimate:

i the initial temperature of the liquid

ii the temperature of the liquid after 25 minutes.

It is suspected that one of the temperature readings was incorrect.

c Re-calculate the model to fit the data, omitting the incorrect reading.

d Use the new model to estimate:

i the initial temperature of the liquid **ii** the temperature of the liquid at $t = 12$.

e If the room temperature is 15°C, find the approximate time at which the cooling of the liquid ceased.

9 The curve with equation $y = ab^x$ passes through the points (1, 1) and (2, 5)

a Find the values of a and b.

b Let $b^x = 10^z$.

i Take logarithms of both sides (base 10) to find z as an expression in x.

ii Find the value of k and a such that $y = a10^{kx}$ passes through the points (1, 1) and (2, 5).

10 **a** Find an exponential model of the form $y = a.b^x$ to fit the following data.

x	0	2	4	5	10
y	2	5	13	20	200

b Express the model you have found in **a** in the form $y = a10^{kx}$.

c Hence find an expression for x in terms of y.

CHAPTER

13

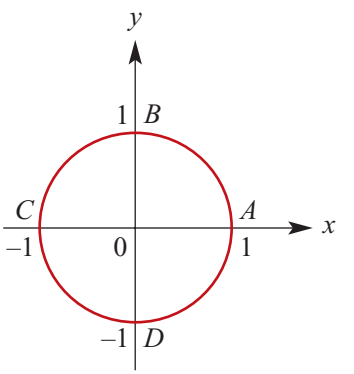
Circular Functions

Objectives

- To use radians and degrees for the measurement of angle.
- To convert radians to degrees and vice versa.
- To define the circular functions sine, cosine and tangent.
- To explore the symmetry properties of circular functions.
- To find standard exact values of circular functions.
- To understand and sketch the graphs of circular functions.

13.1 Measuring angles in degrees and radians

The diagram shows a unit circle, i.e. a circle of radius 1 unit.



The circumference of the unit circle $= 2\pi \times 1$

$= 2\pi$ units

$\therefore$ the distance in an anticlockwise direction around the circle from

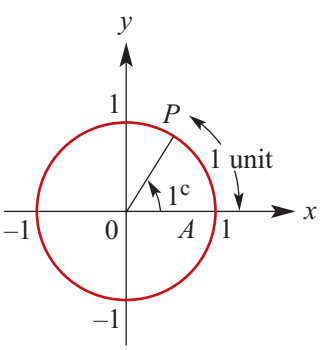
A to $B = \frac{\pi}{2}$ units

A to $C = \pi$ units

A to $D = \frac{3\pi}{2}$ units

Definition of a radian

In moving around the circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



One radian (written 1^c) is the angle **subtended** at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the circumference of the unit circle are defined as **positive**. Those formed by moving in a **clockwise** direction are said to be **negative**.

Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$\therefore \quad 2\pi^c = 360^\circ$

$\therefore \quad \pi^c = 180^\circ$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Example 1

Convert 30° to radians.

Solution

$$1^\circ = \frac{\pi^c}{180}$$

$$\therefore \quad 30^\circ = \frac{30 \times \pi}{180} = \frac{\pi^c}{6}$$

Example 2

Convert $\frac{\pi^c}{4}$ to degrees.

Solution

$$1^c = \frac{180^\circ}{\pi}$$

$$\therefore \quad \frac{\pi^c}{4} = \frac{\pi \times 180}{4 \times \pi} = 45^\circ$$

Note: Often the symbol for radian, ^{c}, is omitted. For example, angle 45° is written as $\frac{\pi}{4}$ rather than $\frac{\pi^c}{4}$.

Using a graphics calculator

When using a graphics calculator to change 32° to radians, have your calculator in **Radian** mode and enter 32°. The degree symbol ($^\circ$) is obtained from the **ANGLE** menu.

To change 0.5 radians to degrees, change your calculator to **Degree** mode and enter 0.5^r. The calculator symbol for radians (^{r}) is obtained from the **ANGLE** menu. In order to express this in degrees, minutes and seconds, select **DMS** from the **ANGLE** menu.

Exercise 13A

Example 1

1 Express the following angles in radian measure in terms of π:

a $60°$ **b** $144°$ **c** $240°$
d $330°$ **e** $420°$ **f** $480°$

Example 2

2 Express, in degrees, the angles with the following radian measures:

a $\frac{2\pi}{3}$ **b** $\frac{5\pi}{6}$ **c** $\frac{7\pi}{6}$ **d** 0.9π
e $\frac{5\pi}{9}$ **f** $\frac{9\pi}{5}$ **g** $\frac{11\pi}{9}$ **h** 1.8π

3 Use a calculator to convert the following angles from radians to degrees:

a 0.6 **b** 1.89 **c** 2.9 **d** 4.31
e 3.72 **f** 5.18 **g** 4.73 **h** 6.00

4 Use a calculator to express the following in radian measure:

a $38°$ **b** $73°$ **c** $107°$ **d** $161°$
e $84.1°$ **f** $228°$ **g** $136.4°$ **h** $329°$

5 Express, in degrees, the angle with the following radian measure:

a $-\frac{\pi}{3}$ **b** -4π **c** -3π **d** $-\pi$
e $\frac{5\pi}{3}$ **f** $\frac{-11\pi}{6}$ **g** $\frac{23\pi}{6}$ **h** $\frac{-23\pi}{6}$

6 Express each of the following in radian measure, in terms of π:

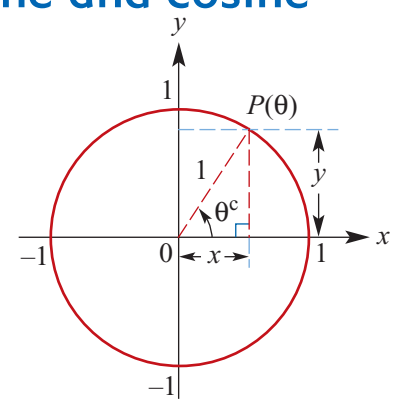
a $-360°$ **b** $-540°$ **c** $-240°$
d $-720°$ **e** $-330°$ **f** $-210°$

13.2 Defining circular functions: sine and cosine

Consider the unit circle.

The position of point P on the circle can be described by relating the cartesian coordinates x and y and the angle θ. The point P on the circumference corresponding to an angle θ is written $P(\theta)$.

Many different angles will give the same point P on the circle, so the relation linking an angle to the coordinates is a many-to-one function. There are, in fact, two functions involved and they are called sine and cosine, and they are defined as follows:

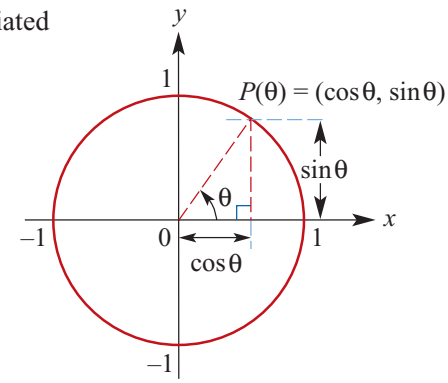


The x-coordinate of P, $x = \text{cosine } \theta$, $\theta \in R$
The y-coordinate of P, $y = \text{sine } \theta$, $\theta \in R$

Note: These functions are usually written in an abbreviated form as follows:

$$x = \cos\theta$$
$$y = \sin\theta$$

Note: $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$, as adding 2π results in a return to the same point on the unit circle.



Example 3

Evaluate $\sin\pi$ and $\cos\pi$.

Solution

In moving through an angle of π, the position is $P(\pi)$, which is $(-1, 0)$

$$\therefore \quad \cos\pi = -1$$
$$\sin\pi = 0$$

Example 4

Evaluate $\sin\left(-\frac{3\pi}{2}\right)$ and $\cos\left(-\frac{\pi}{2}\right)$.

Solution

$$\sin\left(-\frac{3\pi}{2}\right) = 1$$
$$\cos\left(-\frac{\pi}{2}\right) = 0$$

Example 5

Evaluate $\sin\left(\frac{5\pi}{2}\right)$ and $\sin\left(\frac{7\pi}{2}\right)$.

Solution

$$\sin\left(\frac{5\pi}{2}\right) = \sin\left(2\frac{1}{2}\pi\right) = \sin\left(2\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\sin\left(\frac{7\pi}{2}\right) = \sin\left(3\frac{1}{2}\pi\right) = \sin\left(2\pi + \frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$$

Example 6

Evaluate $\sin\left(\frac{9\pi}{2}\right)$ and $\cos(27\pi)$.

Solution

$$\sin\left(\frac{9\pi}{2}\right) = \sin\left(4\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\cos(27\pi) = \cos(26\pi + \pi) = \cos\pi = -1$$

Exercise 13B

Examples 5, 6

1 For each of the following angles, t, determine the values of $\sin t$ and $\cos t$:

a $t = 0$ **b** $t = \frac{3\pi}{2}$ **c** $t = -\frac{3\pi}{2}$ **d** $t = \frac{5\pi}{2}$

e $t = -3\pi$ **f** $t = \frac{9\pi}{2}$ **g** $t = \frac{7\pi}{2}$ **h** $t = 4\pi$

2 Evaluate using your calculator (Check that your calculator is in Rad mode):

a $\sin 1.9$ **b** $\sin 2.3$ **c** $\sin 4.1$ **d** $\cos 0.3$

e $\cos 2.1$ **f** $\cos(-1.6)$ **g** $\sin(-2.1)$ **h** $\sin(-3.8)$

Example 6

3 For each of the following angles, θ, determine the values of $\sin\theta$ and $\cos\theta$:

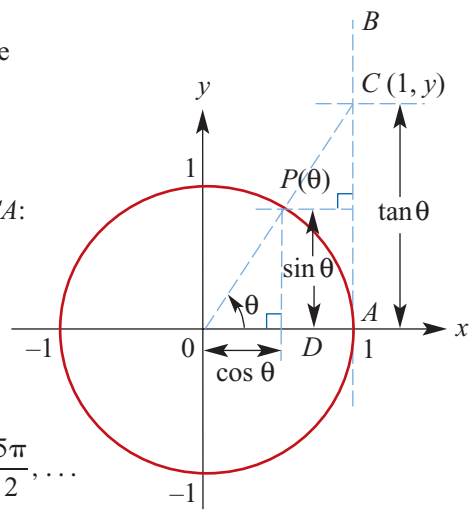
a $\theta = 27\pi$ **b** $\theta = -\frac{5\pi}{2}$ **c** $\theta = \frac{27\pi}{2}$ **d** $\theta = -\frac{9\pi}{2}$

e $\theta = \frac{11\pi}{2}$ **f** $\theta = 57\pi$ **g** $\theta = 211\pi$ **h** $\theta = -53\pi$

13.3 Another circular function: tangent

Again consider the unit circle.

If a tangent to the unit circle at A is drawn then the y-coordinate of C, the point of intersection of the extension of OP and the tangent, is called **tangent** θ (abbreviated to $\tan\theta$).



By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Now when $\cos\theta = 0$, $\tan\theta$ is undefined.

Hence $\tan\theta$ is undefined when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\therefore$ Domain of $\tan = R \setminus \{\theta : \cos\theta = 0\}$

Example 7

Evaluate, using a calculator:

a $\tan 1.3$ **b** $\tan 1.9$ **c** $\tan(-2.8)$ **d** $\tan 59°$ **e** $\tan 138°$

Solution

a $\tan 1.3 = 3.6$ (Don't forget calculator must be in RAD mode)
b $\tan 1.9 = -2.93$ (cos 1.9 is negative)
c $\tan(-2.8) = 0.36$ (cos −2.8 and sin −2.8 both negative ∴ tan is positive)
d $\tan 59^\circ = 1.66$ (Calculate in DEG mode)
e $\tan 138^\circ = -0.9$

Exercise 13C

1 Evaluate:

a $\tan \pi$ **b** $\tan(-\pi)$ **c** $\tan\left(\frac{7\pi}{2}\right)$
d $\tan(-2\pi)$ **e** $\tan\left(\frac{5\pi}{2}\right)$ **f** $\tan\left(-\frac{\pi}{2}\right)$

Example 7

2 Use a calculator to find correct to 2 decimal places:

a $\tan 1.6$ **b** $\tan(-1.2)$ **c** $\tan 136^\circ$ **d** $\tan(-54^\circ)$
e $\tan 3.9$ **f** $\tan(-2.5)$ **g** $\tan 239^\circ$

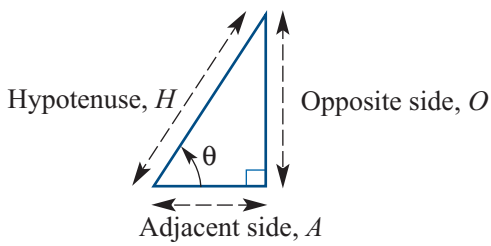
3 For each of the following values of θ find $\tan\theta$:

a $\theta = 180^\circ$ **b** $\theta = 360^\circ$ **c** $\theta = 0$
d $\theta = -180^\circ$ **e** $\theta = -540^\circ$ **f** $\theta = 720^\circ$

13.4 Reviewing trigonometric ratios

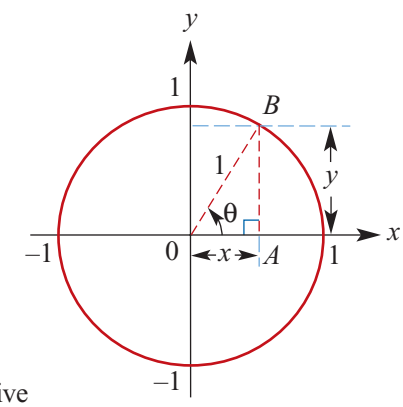
For right-angled triangles:

$$\sin\theta = \frac{O}{H}$$
$$\cos\theta = \frac{A}{H}$$
$$\tan\theta = \frac{O}{A}$$



Applying these trigonometric ratios to the right-angled triangle, OAB, in the unit circle:

$$\sin\theta = \frac{O}{H} = \frac{y}{1} = y$$
$$\cos\theta = \frac{A}{H} = \frac{x}{1} = x$$
$$\tan\theta = \frac{O}{A} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

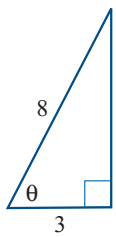


For $0 < \theta < \frac{\pi}{2}$, the functions sin, cos and tan are defined by the trigonometric ratios and are the same as the respective circular functions introduced earlier.

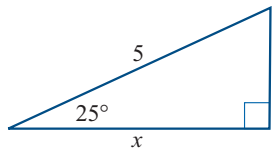
Exercise 13D

1 Find the value of the pronumeral for each of the following:

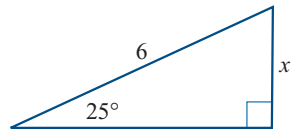
a



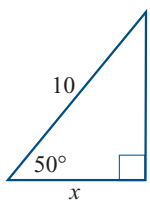
b



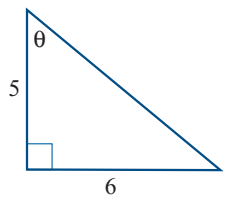
c



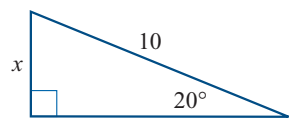
d



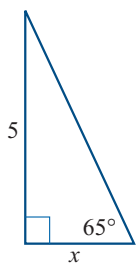
e



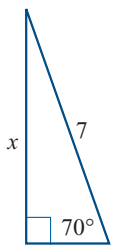
f



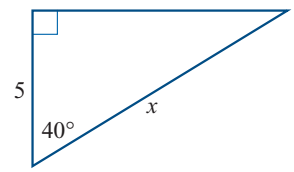
g



h



i

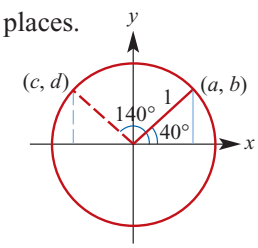


2 **a** Use your calculator to find a and b correct to 4 decimal places.

b Hence find the values of c and d.

c **i** Use your calculator to find cos 140° and sin 140°.

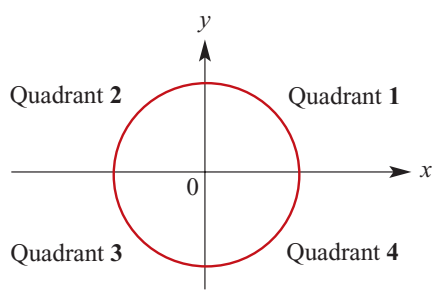
ii Write cos 140° in terms of cos 40°.



13.5 Symmetry properties of circular functions

The coordinate axes divide the unit circle into four quadrants. The quadrants can be numbered, anticlockwise from the positive direction of the x-axis, as shown.

Relationships, based on symmetry, between circular functions for angles in different quadrants can be determined.



Quadrant 2

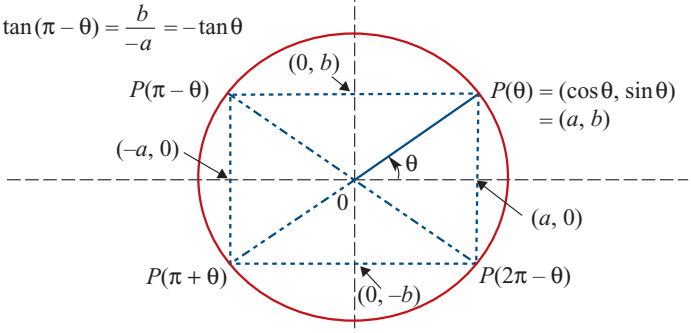
By symmetry:

$\sin(\pi - \theta) = b = \sin\theta$

$\cos(\pi - \theta) = -a = -\cos\theta$

$\tan(\pi - \theta) = \dfrac{b}{-a} = -\tan\theta$

Quadrant 1



Quadrant 3

$\sin(\pi + \theta) = -b = -\sin\theta$

$\cos(\pi + \theta) = -a = -\cos\theta$

$\tan(\pi + \theta) = \dfrac{-b}{-a} = \tan\theta$

Quadrant 4

$\sin(2\pi - \theta) = -b = -\sin\theta$

$\cos(2\pi - \theta) = a = \cos\theta$

$\tan(2\pi - \theta) = -\dfrac{b}{a} = -\tan\theta$

Note: These relationships are true for all values of θ.

Signs of circular functions

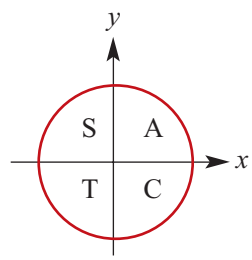
These symmetry properties can be summarised for the signs of sin, cos and tan for the four quadrants as follows:

1st quadrant: All are positive (A)

2nd quadrant: Sin is positive (S)

3rd quadrant: Tan is positive (T)

4th quadrant: Cos is positive (C).



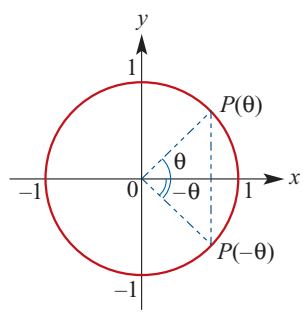
Negative of angles

By symmetry:

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Example 8

If $\sin x = 0.6$, find the value of:

a $\sin(\pi - x)$ **b** $\sin(\pi + x)$ **c** $\sin(2\pi - x)$ **d** $\sin(-x)$

Solution

a $\sin(\pi - x)$
$= \sin x$
$= 0.6$

b $\sin(\pi + x)$
$= -\sin x$
$= -0.6$

c $\sin(2\pi - x)$
$= -\sin x$
$= -0.6$

d $\sin(-x)$
$= -\sin x$
$= -0.6$

Example 9

If $\cos x^\circ = 0.8$, find the value of:

a $\cos(180 - x)^\circ$ **b** $\cos(180 + x)^\circ$ **c** $\cos(360 - x)^\circ$ **d** $\cos(-x)^\circ$

Solution

a $\cos(180 - x)^\circ$
$= -\cos x^\circ$
$= -0.8$

b $\cos(180 + x)^\circ$
$= -\cos x^\circ$
$= -0.8$

c $\cos(360 - x)^\circ$
$= \cos(x)^\circ$
$= 0.8$

d $\cos(-x)^\circ$
$= \cos x^\circ$
$= 0.8$

Exercise 13E

Example 8

1 If $\sin\theta = 0.42$, $\cos x = 0.7$ and $\tan\alpha = 0.38$, write down the values of:

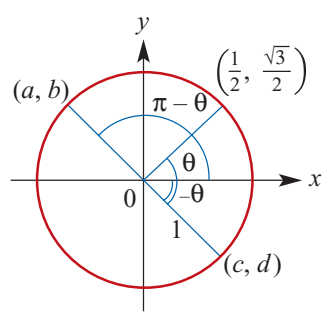
a $\sin(\pi + \theta)$ **b** $\cos(\pi - x)$ **c** $\sin(2\pi - \theta)$ **d** $\tan(\pi - \alpha)$
e $\sin(\pi - \theta)$ **f** $\tan(2\pi - \alpha)$ **g** $\cos(\pi + x)$ **h** $\cos(2\pi - x)$.

2 **a** If $\sin x^\circ = \sin 60^\circ$ and $90^\circ < x^\circ < 180^\circ$, find the value of x.
b If $\sin x^\circ = -\sin 60^\circ$ and $180^\circ < x^\circ < 270^\circ$, find the value of x.
c If $\sin x^\circ = -\sin 60^\circ$ and $-90^\circ < x^\circ < 0^\circ$, find the value of x.
d If $\cos x^\circ = -\cos 60^\circ$ and $90^\circ < x^\circ < 180^\circ$, find the value of x.
e If $\cos x^\circ = -\cos 60^\circ$ and $180^\circ < x^\circ < 270^\circ$, find the value of x.
f If $\cos x^\circ = \cos 60^\circ$ and $270^\circ < x^\circ < 360^\circ$, find the value of x.

3 **a** If $\cos x = -\cos\left(\frac{\pi}{6}\right)$ and $\frac{\pi}{2} < x < \pi$, find the value of x.
b If $\cos x = -\cos\left(\frac{\pi}{6}\right)$ and $\pi < x < \frac{3\pi}{2}$, find the value of x.
c If $\cos x = \cos\left(\frac{\pi}{6}\right)$ and $\frac{3\pi}{2} < x < 2\pi$, find the value of x.

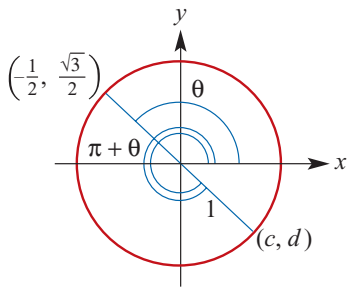
4 Write down the values of:

a $a = \cos(\pi - \theta)$
b $b = \sin(\pi - \theta)$
c $c = \cos(-\theta)$
d $d = \sin(-\theta)$
e $\tan(\pi - \theta)$
f $\tan(-\theta)$



5 Write down the values of:

a $d = \sin(\pi + \theta)$
b $c = \cos(\pi + \theta)$
c $\tan(\pi + \theta)$
d $\sin(2\pi - \theta)$
e $\cos(2\pi - \theta)$



Example 9

6 If $\sin x^\circ = 0.7$, $\cos\theta^\circ = 0.6$ and $\tan\alpha^\circ = 0.4$, write down the values of:

a $\sin(180 + x)^\circ$ **b** $\cos(180 + \theta)^\circ$ **c** $\tan(360 - \alpha)^\circ$ **d** $\cos(180 - \theta)^\circ$
e $\sin(360 - x)^\circ$ **f** $\sin(-x)^\circ$ **g** $\tan(360 + \alpha)^\circ$ **h** $\cos(-\theta)^\circ$

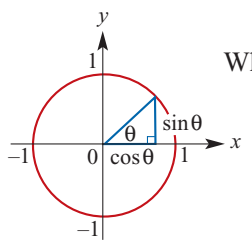
13.6 Exact values of circular functions

A calculator can be used to find the values of the circular functions for different values of θ. For many values of θ the calculator gives an approximation. We consider some values of θ such that sin, cos and tan can be calculated exactly.

Exact values for 0 (0°) and $\frac{\pi}{2}$ (90°)

From the unit circle:

When $\theta = 0$, $\sin\theta = 0$, $\cos\theta = 1$, $\tan\theta = 0$



When $\theta = \frac{\pi}{2}$, $\sin\left(\frac{\pi}{2}\right) = 1$, $\cos\left(\frac{\pi}{2}\right) = 0$, $\tan\left(\frac{\pi}{2}\right)$ is undefined.

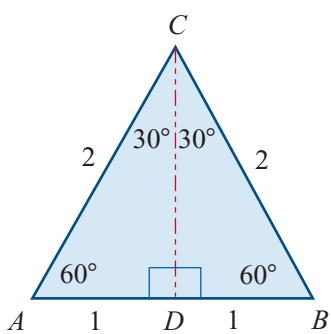
Exact values for $\frac{\pi}{6}$ (30°) and $\frac{\pi}{3}$ (60°)

Consider an equilateral triangle ABC of side length 2 units.

In ΔACD, by Pythagoras' Theorem:

$$DC = \sqrt{AC^2 - AD^2}$$
$$= \sqrt{3}$$

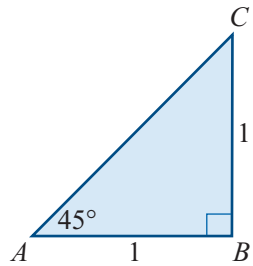
$$\sin 30^\circ = \frac{AD}{AC} = \frac{1}{2}$$
$$\cos 30^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$
$$\tan 30^\circ = \frac{AD}{CD} = \frac{1}{\sqrt{3}}$$



$$\sin 60^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$
$$\cos 60^\circ = \frac{AD}{AC} = \frac{1}{2}$$
$$\tan 60^\circ = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Exact values for $\frac{\pi}{4}$ (45°)

$$AC = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$
$$\cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = \frac{BC}{AB} = 1$$



As an aid to memory, the exact values for circular functions can be tabulated.

Summary

θ (θ°)	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined

Example 10

Evaluate:

a $\cos 150^\circ$ **b** $\sin 690^\circ$

Solution

a $\cos 150^\circ = \cos(180 - 30)^\circ$
$= -\cos 30^\circ$
$= -\frac{\sqrt{3}}{2}$

b $\sin 690^\circ = \sin(2 \times 360 - 30)^\circ$
$= \sin(-30)^\circ$
$= -\frac{1}{2}$

Example 11

Evaluate:

a $\cos\left(\frac{5\pi}{4}\right)$ **b** $\sin\left(\frac{11\pi}{6}\right)$

Solution

a $\cos\left(\frac{5\pi}{4}\right) = \cos\left(\pi + \frac{\pi}{4}\right)$

$= -\cos\left(\frac{\pi}{4}\right)$ (by symmetry)

$= -\frac{1}{\sqrt{2}}$

b $\sin\left(\frac{11\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$

$= -\sin\left(\frac{\pi}{6}\right)$ (by symmetry)

$= -\frac{1}{2}$

Exercise 13F

Example 10

1 Without using a calculator, evaluate the sin, cos and tan of each of the following:

a 120° **b** 135° **c** 210° **d** 240° **e** 315°

f 390° **g** 420° **h** −135° **i** −300° **j** −60°

Example 11

2 Write down the exact values of:

a $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\tan\left(\frac{5\pi}{6}\right)$

d $\sin\left(\frac{7\pi}{6}\right)$ **e** $\cos\left(\frac{5\pi}{4}\right)$ **f** $\tan\left(\frac{4\pi}{3}\right)$

g $\sin\left(\frac{5\pi}{3}\right)$ **h** $\cos\left(\frac{7\pi}{4}\right)$ **i** $\tan\left(\frac{11\pi}{6}\right)$

3 Write down the exact values of:

a $\sin\left(-\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{11\pi}{4}\right)$ **c** $\tan\left(\frac{13\pi}{6}\right)$ **d** $\tan\left(\frac{15\pi}{6}\right)$

e $\cos\left(\frac{14\pi}{4}\right)$ **f** $\cos\left(-\frac{3\pi}{4}\right)$ **g** $\sin\left(\frac{11\pi}{4}\right)$ **h** $\cos\left(-\frac{21\pi}{3}\right)$

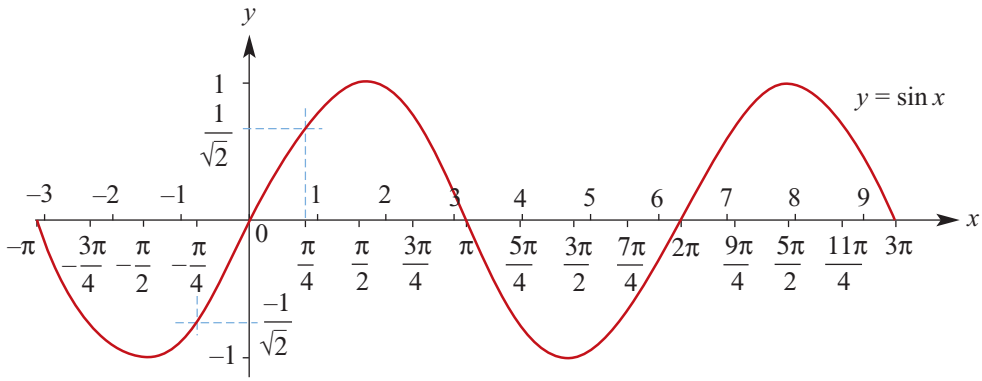
13.7 Graphs of sine and cosine

Graphs of sine functions

A table of values for $y = \sin x$ is given below.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	$-\frac{1}{\sqrt{2}}$	−1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	−1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0

A graphics calculator can be used to plot the graph of $y = \sin x$, $(-\pi \leq x \leq 3\pi)$. Note that **Radian** mode must be selected.



Observations from the graph of $y = \sin x$

- The graph repeats itself after an interval of 2π units. A function which repeats itself regularly is called a **periodic** function and the interval between the repetitions is called the **period** of the function (also called the wavelength).
  Thus $\sin x$ has a period of 2π units.
- The maximum and minimum values of $\sin x$ are 1 and -1 respectively.
  The distance between the mean position and the maximum position is called the **amplitude**. The graph of $y = \sin x$ has an amplitude of 1.

Graphs of cosine functions

A table of exact values for $\cos x$, $-\pi \leq x \leq 3\pi$, is as shown:

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1

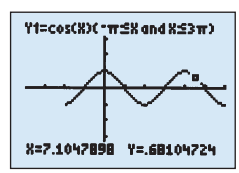
Using a graphics calculator

A graph of $y = \cos x$ for $-\pi \leq x \leq 3\pi$ can be plotted on a graphics calculator.

Y1 $= \cos(x)$ $(-\pi \leq x$ and $x \leq 3\pi)$ is entered in the **Y=** window. Note that the inequalities are obtained from the **TEST** menu, and from **CATALOG**.

Dot mode has been selected and **TRACE** activated.

The syntax **Y1** $= \cos(x)/(-\pi \leq x$ and $x \leq 3\pi)$ also results in this graph with **Connected** mode being used.



Observations from the graph of $y = \cos x$

- Period $= 2\pi$
- Amplitude $= 1$
- The graph of $y = \cos x$ is the graph of $y = \sin x$ translated $\frac{\pi}{2}$ units to the left and parallel to the x-axis.

Sketch graphs of $y = a\sin(nt)$, $y = a\cos(nt)$

Example 12

On separate axes draw graphs of the functions:

a $y = 3\sin(2t), 0 \le t \le \pi$ **b** $y = 2\cos(3t), 0 \le t \le \frac{2\pi}{3}$

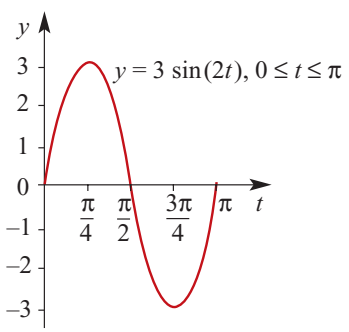
Solution

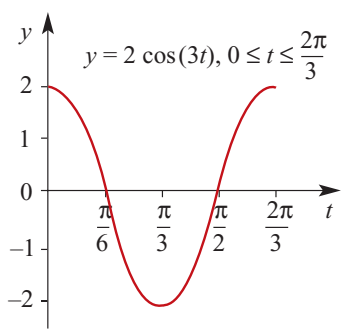
a

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = 3\sin(2t)$	0	3	0	-3	0

b

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$y = 2\cos(3t)$	2	0	-2	0	2





Observations

Function	*Amplitude*	*Period*
$y = 3\sin(2t)$	3	π
$y = 2\cos(3t)$	2	$\frac{2\pi}{3}$

Comparing these results with those for $y = \sin t$ and $y = \cos t$, the following general rules can be stated for a and n positive:

Function	*Amplitude*	*Period*
$y = a \sin(nt)$	a	$\frac{2\pi}{n}$
$y = a \cos(nt)$	a	$\frac{2\pi}{n}$

From the above it can be seen the transformation which takes the graph of $y = \sin t$ to the graph of $y = 3 \sin(2t)$ has the following result on some important points of the graph of $y = \sin t$:

$$(0, 0) \to (0, 0); \left(\frac{\pi}{2}, 1\right) \to \left(\frac{\pi}{4}, 3\right); (\pi, 0) \to \left(\frac{\pi}{2}, 0\right);$$
$$\left(\frac{3\pi}{2}, 1\right) \to \left(\frac{3\pi}{4}, 3\right); (2\pi, 0) \to (\pi, 0)$$

Also, it can be seen that the transformation which takes the graph of $y = \cos t$ to $y = 2 \cos(3t)$ has the following result on some important points of the graph of $y = \cos t$:

$$(0, 1) \to (0, 2); \left(\frac{\pi}{2}, 0\right) \to \left(\frac{\pi}{6}, 0\right); (\pi, -1) \to \left(\frac{\pi}{3}, -2\right);$$
$$\left(\frac{3\pi}{2}, 0\right) \to \left(\frac{\pi}{2}, 0\right); (2\pi, 1) \to \left(\frac{2\pi}{3}, 2\right)$$

Note: The graph of $y = 3 \sin(2t)$ can be obtained from the graph of $y = \sin t$ by applying two dilations. If $f(t) = \sin t$, the graph of $y = f(t)$ is transformed to the graph of $y = 3f(2t)$. From this it can be recognised that the sequence of transformations is:

- dilation of factor $\frac{1}{2}$ from the y-axis
- dilation of factor 3 from the t-axis

The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{2}, 3y\right)$

In general, for a and n positive numbers, the following are important properties of the functions $f(t) = a \sin(nt)$ and $g(t) = a \cos(nt)$:

- The maximal domain of each of the functions is R.
- The amplitude of each of the functions is a.
- The period of each of the functions is $\frac{2\pi}{n}$.

- The graph of $y = a \sin(nt)$ ($y = a \cos(nt)$) is obtained from the graph of $y = \sin t$ ($y = \cos t$) by a dilation of factor a from the t-axis and a factor of $\frac{1}{n}$ from the y-axis. The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{n}, ay\right)$.
- The range of each function is $[-a, a]$.

Example 13

For each of the following functions with domain R state the amplitude and period:

a $f(t) = 2\sin(3t)$ **b** $f(t) = -\frac{1}{2}\sin\left(\frac{t}{2}\right)$ **c** $f(t) = 4\cos(3\pi t)$

Solution

a Amplitude is 2

Period $= \frac{2\pi}{3}$

b Amplitude is $\frac{1}{2}$

Period $= 2\pi \div \frac{1}{2} = 4\pi$

c Amplitude is 4

Period $= \frac{2\pi}{3\pi} = \frac{2}{3}$

Using a graphics calculator

Example 14

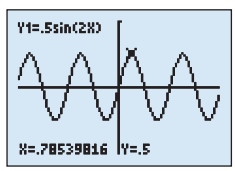
Plot $y = \frac{1}{2}\sin(2x)$ on a graphics calculator.

Solution

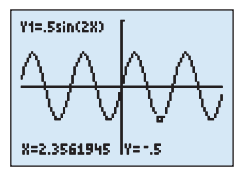
Radian measure is assumed. From the **MODE** menu select **Radian**.

Enter **Y1** $= \frac{1}{2}\sin(2X)$ in the **Y=** window.

Select **7:ZTrig** from the **ZOOM** menu. When **ZTrig** is used x, the values $\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}$, etc. may be identified through **TRACE**.



$\frac{\pi}{4} \approx 0.785\,398\,16$



$\frac{3\pi}{4} \approx 2.356\,194$

Example 15

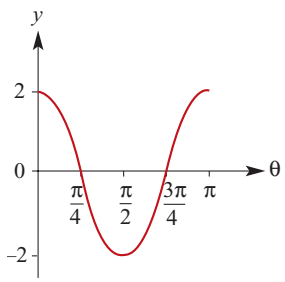
Sketch the graphs of:

a $y = 2\cos(2\theta)$ **b** $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

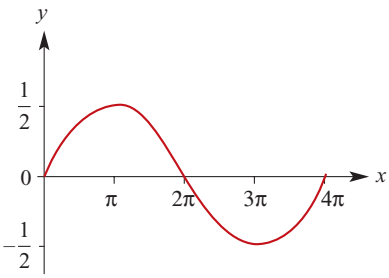
Show one complete cycle.

Solution

a The graph of $y = 2\cos(2\theta)$ is obtained from the graph of $y = \cos\theta$ by a dilation of factor 2 from the θ-axis and by a dilation of factor $\frac{1}{2}$ from the y-axis.
The period $= \frac{2\pi}{2} = \pi$ and the amplitude is 2.



b The graph of $y = \frac{1}{2}\sin\frac{x}{2}$ is obtained from the graph of $y = \cos x$ by a dilation of factor $\frac{1}{2}$ from the x-axis and by a dilation of factor 2 from the y-axis. The period $= 2\pi \div \frac{1}{2} = 4\pi$ and the amplitude is $\frac{1}{2}$.



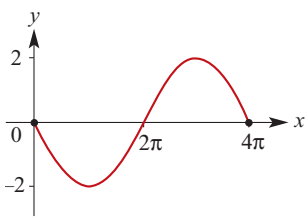
Example 16

Sketch the following graphs for $x \in [0, 4\pi]$:

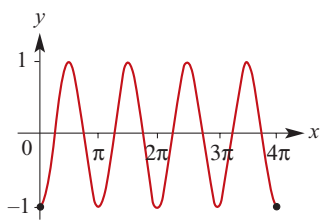
a $f(x) = -2\sin\left(\frac{x}{2}\right)$ **b** $y = -\cos(2x)$

Solution

a The graph of $f(x) = -2\sin\left(\frac{x}{2}\right)$ is obtained from the graph of $y = 2\sin\left(\frac{x}{2}\right)$ by a reflection in the x-axis.
The period is 4π and the amplitude is 2.



b The graph of $y = -\cos(2x)$ is obtained from the graph of $y = \cos(2x)$ by a reflection in the x-axis.
The period is π and the amplitude is 1.



In general, for a and n positive numbers, the following are important properties of the functions $f(t) = -a\sin(nt)$ and $g(t) = -a\cos(nt)$:

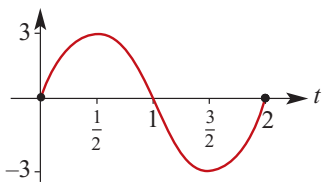
- The amplitude of each of the functions is a.
- The period of each of the functions is $\frac{2\pi}{n}$.
- The graph of $y = -a\sin(nt)$ ($y = -a\cos(nt)$) is obtained from the graph of $y = a\sin(nt)$ ($a\cos(nt)$) by a reflection in the t-axis.
- The range of each function is $[-a, a]$.

Remember that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Hence when reflected in the y-axis the graph of $y = \cos x$ transforms onto itself and the graph of $y = \sin x$ transforms onto the graph of $y = -\sin x$.

Example 17

Sketch the graph of $f: [0, 2] \to R, f(t) = 3\sin(\pi t)$

Solution



Exercise 13G

Example 13 **1** Write down **i** the period and **ii** the amplitude of each of the following:

a $2\sin\theta$ **b** $3\sin(2\theta)$ **c** $\frac{1}{2}\cos(3\theta)$

d $3\sin\left(\frac{1}{2}\theta\right)$ **e** $4\cos(3\theta)$ **f** $-\frac{1}{2}\sin(4\theta)$

g $-2\cos\left(\frac{1}{2}\theta\right)$ **h** $2\cos(\pi t)$ **i** $-3\sin\left(\frac{\pi t}{2}\right)$

Example 15 **2** Sketch the graph of each of the following, showing one complete cycle. State the amplitude and period.

a $y = 3\sin(2x)$ **b** $y = 2\cos(3\theta)$ **c** $y = 4\sin\left(\frac{\theta}{2}\right)$

d $y = \frac{1}{2}\cos(3x)$ **e** $y = 4\sin(3x)$ **f** $y = 5\cos(2x)$

g $y = -3\cos\left(\frac{\theta}{2}\right)$ **h** $y = 2\cos(4\theta)$ **i** $y = -2\sin\left(\frac{\theta}{3}\right)$

Example 16 **3** Sketch the graph of:

a $f(x) = \sin(2x)$ for $x \in [-2\pi, 2\pi]$ **b** $f(x) = 2\sin\left(\frac{x}{3}\right)$ for $x \in [-6\pi, 6\pi]$

c $f(x) = 2\cos(3x)$ for $x \in [0, 2\pi]$ **d** $f(x) = -2\sin(3x)$ for $x \in [0, 2\pi]$

4 Sketch the graph of $f: [0, 2\pi] \to R, f(x) = \frac{5}{2}\cos\left(\frac{2x}{3}\right)$. Hint: For endpoints find $f(0)$ and $f(2\pi)$.

5 For each of the following give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 3\sin x$ **b** $g(x) = \sin(5x)$ **c** $g(x) = \sin\left(\frac{x}{3}\right)$

d $g(x) = 2\sin(5x)$ **e** $g(x) = -\sin(5x)$ **f** $g(x) = \sin(-x)$

g $g(x) = 2\sin\left(\frac{x}{3}\right)$ **h** $g(x) = -4\sin\left(\frac{x}{2}\right)$ **i** $g(x) = 2\sin\left(\frac{-x}{3}\right)$

Example 17 **6** Sketch the graph of:

a $f: [0, 2] \to R,\ f(t) = 2\cos(\pi t)$ **b** $f: [0, 2] \to R,\ f(t) = 3\sin(2\pi t)$

7 **a** On the one set of axes sketch the graph of $f: [0, 2\pi] \to R, f(x) = \sin x$ and $g: [0, 2\pi] \to R, g(x) = \cos x$.

b By inspection from the graph state the values of x for which $\sin x = \cos x$.

13.8 Sketch graphs of $y = a\sin n(t \pm \varepsilon)$ and $y = a\cos n(t \pm \varepsilon)$

In this section translations of graphs of functions of the form $f(t) = a\sin(nt)$ and $g(t) = a\cos(nt)$ in the direction of the t-axis are considered. When a translation of $\frac{\pi}{4}$ units in the positive direction of the t-axis is applied to the graph of $y = f(t)$, the resulting image has equation $y = f\left(t - \frac{\pi}{4}\right)$. That is, the graph of $f(t) = a\sin(nt)$ is mapped to the graph with equation $y = a\sin n\left(t - \frac{\pi}{4}\right)$.

Example 18

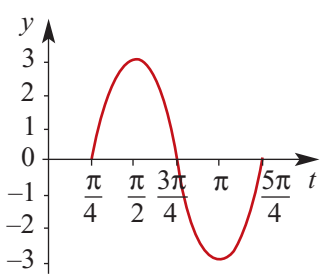
On separate axes draw the graphs of the following functions. Use a graphics calculator to help establish the shape. Set the window appropriately by noting the range and period.

a $y = 3\sin 2\left(t - \frac{\pi}{4}\right),\ \frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right),\ -\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

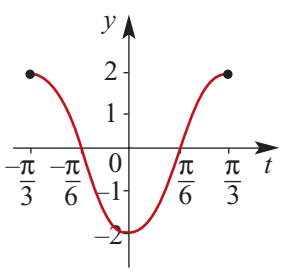
a $y = 3\sin 2\left(t - \frac{\pi}{4}\right),\ \frac{\pi}{4} \le t \le \frac{5\pi}{4}$

Note that the range is $[-3, 3]$ and the period is π.



b $y = 2\cos 3\left(t + \frac{\pi}{3}\right),\ -\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Note that the range is $[-2, 2]$ and the period is $\frac{2\pi}{3}$.



Observations

1 For $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$, amplitude $= 3$, period $= \pi$. The graph is the same shape as $y = 3\sin(2t)$ but is translated $\frac{\pi}{4}$ units in the positive direction of the t-axis.

2 For $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$, amplitude $= 2$, period $= \frac{2\pi}{3}$. The graph is the same shape as $y = 2\cos(3t)$ but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis.

In each case ε has the effect of translating the graph parallel to the t-axis (ε is called the phase).

Note: To determine the sequence of transformations needed, the techniques of Section 13.5 can also be used.

The graph of $y = \sin t$ is transformed to the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$.

Write the second equation as $\frac{y'}{3} = \sin 2\left(t' - \frac{\pi}{4}\right)$. From this it can be seen that $y = \frac{y'}{3}$ and $t = 2\left(t' - \frac{\pi}{4}\right)$.

Hence $y' = 3y$ and $t' = \frac{t}{2} + \frac{\pi}{4}$. Hence the sequence of transformations is:

- dilation of factor 3 from the t-axis
- dilation of factor $\frac{1}{2}$ from the y-axis and
- translation of $\frac{\pi}{4}$ units in the positive direction of the t-axis.

The observation that the graph of $y = f(t)$ is transformed to the graph of $y = 3f\left(2\left(t - \frac{\pi}{4}\right)\right)$, where $f(t) = \sin t$, also yields this information.

Exercise 13H

Example 18

1 Sketch the graph of each of the following, showing one complete cycle. State the period and amplitude, and the greatest and least values of y.

a $y = 3\sin\left(\theta - \frac{\pi}{2}\right)$ **b** $y = \sin 2(\theta + \pi)$ **c** $y = 2\sin 3\left(\theta + \frac{\pi}{4}\right)$

d $y = \sqrt{3}\sin 2\left(\theta - \frac{\pi}{2}\right)$ **e** $y = 3\sin(2x)$ **f** $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right)$

g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{3}\right)$ **h** $y = -3\sin(2x)$ **i** $y = -3\cos 2\left(\theta + \frac{\pi}{2}\right)$

2 For the function $f: [0, 2\pi] \to R, f(x) = \cos\left(x - \frac{\pi}{3}\right)$:

a find $f(0), f(2\pi)$ **b** sketch the graph of f

3 For the function $f: [0, 2\pi] \to R, f(x) = \sin 2\left(x - \frac{\pi}{3}\right)$:

a find $f(0), f(2\pi)$ **b** sketch the graph of f

4 For the function $f: [-\pi, \pi] \to R, f(x) = \sin 3\left(x + \frac{\pi}{4}\right)$:

a find $f(-\pi), f(\pi)$ **b** sketch the graph of f

5 Find the equation of the image of $y = \sin x$ for each of the following transformations:

a Dilation of factor 2 from the y-axis followed by dilation of factor 3 from the x-axis

b Dilation of factor $\frac{1}{2}$ from the y-axis followed by dilation of factor 3 from the x-axis

c Dilation of factor 3 from the y-axis followed by dilation of factor 2 from the x-axis

d Dilation of factor $\frac{1}{2}$ from the y-axis followed by a translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis

e Dilation of factor 2 from the y-axis followed by a translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis

13.9 Solution of trigonometric equations

Example 19

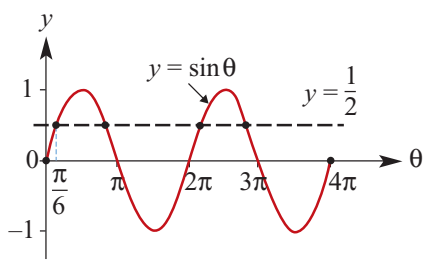
Find all solutions to the equation $\sin\theta = \frac{1}{2}$ for $\theta \in [0, 4\pi]$.

Solution

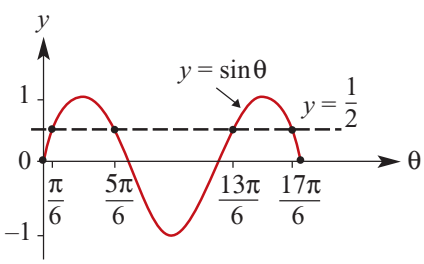
It is clear from the graph that there are four solutions in the interval $[0, 4\pi]$.

The solution for $x \in \left[0, \frac{\pi}{2}\right]$ is $x = \frac{\pi}{6}$.

This solution can be obtained from a knowledge of exact values or by using $\sin^{-1}$ on your calculator.



The second solution is obtained by symmetry. The function is positive in the second quadrant and $\sin(\pi - \theta) = \sin\theta$. Therefore $x = \frac{5\pi}{6}$ is the second solution.



It can be seen that further solutions can be achieved by adding 2π, as $\sin\theta = \sin(\theta + 2\pi)$.

Thus $\theta = \frac{13\pi}{6}$ and $\frac{17\pi}{6}$ are also solutions.

Example 20

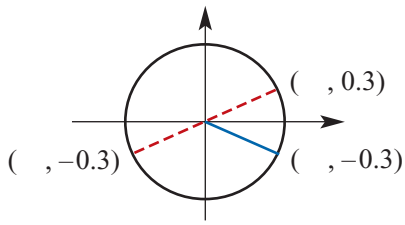
Find two values of x:

a $\sin x = -0.3$ in the range $0 \leq x \leq 2\pi$ **b** $\cos x^\circ = -0.7$ in the range $0 \leq x \leq 360$

Solution

a First we must solve the equation $\sin x = 0.3$.
Use your calculator to find the solution for $x \in \left[0, \frac{\pi}{2}\right]$; $x = 0.30469\ldots$
Now the value of sin is negative for $P(x)$ in the 3rd and 4th quadrants. From the symmetry relationships (or from the graph of $y = \sin x$):

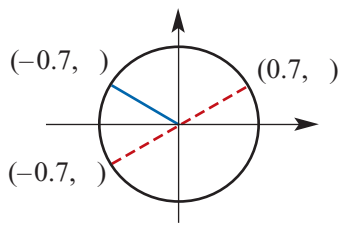
3rd quadrant: $x = \pi + 0.30469\ldots$
$= 3.446$ (correct to 3 decimal places)
4th quadrant: $x = 2\pi - 0.30469\ldots$
$= 5.978$ (correct to 3 decimal places)



$\therefore$ if $\sin x = -0.3$, $x = 1.875$, or $x = 5.978$

b First we solve the equation $\cos x^\circ = 0.7$.
Use your calculator to find the solution for $x \in [0^\circ, 90^\circ]$.
Now the value of cos is negative for $P(x)$ in the 2nd and 3rd quadrants.

2nd quadrant: $x = 180 - 45.57$
$= 134.43$
3rd quadrant: $x = 180 + 45.57$
$= 225.57$



$\therefore$ if $\cos x^\circ = -0.7$, $x = 134.43, 225.57$

Example 21

Find all the values of θ between 0 and 360 for which:

a $\cos\theta^\circ = \frac{\sqrt{3}}{2}$ **b** $\sin\theta^\circ = -\frac{1}{2}$ **c** $\cos\theta^\circ - \frac{1}{\sqrt{2}} = 0$

CAS APPENDIX

Solution

a $\cos\theta^\circ$ is positive, $\therefore P(\theta^\circ)$ lies in the 1st or 4th quadrants.

$$\cos\theta^\circ = \frac{\sqrt{3}}{2}$$
$$\theta = 30 \quad \text{or} \quad 360 - 30$$
$$\theta = 30 \quad \text{or} \quad 330$$

b $\sin\theta^\circ$ is negative, $\therefore P(\theta^\circ)$ is in the 3rd or 4th quadrants.

$$\sin\theta^\circ = -\frac{1}{2}$$
$$\theta = 180 + 30 \quad \text{or} \quad 360 - 30$$
$$\theta = 210 \quad \text{or} \quad 330$$

c $\cos\theta^\circ - \frac{1}{\sqrt{2}} = 0$

$\therefore \cos\theta^\circ = \frac{1}{\sqrt{2}}$

and since $\cos\theta^\circ$ is positive, $P(\theta^\circ)$ lies in the 1st or 4th quadrants.

$\therefore \quad \cos\theta^\circ = \frac{1}{\sqrt{2}}$

$\theta = 45 \quad \text{or} \quad \theta = 360 - 45$

$\theta = 45 \quad \text{or} \quad 315$

CAS APPENDIX

Example 22

Solve the equation $\sin(2\theta) = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$.

Solution

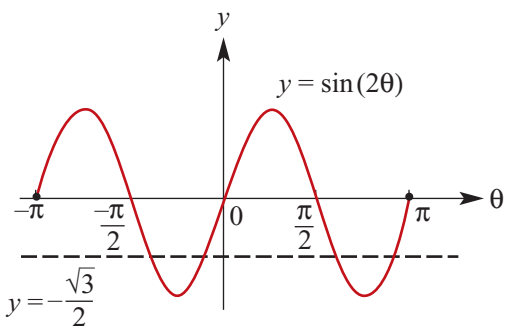
It is clear that there are four solutions.

To solve the equation, let $x = 2\theta$.

Note:

If $\theta \in [-\pi, \pi]$

then $2\theta = x \in [-2\pi, 2\pi]$



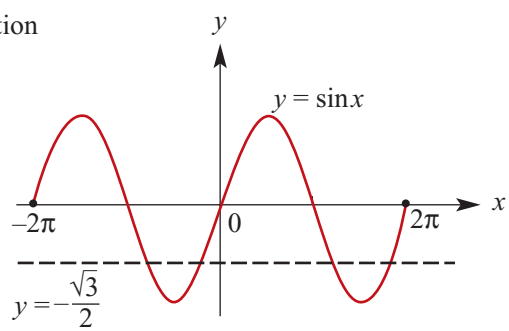
Consider the equation $\sin x = -\frac{\sqrt{3}}{2}$ for $x \in [-2\pi, 2\pi]$.

The first quadrant solution to the equation $\sin x = \frac{\sqrt{3}}{2}$ is $x = \frac{\pi}{3}$.

Symmetry gives the solutions to $\sin x = -\frac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$ as

$x = \pi + \frac{\pi}{3}$ and $x = 2\pi - \frac{\pi}{3}$

i.e. $x = \frac{4\pi}{3}$ or $x = \frac{5\pi}{3}$



The other two solutions are obtained by subtracting 2π: $\frac{4\pi}{3} - 2\pi$ and $\frac{5\pi}{3} - 2\pi$

$\therefore$ the required solutions for x are $-\frac{2\pi}{3}$ or $-\frac{\pi}{3}$ or $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$

$\therefore$ the required solutions for θ are $-\frac{\pi}{3}$ or $-\frac{\pi}{6}$ or $\frac{2\pi}{3}$ or $\frac{5\pi}{6}$

Exercise 13I

Example 19

1 Without using a calculator, find all the values of x between 0 and 2π for each of the following:

a $\sqrt{2}\sin(x) + 1 = 0$ **b** $\sqrt{2}\cos(x) - 1 = 0$

Example 20a

2 Find, correct to 2 decimal places, all the values of x between 0 and 2π for which:

a $\sin x = 0.8$ **b** $\cos x = -0.4$ **c** $\sin x = -0.35$

d $\sin x = 0.4$ **e** $\cos x = -0.7$ **f** $\cos x = -0.2$

Example 21

3 Without using a calculator, find all the values of θ between 0° and 360° for each of the following:

a $\cos\theta^\circ = -\frac{\sqrt{3}}{2}$ **b** $\sin\theta^\circ = \frac{1}{2}$ **c** $\cos\theta^\circ = -\frac{1}{2}$

d $2\cos(\theta^\circ) + 1 = 0$ **e** $2\sin\theta^\circ = \sqrt{3}$ **f** $\sqrt{2}\sin(\theta^\circ) - 1 = 0$

4 Find all the values of x between 0 and 4π for which:

a $\sin x = 0.6$ **b** $\sin x = -\frac{1}{\sqrt{2}}$ **c** $\sin x = \frac{\sqrt{3}}{2}$

5 Find all the values of x between $-\pi$ and π for which:

a $\cos x = -\frac{1}{\sqrt{2}}$ **b** $\sin x = \frac{\sqrt{3}}{2}$ **c** $\cos x = -\frac{1}{2}$

6 **a** Sketch the graph of $f: [-2\pi, 2\pi] \to R, f(x) = \cos x$.

b On the graph mark the points which have y-coordinate $\frac{1}{2}$ and give the associated x-values.

c On the graph mark the points which have y-coordinate $-\frac{1}{2}$ and give the associated x-values.

Example 22

7 Solve the following equation for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -\frac{1}{2}$ **b** $\cos(2\theta) = \frac{\sqrt{3}}{2}$ **c** $\sin(2\theta) = \frac{1}{2}$

d $\sin(3\theta) = -\frac{1}{\sqrt{2}}$ **e** $\cos(2\theta) = -\frac{\sqrt{3}}{2}$ **f** $\sin(2\theta) = -\frac{1}{\sqrt{2}}$

8 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -0.8$ **b** $\sin(2\theta) = -0.6$

c $\cos(2\theta) = 0.4$ **d** $\cos(3\theta) = 0.6$

13.10 Sketch graphs of $y = a\sin n(t \pm \varepsilon) \pm b$ and $y = a\cos n(t \pm \varepsilon) \pm b$

Translations parallel to the y-axis are now considered.

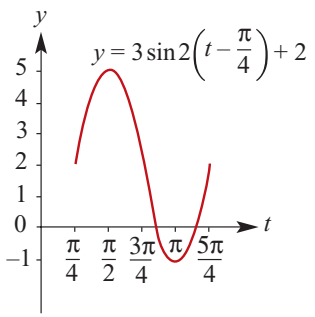
Example 23

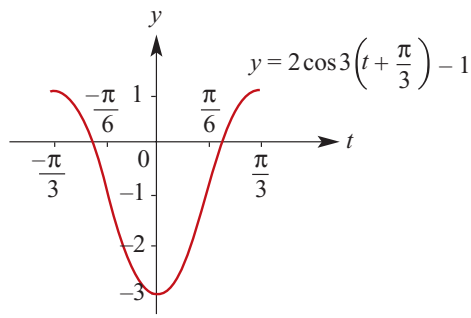
Sketch each of the following graphs. Use a graphics calculator to help establish the shape.

a $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2$ for $\frac{\pi}{4} \leq t \leq \frac{5\pi}{4}$

b $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$ for $-\frac{\pi}{3} \leq t \leq \frac{\pi}{3}$

Solution





Observations

1 The graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2$ is the same shape as the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$ but it is translated 2 units in the positive direction of the y-axis.

2 Similarly, the graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$ is the same shape as the graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$ but it is translated 1 unit in the negative direction of the y-axis.

In general, the effect of $\pm b$ is to translate the graph $\pm b$ units parallel to the y-axis.

Finding axis intercepts

Example 24

Sketch the graphs of each of the following for $x \in [0, 2\pi]$. Clearly indicate axis intercepts.

a $y = \sqrt{2}\sin(x) + 1$ **b** $y = 2\cos(2x) - 1$ **c** $y = 2\sin 2\left(x - \frac{\pi}{3}\right) - \sqrt{3}$

Solution

a To determine the axis intercepts, the equation $\sqrt{2}\sin(x) + 1 = 0$ must be solved.

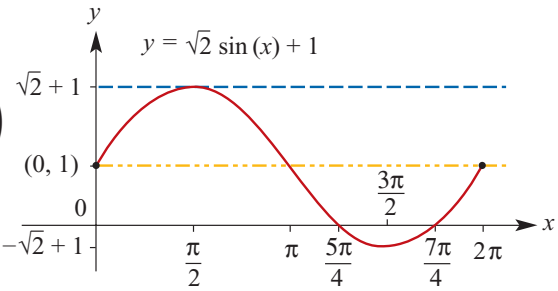
$\sqrt{2}\sin(x) + 1 = 0$

$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}$

$\therefore \quad x = \pi + \frac{\pi}{4}$ or $2\pi - \frac{\pi}{4}$

$\therefore \quad x = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$

$\therefore$ Intercepts: $\left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$



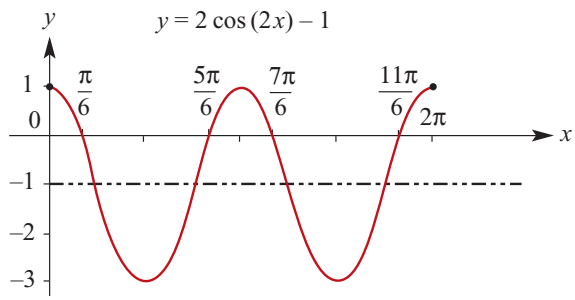
b $2\cos(2x) - 1 = 0$

$\therefore \quad \cos(2x) = \frac{1}{2}$

$\therefore \quad 2x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \text{ or } \frac{7\pi}{3} \text{ or } \frac{11\pi}{3}$

$\therefore \quad x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$

$\therefore$ Intercepts: $\left(\frac{\pi}{6}, 0\right), \left(\frac{5\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$



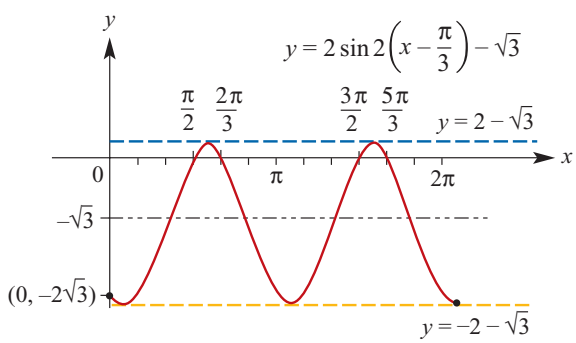
c $\sin 2\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\therefore \quad 2\left(x - \frac{\pi}{3}\right) = \frac{\pi}{3} \text{ or } \frac{2\pi}{3} \text{ or } \frac{7\pi}{3} \text{ or } \frac{8\pi}{3}$

$\therefore \quad x - \frac{\pi}{3} = \frac{\pi}{6} \text{ or } \frac{\pi}{3} \text{ or } \frac{7\pi}{6} \text{ or } \frac{4\pi}{3}$

$\therefore \quad x = \frac{\pi}{2} \text{ or } \frac{2\pi}{3} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{3}$

$\therefore$ Intercepts: $\left(\frac{\pi}{2}, 0\right), \left(\frac{2\pi}{3}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{5\pi}{3}, 0\right)$



Skillsheet TEACHER

Exercise 13J

Example 24

1 Sketch the graphs of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval:

a $y = 2\sin(x) + 1$ **b** $y = 2\sin(2x) - \sqrt{3}$ **c** $y = \sqrt{2}\cos(x) + 1$

d $y = 2\sin(2x) - 2$ **e** $y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1$

2 Sketch the graphs of each of the following for $x \in [-2\pi, 2\pi]$:

a $y = 2\sin(3x) - 2$ **b** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$

c $y = 2\sin(2x) - 3$ **d** $y = 2\cos(2x) + 1$

e $y = 2\cos 2\left(x - \frac{\pi}{3}\right) - 1$ **f** $y = 2\sin 2\left(x + \frac{\pi}{6}\right) + 1$

3 Sketch the graphs of each of the following for $x \in [-\pi, \pi]$:

a $y = 2\sin 2\left(x + \frac{\pi}{3}\right) + 1$ **b** $y = -2\sin 2\left(x + \frac{\pi}{6}\right) + 1$

c $y = 2\cos 2\left(x + \frac{\pi}{4}\right) + \sqrt{3}$

13.11 Further symmetry properties and the Pythagorean identity

Complementary relationships

$$\sin\left(\frac{\pi}{2} - \theta\right) = a$$

and since $a = \cos\theta$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

Similarly

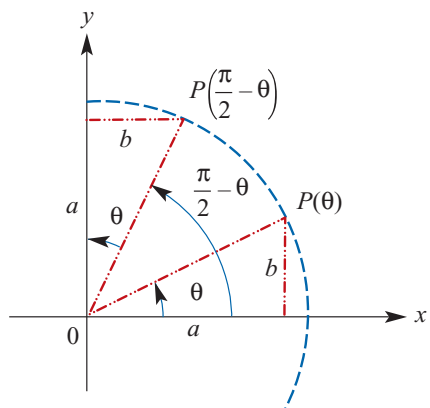
$$\cos\left(\frac{\pi}{2} - \theta\right) = b$$

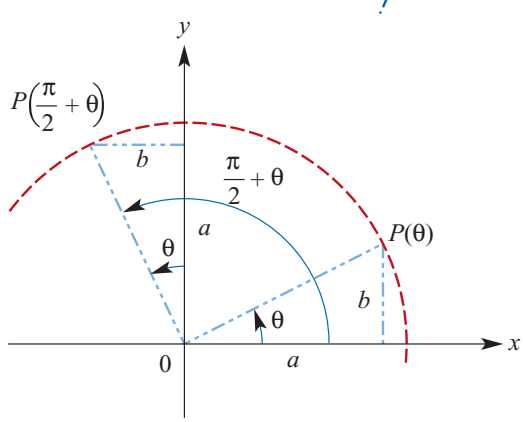
and since $b = \sin\theta$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$





Example 25

If $\sin\theta = 0.3$ and $\cos\alpha = 0.8$ find the values of:

a $\sin\left(\frac{\pi}{2} - \alpha\right)$ **b** $\cos\left(\frac{\pi}{2} + \theta\right)$ **c** $\sin(-\theta)$

Solution

a $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$
$= 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
$= -0.3$

c $\sin(-\theta) = -\sin\theta$
$= -0.3$

The Pythagorean identity

Consider a point, $P(\theta)$ on the unit circle.

By Pythagoras' Theorem

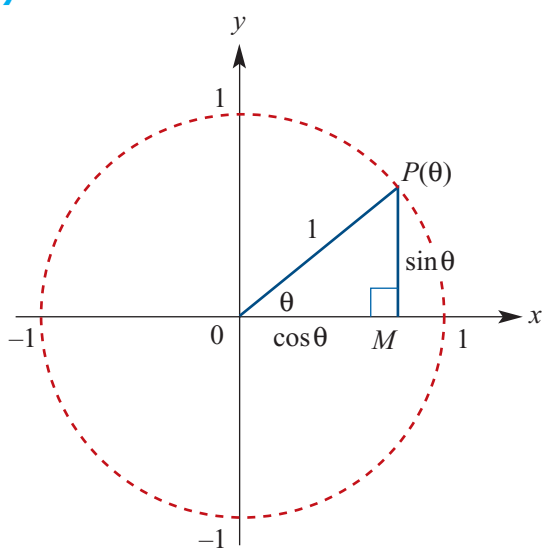
$$OP^2 = OM^2 + MP^2$$
$$\therefore 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Now $(\cos\theta)^2$ and $(\sin\theta)^2$ may be written as $\cos^2\theta$ and $\sin^2\theta$.

$$\therefore 1 = \cos^2\theta + \sin^2\theta$$

As this is true for all values of θ it is called an identity. In particular this is called the Pythagorean identity.

$$\cos^2\theta + \sin^2\theta = 1$$



Example 26

Given that $\sin x = \frac{3}{5}$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

Solution

$1 = \cos^2 x + \sin^2 x$.

For $\sin x = \frac{3}{5}$, $1 = \cos^2 x + \frac{9}{25}$

Then $\cos^2 x = 1 - \frac{9}{25}$
$= \frac{16}{25}$

Therefore $\cos x = \pm\frac{4}{5}$

But x is in the second quadrant, hence $\cos x = -\frac{4}{5}$.

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x}\\ &= \frac{3}{5} \div -\frac{4}{5}\\ &= \frac{3}{5} \times -\frac{5}{4}\\ &= -\frac{3}{4}\end{aligned}$$

Exercise 13K

Example 25

1 If $\sin x = 0.3$, $\cos \alpha = 0.6$ and $\tan \theta = 0.7$, find the values of:

a $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **c** $\tan(-\theta)$ **d** $\cos\left(\frac{\pi}{2} - x\right)$

e $\sin(-x)$ **f** $\tan\left(\frac{\pi}{2} - \theta\right)$ **g** $\cos\left(\frac{\pi}{2} + x\right)$ **h** $\sin\left(\frac{\pi}{2} - \alpha\right)$

i $\sin\left(\frac{3\pi}{2} + \alpha\right)$ **j** $\cos\left(\frac{3\pi}{2} - x\right)$

2 **a** Given that $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \sin\frac{\pi}{6}$, find the value of θ.

b Given that $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \cos\frac{\pi}{6}$, find the value of θ.

c Given that $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \sin\frac{\pi}{12}$, find the value of θ.

d Given that $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \cos\frac{3\pi}{7}$, find the value of θ.

Example 26

3 Given that $\cos x = \frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

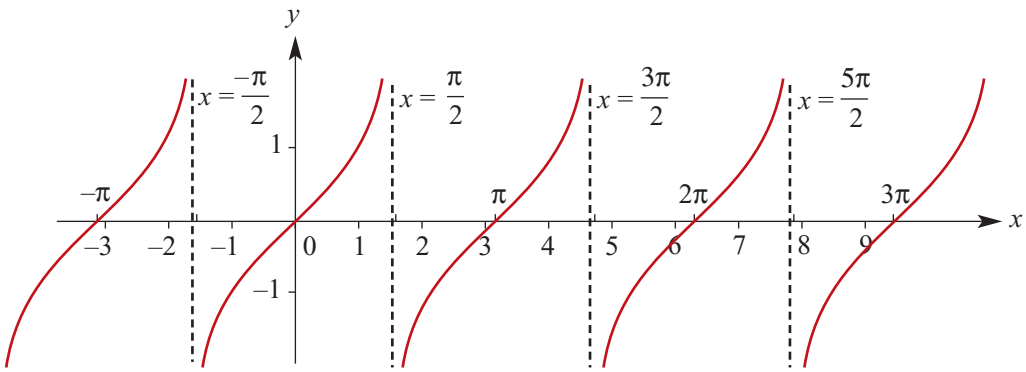
4 Given that $\sin x = \frac{5}{13}$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

5 Given that $\cos x = \frac{1}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

13.12 The tangent function

A table of values for $y = \tan x$ is given below. Use a graphics calculator to check these values, and plot the graph of $y = \tan(x)$.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	1	ud	-1	0	1	ud	-1	0	1	ud	-1	0	1	ud	-1	0



Note: $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ and $\frac{5\pi}{2}$ are asymptotes.
The x-axes intercepts where $\sin x = 0$, are $x = 0$ or π or 2π etc.

In general, $x = k\pi$, where k is an integer.

Observations from the graph

- The graph repeats itself every π units, i.e. the period of tan is π.
- Range of tan is R.
- The equation of the asymptotes are of the form $y = \frac{(2k+1)\pi}{2}$, where k is an integer.
- The x-axis intercepts occur for $x = k\pi$, where k is an integer.

Transformations of $y = \tan x$

Consider a dilation of factor $\frac{1}{2}$ from the y-axis and a dilation of factor 3 from the x-axis being applied to the graph of $y = \tan x$.

$(x, y) \rightarrow \left(\frac{1}{2}x, 3y\right)$. If the image of (x, y) under the transformation is (x', y') then $x' = \frac{1}{2}x$ and $y' = 3y$.

Hence $x = 2x'$ and $y = \frac{y'}{3}$. Thus the graph of $y = \tan x$ is transformed to the graph of $\frac{y'}{3} = \tan 2x'$; that is, it is transformed to the graph of $y = 3\tan 2x$.

The period of the graph will be $\frac{\pi}{2}$.

In general, for a and n positive numbers, the following are important properties of the function $f(t) = a\tan(nt)$:

- The period of the function is $\frac{\pi}{n}$.
- The graph of $y = a\tan(nt)$ is obtained from the graph of $y = \tan t$ by a dilation of factor a from the t-axis and a factor of $\frac{1}{n}$ from the y-axis.
- The range of the function is R.
- The asymptotes have equations $x = \frac{(2k+1)\pi}{2n}$, where k is an integer.
- The t-axis intercepts are $x = \frac{k\pi}{n}$, where k is an integer.

Example 27

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3\tan(2x)$ **b** $y = -2\tan(3x)$

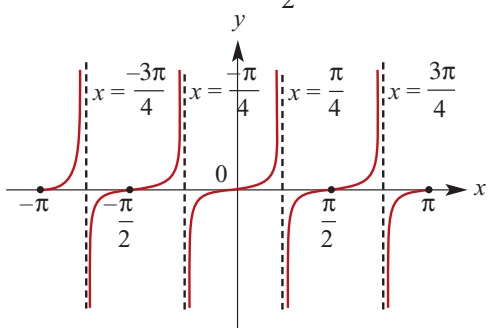
Solution

a $y = 3\tan(2x)$

Period $= \dfrac{\pi}{n} = \dfrac{\pi}{2}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{4}, k \in Z$

Axes intercepts: $x = \dfrac{k\pi}{2}, k \in Z$

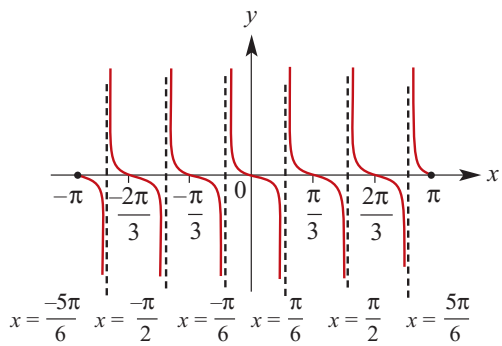


b $y = -2\tan(3x)$

Period $= \dfrac{\pi}{n} = \dfrac{\pi}{3}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{6}, k \in Z$

Axes intercepts: $x = \dfrac{k\pi}{3}, k \in Z$



Solution of equations

Example 28

Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\tan x = -1$ **b** $\tan(2x) = \sqrt{3}$ **c** $2\tan(3x) = 0$

Solution

a $\tan x = -1$

$x = \dfrac{3\pi}{4}$ or $\dfrac{-\pi}{4}$

b $\tan(2x) = \sqrt{3}$

$2x = \dfrac{\pi}{3}$ or $\dfrac{4\pi}{3}$ or $\dfrac{-2\pi}{3}$ or $\dfrac{-5\pi}{3}$

$x = \dfrac{\pi}{6}$ or $\dfrac{4\pi}{6}$ or $\dfrac{-2\pi}{6}$ or $\dfrac{-5\pi}{6}$

$= \dfrac{\pi}{6}$ or $\dfrac{2\pi}{3}$ or $\dfrac{-\pi}{3}$ or $\dfrac{-5\pi}{6}$

c $2\tan(3x) = 0$

$3x = -3\pi$ or -2π or $-\pi$ or 0 or π or 2π or 3π

Hence $x = -\pi$ or $\dfrac{-2\pi}{3}$ or $\dfrac{-\pi}{3}$ or 0 or $\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$ or π

Example 29

Sketch the graph of $y = \tan(2x) + 1$ for $x \in [-\pi, \pi]$.

Solution

The graph of $y = \tan(2x) + 1$ is formed from the graph of $y = \tan(2x)$ by a translation of 1 unit in the positive direction of the y-axis.

The y-axis intercept occurs when $x = 0$. When $x = 0$ then $y = 0$.

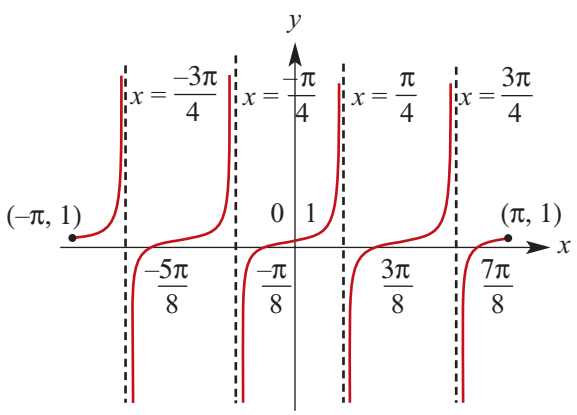
For the x-axis intercepts consider $\tan(2x) + 1 = 0$

This implies $\tan(2x) = -1$

Hence $2x = \frac{3\pi}{4}$ or $\frac{-\pi}{4}$ or $\frac{7\pi}{4}$ or $\frac{-5\pi}{4}$

and $x = \frac{3\pi}{8}$ or $\frac{-\pi}{8}$ or $\frac{7\pi}{8}$ or $\frac{-5\pi}{8}$

The asymptotes are the same as those for $y = \tan(2x)$. That is, $x = \frac{(2k+1)\pi}{4}$, $k \in Z$.



Exercise 13L

1 For each of the following state the period:

a $y = \tan(4x)$ **b** $y = \tan\left(\frac{2x}{3}\right)$ **c** $y = -3\tan(2x)$

Example 27 **2** Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = \tan(2x)$ **b** $y = 2\tan(3x)$ **c** $y = -2\tan(3x)$

Example 28 **3** Solve each of the following equations for $x \in [-\pi, \pi]$:

a $2\tan(2x) = 2$ **b** $3\tan(3x) = \sqrt{3}$

c $2\tan(2x) = 2\sqrt{3}$ **d** $3\tan(3x) = -\sqrt{3}$

Example 29 **4** Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3\tan(x) + \sqrt{3}$ **b** $y = 2\tan(x) + 2$ **c** $y = 3\tan(x) - 3$

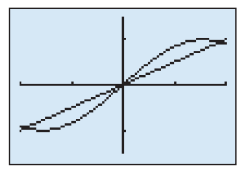
Using a graphics calculator

Example 30

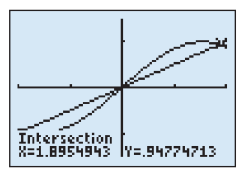
Solve the equation $\frac{x}{2} = \sin x$, giving your answer correct to 2 decimal places.

Solution

Plot the graphs of $y = \frac{x}{2}$ and $y = \sin x$ in a suitable window. There will be only three points of intersection. Use **5:intersect** from the **CALC** menu to find the coordinates of one point of intersection and use symmetry ($\sin(-x) = -\sin x$) to find the other.

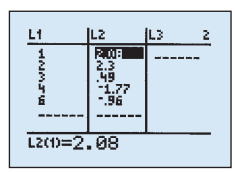


The solutions are $x = 1.90$ and $x = -1.90$ correct to 2 decimal places.

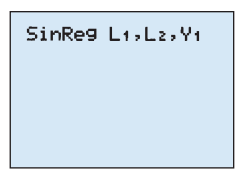


Fitting data

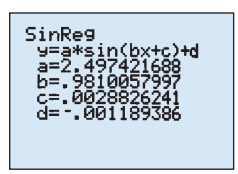
Consider the points (1, 2.08), (2, 2.3), (3, 0.49), (4, −1.77), (6, −0.96). Enter the data in **L1** and **L2**.



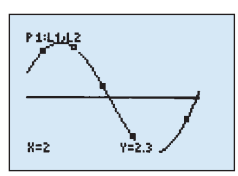
Select **SinReg** from the **CALC** submenu of **STAT** and complete in the Home screen as shown.



The result is as shown.



Activating **Plot1** and using **ZoomStat** from the **ZOOM** menu results in the graph opposite.



Exercise 13M

Example 30

1 Solve each of the following equations for x correct to 2 decimal places:

a $\cos x = x$ **b** $\sin x = 1 - x$

c $\cos x = x^2$ **d** $\sin x = x^2$

2 For each of the following sets of data find a suitable trigonometric rule (model).

a

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	1	2.4	-1	2.4	1

b

θ	0	0.2	0.4	0.6	0.8
y	0	1.77	2.85	2.85	1.77

c

θ	0	0.2	0.4	0.6	0.8
y	5	2.18	0.34	0.13	1.62

13.13 Applications of trigonometric functions

Example 31

It is suggested that the height $h(t)$ metres of the tide above mean sea level on 1 January at Warnung is given approximately by the rule $h(t) = 4\sin\left(\frac{\pi}{6}t\right)$, where t is the number of hours after midnight.

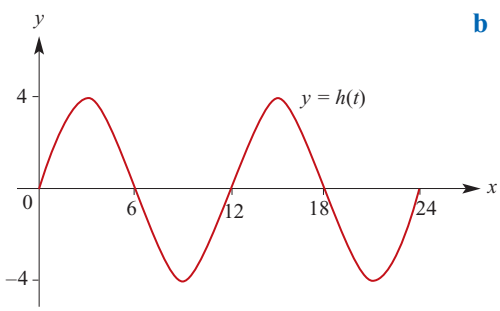
a Draw the graph of $y = h(t)$ for $0 \leq t \leq 24$. **b** When was high tide?

c What was the height of the high tide? **d** What was the height of the tide at 8 am?

e A boat can only cross the harbour bar when the tide is at least 1 metre above mean sea level. When could the boat cross the harbour bar on 1 January?

Solution

a



We note: period $= 2\pi \div \frac{\pi}{6} = 12$

b High tide occurs when $h(t) = 4$

$$4\sin\left(\frac{\pi}{6}t\right) = 4$$

$$\text{implies } \sin\left(\frac{\pi}{6}t\right) = 1$$

$$\therefore \quad \frac{\pi}{6}t = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore \quad t = 3, 15$$

i.e. high tide occurs at 03.00 and 15.00 (3 pm).

c The high tide has height 4 metres above the mean height.

d $h(8) = 4\sin\left(\frac{8\pi}{6}\right) = 4\sin\left(\frac{4\pi}{3}\right) = 4 \times \frac{-\sqrt{3}}{2} = -2\sqrt{3}$.

The water is $2\sqrt{3}$ metres below the mean height at 8 am.

e We first consider $4\sin\frac{\pi}{6}t = 1$.

Thus $\sin\frac{\pi}{6}t = \frac{1}{4}$

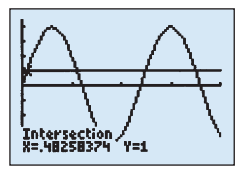
$\therefore \frac{\pi}{6}t = 0.2526, 2.889, 6.5358, 9.172$

$\therefore \quad t = 0.4824, 5.5176, 12.4824, 17.5173$

i.e. the water is at height 1 metre at 00:29, 05:31, 12:29, 17:31.
Thus the boat can pass across the harbour bar between 00:29 and 05:31 and between 12:29 and 17:31.

A graphics calculator can be used to obtain the solutions for part e.
Enter **Y1** $= 4\sin\left(\frac{\pi}{6}x\right)$ and **Y2** $= 1$ in the **Y=** window.

Use **5:intersect** from the **CALC** menu to obtain solutions to the equation $4\sin\left(\frac{\pi}{6}t\right) = 1$.



Exercise 13N

1 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$, $0 \leq t \leq 24$.

a Sketch the graph of $D(t)$ for $0 \leq t \leq 24$. **b** Find the value of t for which $D(t) \geq 8.5$.

c Boats which need a depth of w metres are permitted to enter the harbour only if the depth of the water at the entrance is at least w metres for a continuous period of 1 hour. Find, correct to 1 decimal place, the largest value of w which satisfies this condition.

2 The depth of water at the entrance to a harbour t hours after high tide is D metres, where $D = p + q\cos(rt)^\circ$ for suitable constants p, q, r. At high tide the depth is 7 m; at low tide, 6 hours later, the depth is 3 m.

a Show that $r = 30$ and find the values of p and q.

b Sketch the graph of D against t for $0 \leq t \leq 12$.

c Find how soon after low tide a ship that requires a depth of at least 4 m of water will be able to enter the harbour.

3 A particle moves on a straight line, OX, and its distance x metres from O at time t (s) is given by $x = 3 + 2\sin(3t)$.

a Find its greatest distance from O. **b** Find its least distance from O.

c Find the times at which it is 5 metres from O for $0 \leq t \leq 5$.

d Find the times at which it is 3 metres from O for $0 \leq t \leq 3$.

e Describe the motion of the particle.

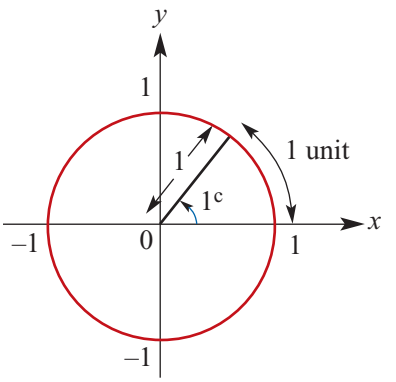
Review

Chapter summary

- Definition of a radian

One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

$$1^c = \frac{180^\circ}{\pi} \qquad 1^\circ = \frac{\pi^c}{180}$$



- Sine and cosine

x-coordinate of $P(\theta)$ in unit circle,

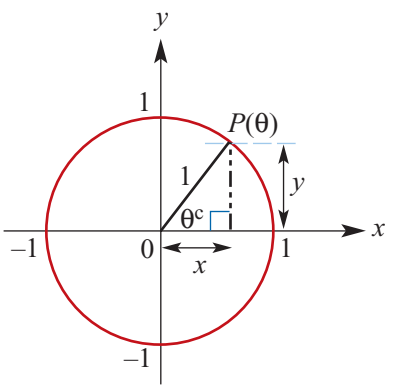
$x =$ cosine θ, $\theta \in R$

y-coordinate of $P(\theta)$ in unit circle,

$y =$ sine θ, $\theta \in R$

Abbreviated to $x = \cos\theta$

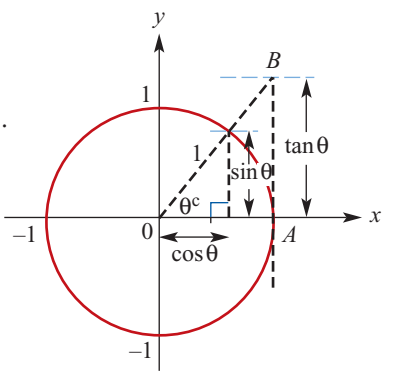
$y = \sin\theta$



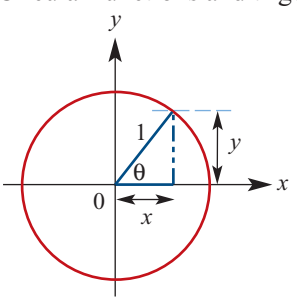
- Tangent

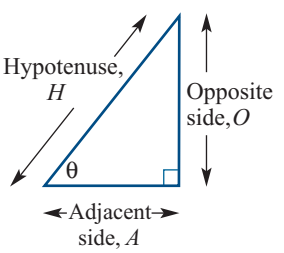
If the tangent to the unit circle at A is drawn then the y-coordinate of B is called tangent θ (abbreviated to $\tan\theta$). Also by using similar triangles:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$



- Circular functions and trigonometric ratios





$$\sin\theta = \frac{O}{H} = \frac{y}{1} = y$$

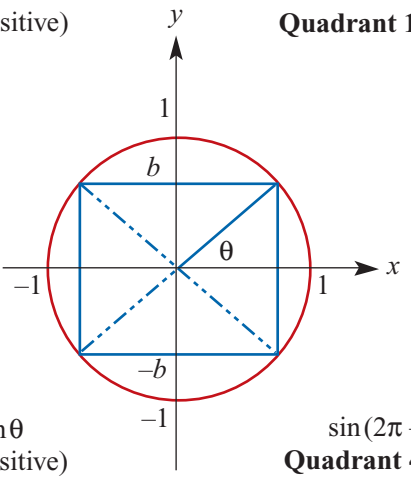
$$\cos\theta = \frac{A}{H} = \frac{x}{1} = x$$

$$\tan\theta = \frac{O}{A} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

- Symmetry properties of circular functions

Quadrant 2 (sin is positive)
$\sin(\pi - \theta) = b = \sin\theta$

Quadrant 1 (all functions are positive)
e.g. $\sin\theta = b$



$\sin(\pi + \theta) = -b = -\sin\theta$
Quadrant 3 (tan is positive)

$\sin(2\pi - \theta) = -b = -\sin\theta$
Quadrant 4 (cos is positive)

- Solutions of trigonometric equations of the type $\sin x^\circ = a$ and $\cos x^\circ = a$

e.g. If $\cos x^\circ = -0.7$, find the two values of x in the range $0 \leq x \leq 360$.

If $\cos x^\circ = 0.7$ then $x = 45.6$

cos is negative in the 2nd and 3rd quadrants

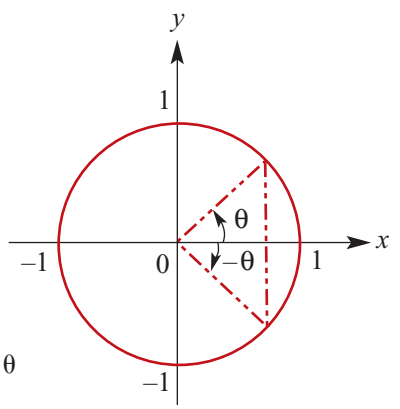
$\therefore \quad x = 180 - 45.6 = 134.4$

and $\quad x = 180 + 45.6 = 225.6$

- Further symmetry properties

Negative angles:

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\tan(-\theta) = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$



Complementary angles:

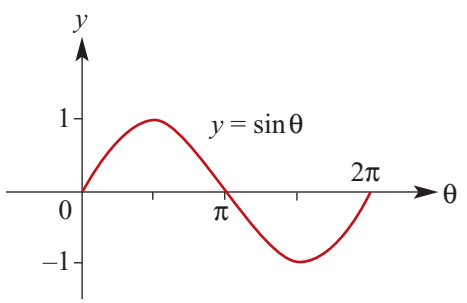
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

Review

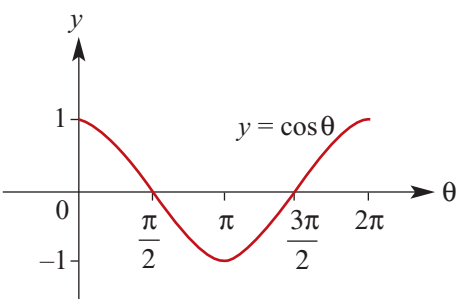
- Exact values of circular functions

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

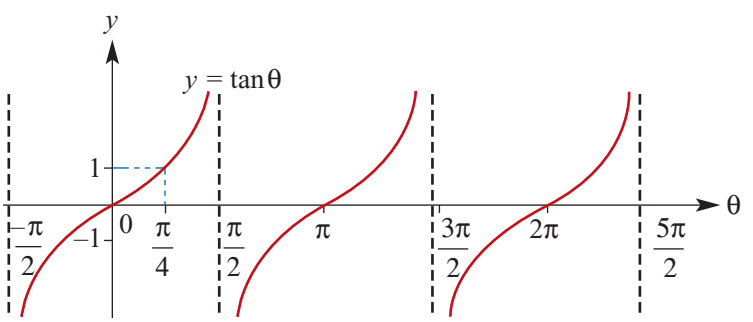
- Graphs of circular functions



Amplitude = 1
Period = 2π



Amplitude = 1
Period = 2π



Amplitude is undefined
Period = π

- Graphs of circular functions of the type $y = a\sin n(t \pm \varepsilon) \pm b$ and $y = a\cos n(t \pm \varepsilon) \pm b$

  e.g. $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$

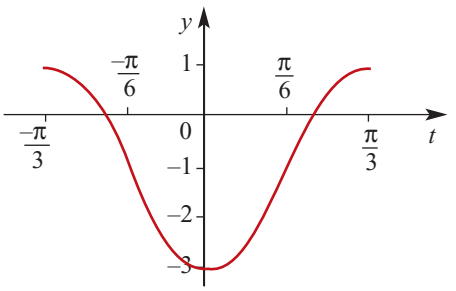
  Amplitude, $a = 2$

  Period $= \frac{2\pi}{n} = \frac{2\pi}{3}$

  The graph is the same shape as $y = 2\cos(3t)$ but is translated:

  i $\frac{\pi}{3}$ units in the negative direction of the t-axis, and

  ii 1 unit in the negative direction of the y-axis.



- Pythagorean identity

  $\cos^2\theta + \sin^2\theta = 1$

Multiple-choice questions

1 In a right-angled triangle, the two shorter side lengths are 3 cm and 4 cm. To the nearest degree, the value of the smallest angle is

A 1° **B** 23° **C** 37° **D** 53° **E** 92°

2 The minimum value of $3 - 10\cos(2x)$ is

A -13 **B** -17 **C** -23 **D** -7 **E** -10

3 The range of the function $f: [0, 2\pi] \to R, f(x) = 4\sin\left(2x - \frac{\pi}{2}\right)$ is

A R **B** $[0, 4]$ **C** $[-4, 0]$ **D** $[0, 8]$ **E** $[-4, 4]$

4 The period of the graph of $y = 3\sin\left(\frac{1}{2}x - \pi\right) + 4$ is

A π **B** 3 **C** 4π **D** $\pi + 4$ **E** 2π

5 The graph of $y = \sin x$ is dilated by factor $\frac{1}{2}$ from the y-axis and translated $\frac{\pi}{4}$ units in the positive direction of the x-axis. The equation of the image is

A $y = \sin\left(\frac{1}{2}x + \frac{\pi}{4}\right)$ **B** $y = \sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ **C** $y = 2\sin\left(x - \frac{\pi}{4}\right)$

D $y = \sin\left(2x - \frac{\pi}{4}\right)$ **E** $y = \sin\left(2\left(x - \frac{\pi}{4}\right)\right)$

6 The period of the function $f: R \to R$, where $f(x) = a\sin(bx) + c$ and a, b and c are positive constants, is

A a **B** b **C** $\frac{2\pi}{a}$ **D** $\frac{2\pi}{b}$ **E** $\frac{b}{2\pi}$

7 One cycle of the graph of function with equation $y = \tan ax$ has vertical asymptotes at $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{6}$. A possible value of a is

A 6 **B** π **C** $\frac{\pi}{6}$ **D** $\frac{1}{3}$ **E** 3

Review

8 The equation $3\sin(x) + 1 = b$, where b is a positive real number, has one solution in the interval $[0, 2\pi]$. The value of b is

A 1 **B** 1.5 **C** 2 **D** 3 **E** 4

9 The number of solutions of the equation $b = a\sin x$, where $x \in [-2\pi, 2\pi]$ and a and b are positive real numbers with $a > b$, is

A 2 **B** 3 **C** 4 **D** 5 **E** 6

10 The depth of water, in metres, in a harbour at a certain point at time t hours is given by $D(t) = 8 + 2\sin\left(\frac{\pi t}{6}\right)$, $0 \leq t \leq 24$. The depth of the water is first 9 m at

A $t = 0$ **B** $t = 1$ **C** $t = 2$ **D** $t = 3$ **E** $t = 4$

Short-answer questions (technology-free)

1 Change each of the following to radian measure in terms of π:

a 330° **b** 810° **c** 1080° **d** 1035° **e** 135°

f 405° **g** 390° **h** 420° **i** 80°

2 Change each of the following to degree measure:

a $\frac{5\pi^c}{6}$ **b** $\frac{7\pi^c}{4}$ **c** $\frac{11\pi^c}{4}$ **d** $\frac{3\pi^c}{12}$ **e** $\frac{15\pi^c}{2}$

f $-\frac{3\pi^c}{4}$ **g** $-\frac{\pi^c}{4}$ **h** $-\frac{11\pi^c}{4}$ **i** $-\frac{23\pi^c}{4}$

3 Give exact values of each of the following:

a $\sin\left(\frac{11\pi}{4}\right)$ **b** $\cos\left(-\frac{7\pi}{4}\right)$ **c** $\sin\left(\frac{11\pi}{6}\right)$ **d** $\cos\left(-\frac{7\pi}{6}\right)$

e $\cos\left(\frac{13\pi}{6}\right)$ **f** $\sin\left(\frac{23\pi}{6}\right)$ **g** $\cos\left(-\frac{23}{3}\pi\right)$ **h** $\sin\left(-\frac{17}{4}\pi\right)$

4 State the amplitude and period of each of the following:

a $2\sin\left(\frac{\theta}{2}\right)$ **b** $-3\sin(4\theta)$ **c** $\frac{1}{2}\sin(3\theta)$

d $-3\cos(2x)$ **e** $-4\sin\left(\frac{x}{3}\right)$ **f** $\frac{2}{3}\sin\left(\frac{2x}{3}\right)$

5 Sketch the graphs of each of the following (showing one cycle):

a $y = 2\sin(2x)$ **b** $y = -3\cos\left(\frac{x}{3}\right)$ **c** $y = -2\sin(3x)$

d $y = 2\sin\left(\frac{x}{3}\right)$ **e** $y = \sin\left(x - \frac{\pi}{4}\right)$ **f** $y = \sin\left(x + \frac{2\pi}{3}\right)$

g $y = 2\cos\left(x - \frac{5\pi}{6}\right)$ **h** $y = -3\cos\left(x + \frac{\pi}{6}\right)$

6 Solve each of the following equations for R:

a $\sin\theta = -\frac{\sqrt{3}}{2}, \theta \in [-\pi, \pi]$

b $\sin(2\theta) = -\frac{\sqrt{3}}{2}, \theta \in [-\pi, \pi]$

c $\sin\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2}, \theta \in [0, 2\pi]$

d $\sin\left(\theta + \frac{\pi}{3}\right) = -1, \theta \in [0, 2\pi]$

e $\sin\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2}, \theta \in [0, 2\pi]$

Extended-response questions

1 The number of hours of daylight at a point on the Antarctic Circle is given approximately by $d = 12 + 12\cos\frac{1}{6}\pi\left(t + \frac{1}{3}\right)$, where t is the number of months which have elapsed since 1 January.

a Find d:

i on 21 June ($t \approx 5.7$)

ii on 21 March ($t \approx 2.7$)

b When will there be 5 hours of daylight?

2 The temperature $A°$C inside a house at t hours after 4 am is given by $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$ for $0 \leq t \leq 24$, and the temperature $B°$C outside the house at the same time is given by $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$ for $0 \leq t \leq 24$.

a Find the temperature inside the house at 8 am.

b Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.

c Sketch the graph of D for $0 \leq t \leq 24$.

d Determine when the inside temperature is less than the outside temperature.

3 At a certain time of the year the depth of water d m in the harbour at Bunk Island is given by the rule $d = 3 + 1.8\cos\left(\frac{\pi}{6}t\right)$, where t is the time in hours after 3 am.

a Sketch the graph of the function $d = 3 + 1.8\cos\left(\frac{\pi}{6}t\right)$ over a 24-hour period from 3 am to 3 am.

b At what time(s) does high tide occur for $t \in [0, 24]$?

c At what time(s) does low tide occur for $t \in [0, 24]$?

A passenger ferry operates between Main Beach and Bunk Island. It takes 50 minutes to go from Main Beach to Bunk Island. The ferry only runs between the hours of 8 am and 8 pm and is only able to enter the harbour at Bunk Island if the depth of water is at least 2 metres.

d What is the earliest time the ferry should leave Main Beach so that it arrives at Bunk Island and can immediately enter the harbour?

Review

e The time to go from Bunk Island to Main Beach is also 50 minutes. The minimum time the ferry takes at Bunk Island harbour is 5 minutes. The minimum time at Main Beach is also 5 minutes.

i What is the latest time the ferry can leave Main Beach to complete a round trip in 105 minutes?

ii How many complete round trips can the ferry make in a day?

4 The depth of water D at the end of Brighton pier t hours after low tide is given by the rule $D = p - 2\cos(rt)$, where p and r are suitable constants.

At low tide ($t = 0$) the depth is 2 metres; at high tide, which occurs 8 hours later, the depth is 6 metres.

a Show that $r = \frac{\pi}{8}$ and $p = 4$.

b Sketch the graph of $D = 4 - 2\cos\left(\frac{\pi}{8}t\right)$ for $0 \leq t \leq 16$.

c If the first low tide occurs at 4 am, when will the next low tide occur?

d At what times will the depth be equal to 4 metres?

The poles that support the Brighton pier stand 7.5 metres above the sea bed.

e How much of a particular pole is exposed at

i high tide **ii** 2 pm?

Over the years mussels have attached themselves to the pole. A particular mussel is attached 4 metres from the top of the pole so that some of the time it is exposed and some of the time it is covered by water.

f For how long will the mussel be covered by water during the time from one low tide to the next?

CHAPTER

14

Revision

Revision of Chapters 12–13

14.1 Multiple-choice questions

1 If $\log_a 8 = 3$ then a is equal to

A 1 **B** 2 **C** 3 **D** 4 **E** 0

2 $5^{n-1} \times 5^{n+1}$ is equal to

A 5^{n^2} **B** 5^{2n} **C** 10^{2n} **D** 25^{2n} **E** 25^{n^2-1}

3 If $2^x = \frac{1}{64}$ then x is equal to

A 6 **B** -6 **C** 5 **D** -5 **E** $\frac{1}{6}$

4 $125^a \times 5^b$ is equal to

A 625^{a+b} **B** 625^{ab} **C** 125^{a+3b} **D** 5^{a+3b} **E** 5^{3a+b}

5 The solution of the equation $4^x = 10 - 4^{x+1}$ is

A $x = 4$ **B** $x = 2$ **C** $x = \frac{1}{4}$ **D** $x = \frac{1}{2}$ **E** $x = 1$

6 $\frac{7^{n+2} - 35(7^{n-1})}{44(7^{n+2})}$ is equal to

A $\frac{1}{49}$ **B** $\frac{1}{44}$ **C** $\frac{1}{28}$ **D** $\frac{1}{7}$ **E** 7

7 If $f(x) = 2 + 3^x$ then $f(2x) - f(x)$ is equal to

A 3^x **B** 3^{2x} **C** $2 + 3^x$ **D** $3^x(3^x - 1)$ **E** $3^x(3^x + 1)$

8 If $7^{2x} \times 49^{2x-1} = 1$ then x is equal to

A -1 **B** $-\frac{1}{3}$ **C** $\frac{1}{3}$ **D** 1 **E** 3

Revision

9 The graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ have

A the same x-axis intercept **B** the same y-axis intercept
C no point in common **D** two points in common
E three points in common

10 If $f(x) = (2x)^0 + x^{-\frac{2}{3}}$, then $f(8)$ is equal to

A $\frac{5}{4}$ **B** $\frac{65}{4}$ **C** 5 **D** 20 **E** None of these

11 $\log a^2 + \log b^2 - 2 \log ab$ is equal to

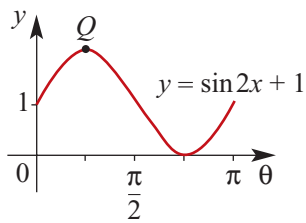
A 0 **B** 1 **C** a **D** b **E** a^2b^2

12 An angle is measured as $2x$ radians. The measure of the angle in degrees is

A $\left(\frac{\pi x}{90}\right)^\circ$ **B** $\left(\frac{90x}{\pi}\right)^\circ$ **C** $\left(\frac{\pi}{180x}\right)^\circ$ **D** $\left(\frac{360x}{\pi}\right)^\circ$ **E** $\left(\frac{\pi x}{360}\right)^\circ$

13 The figure shows the graph of $y = \sin 2x + 1$.
The coordinates of Q are

A $\left(\frac{\pi}{4}, 2\right)$ **B** $\left(\frac{\pi}{12}, 2\right)$ **C** $(\pi, 1)$
D $\left(\frac{\pi}{4}, 1\right)$ **E** $\left(\frac{\pi}{2}, 1\right)$



14 The smallest values of $1 - 3\cos\theta$ is

A -5 **B** -4 **C** -3 **D** -2 **E** -1

15 A rollercoaster is constructed in such a way that any car is y metres above the ground when it is x metres from the starting point, where $y = 16 + 15\sin\left(\frac{\pi}{60}x\right)$.
The height of the car, in metres, when $x = 10$ is

A 31 **B** 1 **C** 16 **D** 23.5 **E** $16 - 5\sqrt{2}$

16 $\sin(\pi + \theta) + \cos(\pi + \theta)$ is equal to

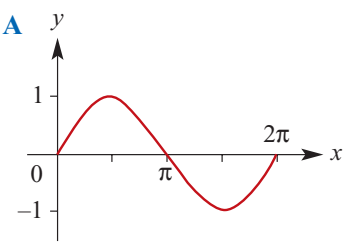
A $\sin\theta + \cos\theta$ **B** $-\sin\theta + \cos\theta$ **C** $\sin\theta - \cos\theta$
D $-\sin\theta - \cos\theta$ **E** $-\sin\theta\cos\theta$

17 For $\theta \in [0, \pi]$, which of the following equations has exactly two solutions?

A $\sin x = 0$ **B** $\cos x = 0$ **C** $\sin x = 1$
D $\sin x = -1$ **E** $\cos x = 1$

18 Which of the following is the graph of $y = \sin\frac{\theta}{2}$ for one cycle?

A



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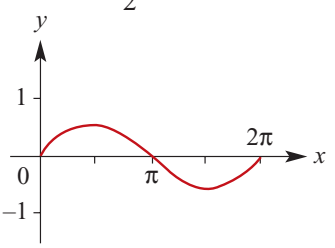
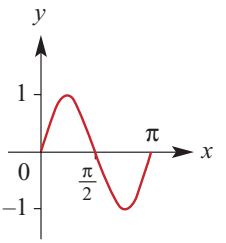
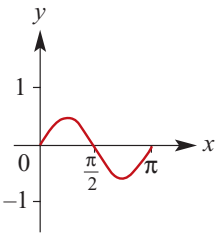


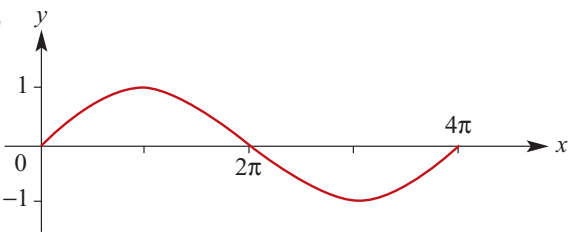
C



D



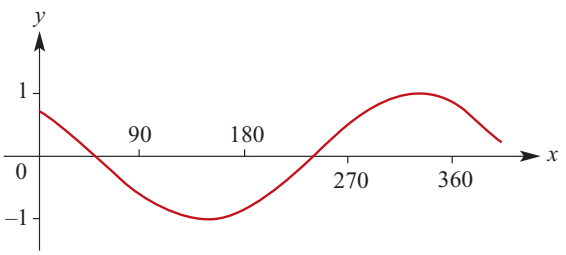
E



19 The minimum value of $2 - 3 \sin \theta$ is

A 2 **B** 1 **C** 0 **D** -1 **E** 3

20



The above is the graph of

A $y = \cos(x - 30)^\circ$ **B** $y = \frac{1}{2}\cos(x + 30)^\circ$ **C** $y = \frac{1}{2}\cos(x - 30)^\circ$

D $y = \cos(x + 30)^\circ$ **E** $y = \frac{1}{2}\cos x$

21 The function $f: R \to R, f(x) = -2 \cos 3x$ has

A amplitude 2 and period π
B amplitude -2 and period $\frac{2\pi}{3}$
C amplitude 2 and period 6π
D amplitude 3 and period π
E amplitude 2 and period $\frac{2\pi}{3}$

22 If $C^d = 3$ then $C^{4d} - 5$ equals

A 76 **B** 7 **C** 22 **D** 86 **E** 35

23 The value of $\log_2 56 - \log_2 7 + \log_2 2$ is

A $\log_2 51$ **B** 1 **C** 2 **D** 3 **E** 4

24 If $\log_b a = c$ and $\log_x b = c$, then $\log_a x$ equals

A a **B** c^{-2} **C** b^2 **D** b **E** abc^2

25 If $\cos \theta - \sin \theta = \frac{1}{4}$, then $\sin \theta \cos \theta$ equals

A $\frac{1}{16}$ **B** $\frac{15}{16}$ **C** $\frac{1}{32}$ **D** $\frac{15}{32}$ **E** $\frac{1}{2}$

26 The coordinates of a point of intersection of the graphs $y = \frac{1}{2}\sin(2x)$ and $y = \frac{1}{2}$ are

A $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ **B** $\left(\frac{\pi}{4}, \frac{1}{2}\right)$ **C** $\left(\frac{\pi}{2}, \frac{1}{2}\right)$ **D** $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ **E** $\left(2\pi, \frac{1}{2}\right)$

Revision

14.2 Extended-response questions

1 The height of the tide, h metres, at a harbour at any time during a 24-hour period is given by the equation $h(t) = 10 + 4\sin(15t)^\circ$, where t is measured in hours.

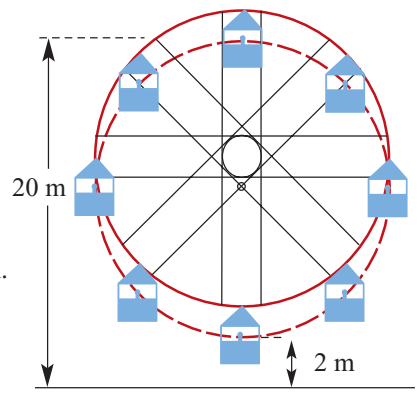
- **a** Sketch the graph of h against t for $0 \leq t \leq 24$.
- **b** Find the times at which $h = 13$ during the 24-hour period.
- **c** A boat can leave the harbour when the height of the tide is at 11 m or more. State the times during the 24 hours when the boat can leave the harbour.

2 Medical researchers studying the growth of a strain of bacteria observe that the number of bacteria present after t hours is given by the formula $N(t) = 40 \times 2^{1.5t}$.

- **a** State the number of bacteria present at the start of the experiment.
- **b** State the number of bacteria present after:
  - **i** 2 hours **ii** 4 hours **iii** 12 hours
- **c** Sketch the graph of N against t.
- **d** How many minutes does it take for the number of bacteria to double?

3 For a ride on a Ferris wheel, the height above the ground of a person, h m, at time t (s) is given by

$$h(t) = 11 + 9\cos\left(\frac{\pi}{30}(t - 10)\right)$$



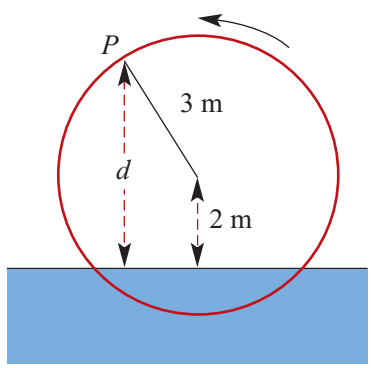
- **a** How long does it take for the Ferris wheel to make one complete revolution?
- **b** Sketch the graph of h against t for one revolution.
- **c** State the range of the function.
- **d** At what time(s) is the person at a height of 2 metres above the ground?
- **e** At what time(s) is the person at a height of 15.5 metres above the ground?

4 The voltage, V, in a circuit after t seconds is given by $V = 120\cos 60\pi t$.

- **a** Sketch the graph of V against t for one cycle.
- **b** Find the first time the voltage is 60.
- **c** Find all times at which the voltage is maximised.

5 The figure shows a waterwheel rotating at 4 revolutions per minute. The distance, d, of a point, P, from the surface of the water as a function of time, t in seconds, can be modelled by a rule of the form

$$d = a + b\sin c(t - h)$$



- **a** Find:
  - **i** the period **ii** the amplitude **iii** c
- **b** If $d = 0$ when $t = 0$ find h.
- **c** Sketch the graph of d against t for $0 \leq t \leq 30$.

6 A forest fire has burnt out 30 hectares by 11.00 am. It then spreads according to the formula

$$h = 30(1.65)^t$$

where t is the time in hours after 11.00 am.

a Find h when:

i $t = 0$ **ii** $t = 1$ **iii** $t = 2$

b Find k such that $h(N + 1) = kh(N)$.

c How long does it take for 900 hectares to be burnt?

d Sketch the graph of h against t.

7 A bowl of water is initially at boiling point (i.e. 100°C). The temperature, θ°C, of the water t minutes after beginning to cool is $\theta = 80(2^{-t}) + 20$.

a Copy and complete this table:

t	0	1	2	3	4	5
θ						

b Draw a graph of θ against t.

c At what time is the temperature 60°C?

d Find θ when $t = 3.5$.

8 A, B and C are three countries. Each of them now has a population of 70 million.

- Country A's population grows uniformly by 30 million in every period of 10 years.
- Country B's population grows uniformly by 50 million in every period of 10 years.
- Country C's population is multiplied by 1.3 every 10 years.

a Give an equation for the population of each country at time t.

b On the same set of axes carefully draw the graphs of the three equations.

c From your graph find out when the population of C overtakes:

i the population of A **ii** the population of B

9 An estimate for the population of the Earth, P in billions, is

$$P = 4(2)^{\frac{(t-1975)}{35}}$$

where t is the year.

a Evaluate P for:

i $t = 1975$ **ii** $t = 1995$ **iii** $t = 2005$

b When will the population be twice that of the Earth in 1997?

Revision

10 Two tanks simultaneously start to leak. Tank A contains $V_1(t)$ litres and tank B $V_2(t)$ litres of water, where

$$V_1(t) = 1000e^{\frac{-t}{10}} \qquad t \geq 0$$
$$V_2(t) = 1000 - 40t \qquad 0 \leq t \leq 25$$

and t is the time after the tanks start to leak.

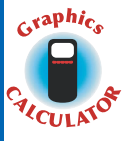
a Find $V_1(0)$ and $V_2(0)$.

b Sketch the graphs of $y = V_1(t)$ and $y = V_2(t)$ for $0 \leq t \leq 25$.

c How much water is in tank A when tank B is empty?

d Find the times at which the two tanks have equal amounts of water in them.

11 A river gate is used to control the height of water in a river.



On one side of the gate the height of the water is subject to tides. The depth of the water in metres on this side is given by

$$h_1(t) = 18 + 10\sin\left(\frac{\pi}{6}t\right)$$

where t is the time in hours past midnight.

On the other side of the gate the water rises according to the rule

$$h_2(t) = 8 + 6t$$

where t is the time in hours past midnight and $h_2(t)$ is the height of the water at time t.

a Sketch the graphs of $y = h_1(t)$ and $y = h_2(t)$ for $0 \leq t \leq 6$, on the one set of axes.

b Find the time at which $h_1(t) = h_2(t)$.

c When the water levels on both sides of the gate are the same, the gate is opened and the height of the water on both sides is given by $y = h_1(t)$.

i The gate is closed again when $h_1(t)$ reaches its minimum value. At what time does this happen?

ii Water flows in on the non tidal side so that the height increases by 6 metres every hour. What will be the height of the river t hours after the gates have been closed?

CHAPTER

15

Rates of Change

Objectives

- To recognise **relationships** between variables.
- To calculate average **rates of change**.
- To estimate **gradients** of curves.
- To estimate instantaneous rates of change.
- To estimate and calculate rates of change of **algebraic functions**.
- To **apply** the estimation and calculation of rates of change to solving problems.

When the relationship between two variables has been considered in chapters 2–6 and chapters 12 and 13, the idea that one variable, y for example, is a function of another variable, such as x, has been used and developed. The variable x is often considered to be the **independent** variable and y the **dependent** variable. The value of y depends on the value of x and is found by applying the rule connecting the two variables for the given value of x. So, the rule that connects x and y enables the value of y to be determined for a given value of x. For example, for the rule $y = x^2 - 3x + 5$, if $x = 3$, then $y = 5$.

Furthermore, such relationships have been represented graphically, and key features such as axis intercepts, turning points and asymptotes have been considered.

Analysis of the graphical representation can also be used to see how that relationship is changing. How the relationship is changing is of critical importance in establishing how accurately a given rule models the relationship between the variables in question. As changes in values of x are considered, the question of how the corresponding values of y are changing arises. If x increases, for example, does y also increase, or does it decrease or remain unaltered? And, if it does change, does it do so consistently, quickly, slowly, indefinitely etc.?

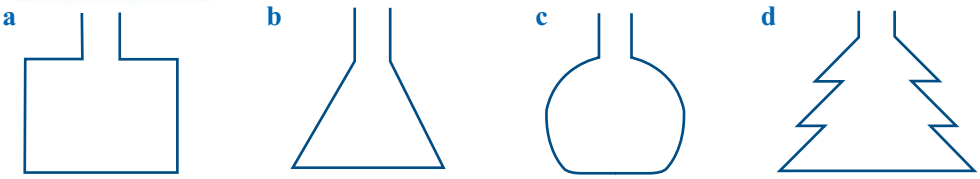
A connection can be made between the manner and the rate at which a function is changing and the slope or gradient of the graph representing the function.

15.1 Recognising relationships

Polynomial, exponential, logarithmic and circular functions have been considered in previous chapters, and it has been shown that many real-life situations may be modelled by these functions. In this first section several real situations concerning two particular variables are considered, and in particular the form of the relationships between the variables through graphs. The algebraic relationship is not established.

Example 1

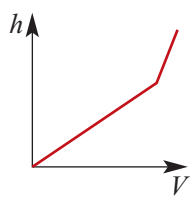
a **b** **c** **d**



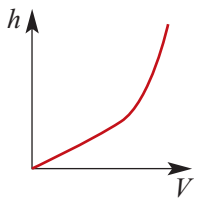
Water is being poured steadily into each of these vessels.
Draw a graph that shows the relationship between the height of the water (h) and the volume (V) that has been poured in.

Solution

a



b



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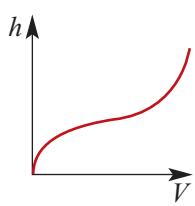

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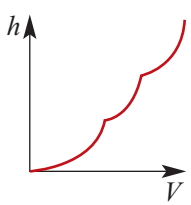
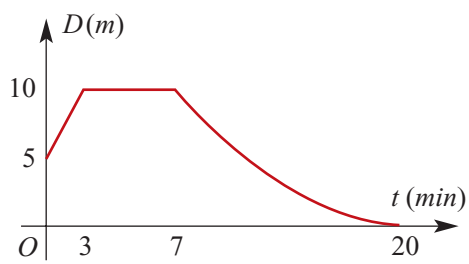


Example 2

The graph shows the displacement, D metres, of a particle from a fixed point O, over a period of 20 minutes.

Describe the motion of the particle.

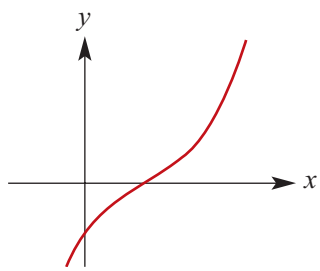


Solution

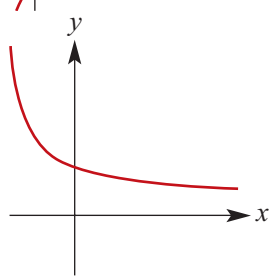
The object is initially 5 m from O. It travels away from O for 3 minutes at a constant speed of $\frac{5}{3}$ m/min. It then remains stationary at a distance of 10 m from O for 4 minutes before returning to O at a speed which is gradually decreasing so that it comes to rest at O at time $t = 20$ minutes.

By examining the graph representing a function, it can be determined whether the rate of change is positive, negative or, in fact, neither.

If the graph of y against x for a function with rule $y = f(x)$, for a given domain, shows an increase in y as x increases, it can be said that the **rate of change** of y with respect to x is **positive** for that domain. It is also said that the function is increasing for that domain. The 'slope' of the curve is positive.



If the graph of y against x for a function with rule $y = f(x)$, for a given domain, shows a decrease in y as x increases, it can be said that the **rate of change** of y with respect to x is **negative** for that domain. It is also said that the function is decreasing for that domain. The 'slope' of the curve is negative.



If y remains the same value as x changes, the corresponding graph is a horizontal line and the rate of change of y with respect to x is said to be **zero**.

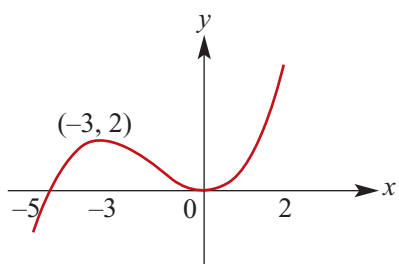
All of this is consistent with the gradient (rate of change) of a linear function, which was discussed in Chapter 2.

Example 3

For the graph shown below for $x \in [-5, 2]$, describe, using interval notation, the set of values of x for which:

a the value of y is increasing as x is increasing

b the rate of change of y with respect to x is positive.



Solution

a The value of y is increasing for $[-5, -3) \cup (0, 2]$.

b The rate of change of y with respect to x is positive over the intervals $[-5, -3) \cup (0, 2]$.

Exercise 15A

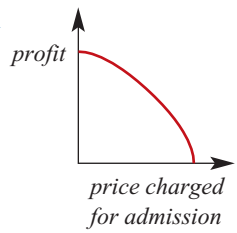
For questions 1–4 there may not be a single correct answer. Your written explanations are an important part of the exercise.

1 The manager of a theatre wishes to know what effect changing the price of admission will have on the profit he makes.

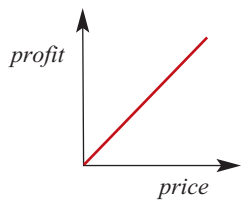
a Which one of the following graphs would show the effect of change.

b Explain your choice, including comments on scales and axes and what the point of intersection of the axes represents.

A



B



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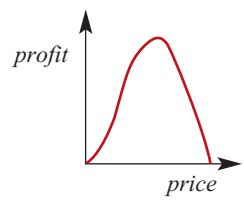
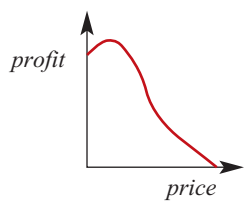
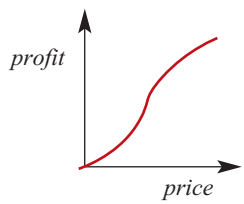


D



E



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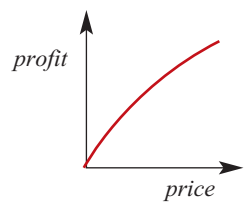
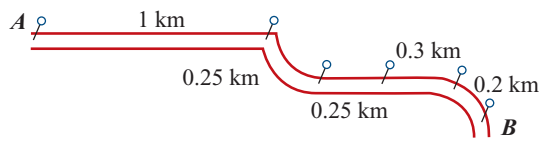


2 Sketch a graph to show how the height of a person might vary with age.

3 A motorist starts a journey at the point marked *A* on a country road, drives 2 km along the route shown, and stops at a point marked *B*. He is able to drive at 100 km/h but must slow down at corners.

a Explain briefly how the car's speed varies along the route.

b Sketch a graph showing how the car's speed varies along the route.



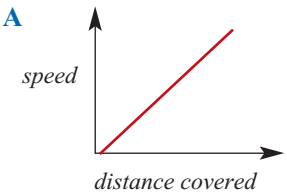
(Label the vertical axis 'Car's speed', and the horizontal axis 'Distance from *A*'.)

4 An athlete is a competitor in a 10 000 m race. Below are a selection of graphs which could show the relationship between the speed of the runner and the distance covered.

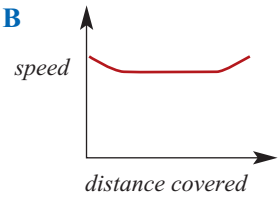
a Explain the meaning of each graph in words.

b Which graph is the most realistic for a winning athlete?

A



B



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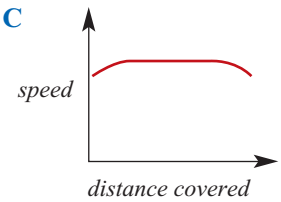
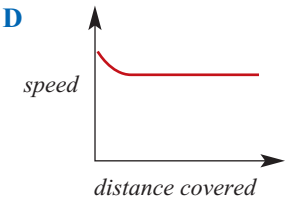


D



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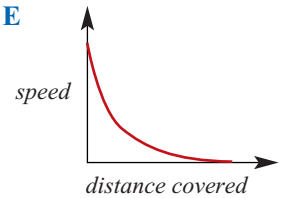

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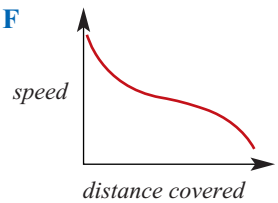


If you do not think any of these graphs are realistic draw your own version and explain it fully.

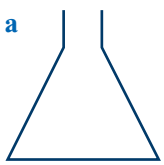
5 A sprinter covers 100 metres at a constant speed of 10 m/s. Sketch:

a the distance–time graph **b** the speed–time graph

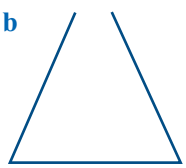
Example 1

6 Water is being poured steadily into these vessels.

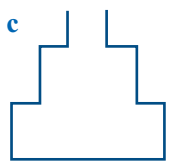
a



b



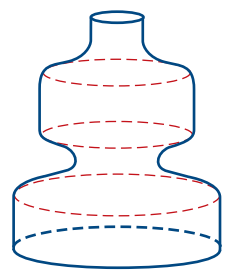
c



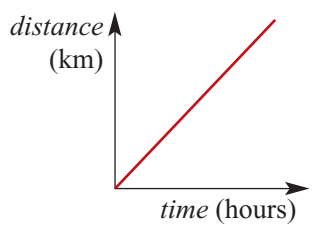
d

Draw a graph that shows the relationship between the height of the water and the volume that has been poured in.

7 For the vessel shown, sketch a reasonable curve for the volume, V, of water in the vessel as a function of the height, h, of the water in the vessel.



8 The graph relating the distance a car travels to the time taken is a straight line as shown. The graph shows that the car is:



A speeding up

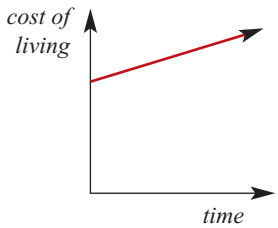
B slowing down

C travelling uphill

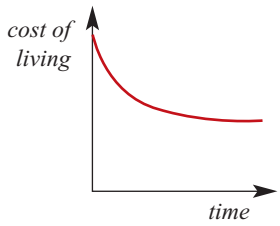
D travelling at a constant speed **E** stationary

9 Which one of these graphs best represents the rate of cost of living slowing down?

A



B



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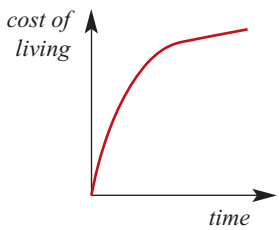
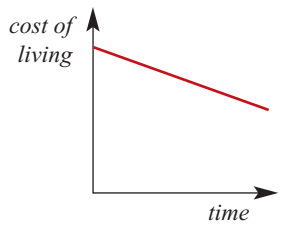


D

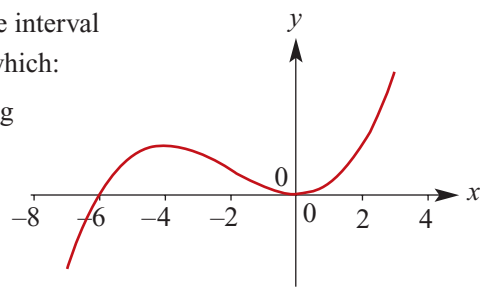


Example 3

10 For the graph shown below for $x \in [-7, 3]$, use interval notation to describe the set of values of x for which:

a the value of y is increasing as x is increasing

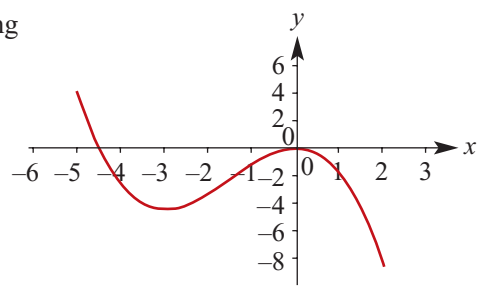
b the rate of change of y with respect to x is positive.



11 For the graph shown below for $x \in [-5, 2]$, describe, using interval notation, the set of values of x for which:

a the value of y is decreasing as x is increasing

b the rate of change of y with respect to x is negative.



15.2 Constant rates of change

Any function that is linear will have a **constant** rate of change. That rate of change is simply the gradient of the graph and, given appropriate information, the rate can be calculated from the graph of the function or simply read from the rule of the function if it is stated.

Example 4

A car travels from Copahunga to Charlegum, a distance of 150 km, in 2 hours (120 minutes). If the car travels at a constant speed, draw a distance–time graph and calculate the speed.

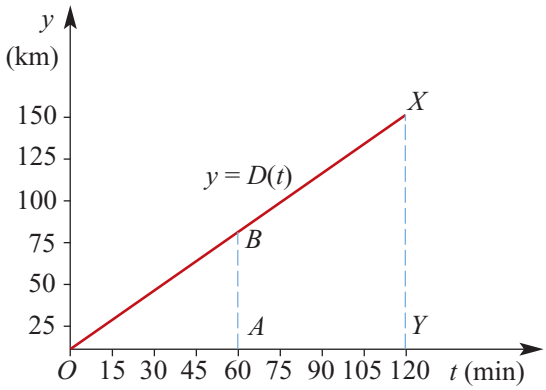
Solution

We denote the function by D.
The graph of the function is shown.

The rule of the function may be written:

$$D(t) = \frac{150}{120}t = \frac{5}{4}t$$

Note that $\frac{XY}{YO} = \frac{BA}{AO} = \frac{5}{4}$



The gradient of the graph gives the speed in kilometres per minute.
Therefore, the speed of this car is $\frac{5}{4}$ kilometres per minute $\left(\frac{5}{4}\text{ km/min}\right)$.
This speed may be expressed in kilometres per hour (km/h).

$$\text{Speed (km/h)} = \frac{5}{4} \times 60 = 75$$

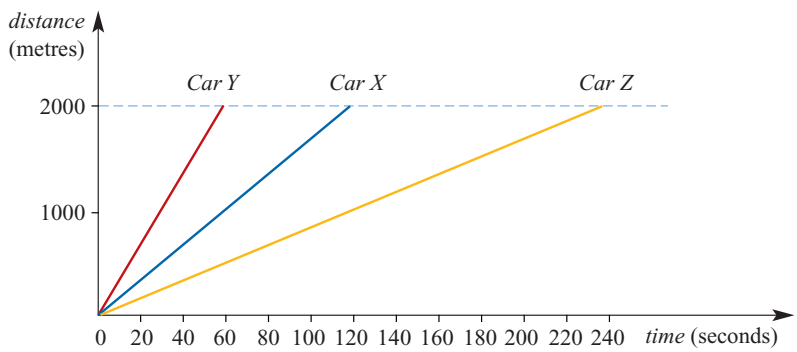
Example 5

Three cars are driven over a 2-kilometre straight track. They are all to go from point A to point B. Each car travels with constant speed. It is not a race as:

- the speed of car Y is twice that of car X, and
- the speed of car Z is half that of car X.

Illustrate this situation with a distance–time graph. Assume that car X travels at 1 km/min.

Solution



Note:

The gradient of the graph for car X is $\frac{2000}{120} = 16\frac{2}{3}$

The gradient of the graph for car Y is $\frac{2000}{60} = 33\frac{1}{3}$

The gradient of the graph for car Z is $\frac{2000}{240} = 8\frac{1}{3}$

An object whose motion can be described by a linear distance–time graph is travelling at a constant speed equal to the gradient of the linear graph.

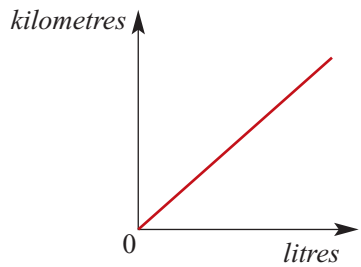
There are many other examples, as well as constant speed motion, in which a real-life situation is usefully modelled by a straight line graph in such a way that the gradient of the graph is meaningful.

In all these situations the gradient of the straight line graph represents a **rate**.

Examples include:

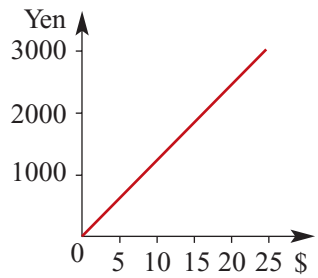
i *Petrol consumption of a car*

A straight line graph has been used as a model, and its gradient represents the rate of consumption of petrol, which is measured in km per litre. Such a model makes fairly large assumptions. What are these?



ii *The exchange rate for currencies*

The gradient of the graph gives the exchange rate of dollars to yen.



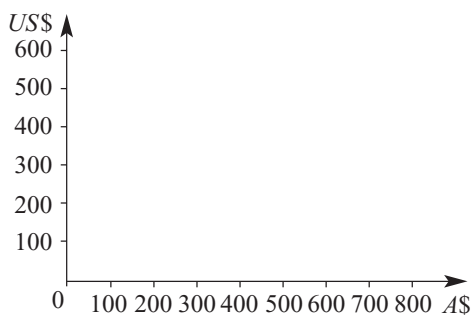
Exercise 15B

Example 4

1 A car travels from Bombay to Pune, a distance of 200 km, at a constant speed. The journey takes 150 minutes. Draw a distance–time graph and calculate the speed.

2 The exchange rate for the Australian dollar in terms of the American dollar is A$1 = US$0.75.

Draw a straight line graph that illustrates this relationship. The axes should be as shown.



3 Find the speed for each of the following (assume constant speed):

a Distance travelled 120 km, time taken 2 hours

b Distance travelled 60 m, time taken 20 seconds

c Distance travelled 8000 m, time taken 20 minutes

d Distance travelled 200 km, time taken 5 hours 40 minutes

e Distance travelled 6542 m, time taken 5 minutes 20 seconds

4 Find the rate of flow from the following taps in litres/min:

a A tap which fills a 40 litre drum in 5 minutes

b A tap which fills a 600 litre tank in 12 minutes

c A tap which takes 17 minutes to fill a 200 litre tank

d A tap which takes 17 minutes 20 seconds to fill a 180 litre tank

5 Water comes out of a tap at the rate of 15 litres/min.

a Copy and complete this table showing the amount which has come out at time t:

Time in minutes t	0	0.5	1	1.5	2	3	4	5
Amount in litres A	0							

b Draw a graph from the table.

6 A worker is paid \$200 for 13 hours work. What is their rate of pay per hour?

7 An aircraft travelling at a constant speed took 24 seconds to travel 5000 metres. What was the speed of the plane in metres per second?

8 A spherical balloon is blown up so that its volume is increasing by 8 cm^3 every second. Sketch a graph to show how the volume of the sphere changes with time.

15.3 Non constant rate of change and average rate of change

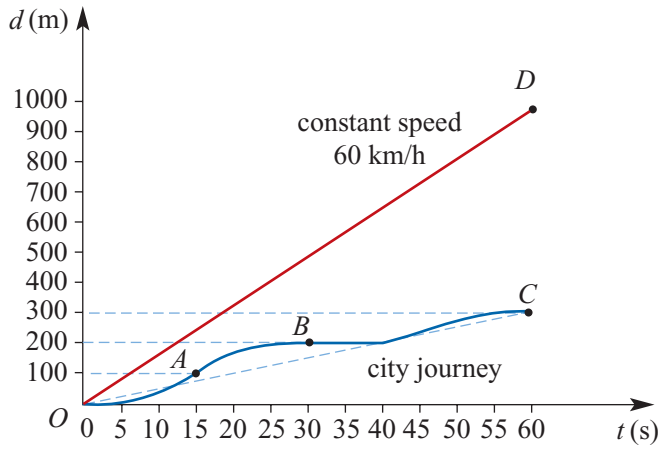
Constant speed is not the only form of motion for an object. Many moving objects do not travel with constant speed. For example, the speedometer of a car being driven in city traffic rarely stays still for long.

Similarly not all functions are linear so not all functions have constant rates of change. For a function that is nonlinear the rate of change of the function varies and may in fact be different for every different point on the graph representing the function.

Before exploring ways of finding the gradient at every different point on the graph, the 'average gradient' of a function or the 'average rate of change' of a function over a specified interval will be considered.

Again a displacement–time graph will be used to illustrate this idea.

The straight line graph through D shows a constant speed of 60 km/h. By comparison, the graph through points A, B and C shows a motorist travelling at varying speeds. The motorist accelerates to reach 60 km/h at A before slowing for the lights at B, where there is a ten-second standstill. There is then another short burst of speed before another standstill at C.



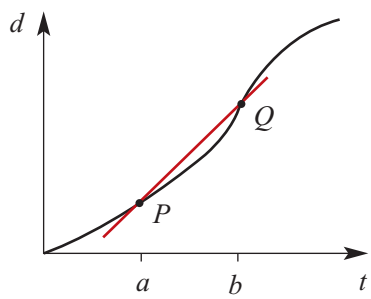
Although we do not know the actual speed of the car at any particular time other than when it is stationary, we can work out the average speed of the car travelling in the city over the full 60 seconds.

The average speed is given by $\dfrac{\text{distance travelled}}{\text{time taken}} = \dfrac{300}{60}$, which gives an average speed of 5 metres per second. The average speed is the gradient of the line OC.

The average may also be calculated for any given time interval. For example, for the time interval $t = 15$ to $t = 30$, the average speed is given by the gradient of the line joining points A and B.

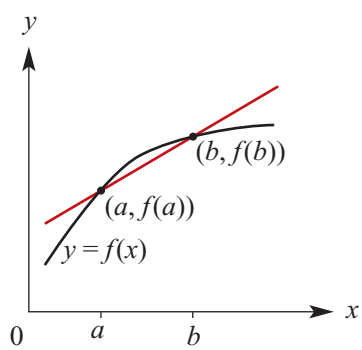
This is $\dfrac{100}{15} = 6\dfrac{2}{3}$ metres per second.

In general, average speed $= \dfrac{\text{total distance travelled}}{\text{total time taken}}$, so the average speed of an object for $a \leq t \leq b$ is given by the gradient of the chord PQ (secant PQ, which is the line passing through points P and Q).



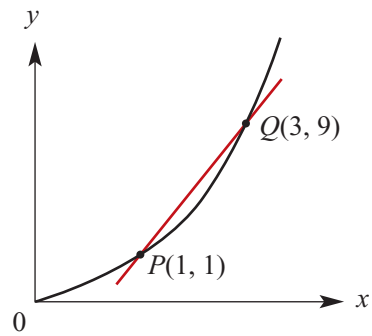
In general, for any function with rule $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the line joining $(a, f(a))$ to $(b, f(b))$,

i.e. average rate of change $= \dfrac{f(b) - f(a)}{b - a}$.



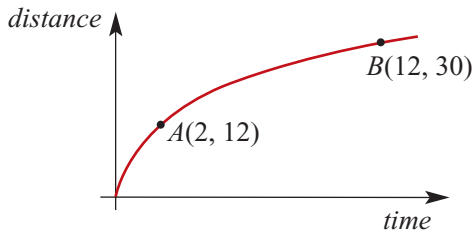
For example, the average rate of change of y with respect to x over the interval [1, 3] for the function with the graph shown is given by the gradient of PQ:

$$\text{gradient} = \frac{9-1}{3-1} = 4$$



Example 6

The distance travelled (metres) against time (seconds) graph for the motion of an object is shown. Find the average speed of the object in m/s over the interval from $t = 2$ to $t = 12$.



Solution

$$\begin{aligned}\text{Average speed} &= \frac{\text{total distance travelled}}{\text{total time taken}} \\ &= \frac{30-12}{12-2} \\ &= \frac{18}{10} \\ &= 1.8 \text{ m/s}\end{aligned}$$

Example 7

Find the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

Solution

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

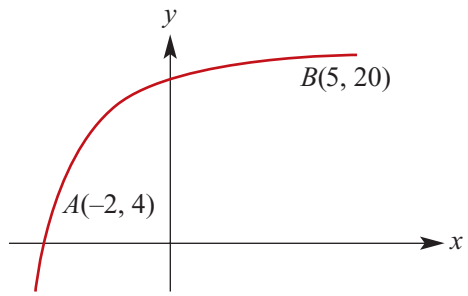
$$f(1) = (1)^2 - 2(1) + 5 = 4$$

$$f(5) = (5)^2 - 2(5) + 5 = 20$$

$$\begin{aligned}\text{Average rate of change} &= \frac{20-4}{5-1} \\ &= 4\end{aligned}$$

Example 8

Find the average rate of change of the function depicted in the graph below for the interval $[-2, 5]$



Solution

Average rate of change for the interval $[-2, 4]$

$$= \frac{20 - 4}{5 - (-2)}$$

$$= \frac{16}{7}$$

Example 9

The air temperature T (°C) at a weather station on a particular day is modelled by the equation $T = \dfrac{600}{t^2 + 2t + 30}$, where t is the time after 6.00 pm.

a Find the temperature at 6.00 pm. **b** Find the temperature at midnight.

c Find the average rate of change of the air temperature from 6.00 pm till midnight.

Solution

a At 6.00 pm $t = 0$

Hence $T = \dfrac{600}{(0)^2 + 2(0) + 30} = 20°\text{C}$

b At midnight, $t = 6$

Hence $T = \dfrac{600}{(6)^2 + 2(6) + 30} = 7.69°\text{C}$ (correct to 2 decimal places)

c Average rate of change of temperature $= \dfrac{7.69 - 20}{6 - 0} = -2.05\,°\text{C/hour}$

Exercise 15C

Example 7

1 For each of the following functions, find the average rate of change over the stated interval:

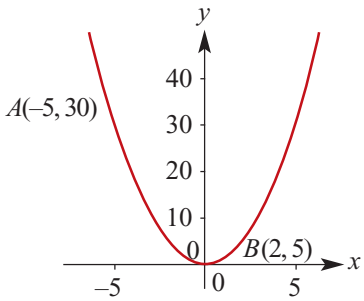
a $f(x) = 2x + 5, \quad x \in [0, 3]$ **b** $f(x) = 3x^2 + 4x - 2, \quad x \in [-1, 2]$

c $f(x) = \dfrac{2}{(x - 3)} + 4, \quad x \in [4, 7]$ **d** $f(x) = \sqrt{5 - x}, \quad x \in [0, 4]$

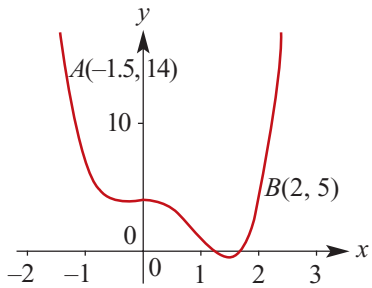
Example 8

2 Find the average rate of change of y with respect to x from point A to point B for each of the following graphs.

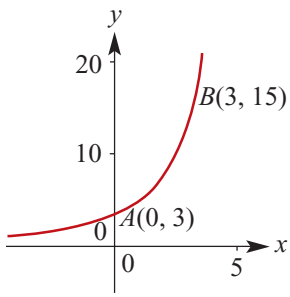
a



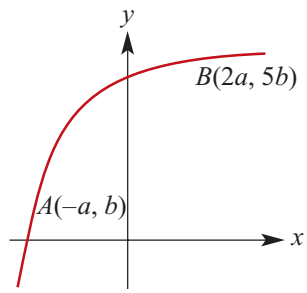
b



c



d



3 The displacement, S m, of an object t seconds after it starts to move is given by the function $S(t) = t^3 + t^2 - 2t, t > 0$.
Find the average rate of change of displacement (speed) of the object:

a in the first 2 seconds **b** in the next 2 seconds

4 A person invests $2000 dollars, which will increase in value by 7% per year for the next 3 years.

a Calculate the value of the investment after 3 years.

b Calculate the average rate of change in the value of the investment over that time.

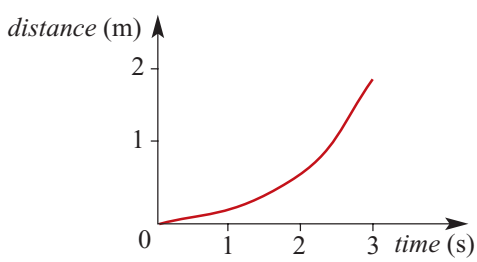
Example 9

5 The depth, d cm, of water in a bath tub t minutes after the tap is turned on is modelled by the function $d(t) = \dfrac{-300}{(t+6)} + 50, t > 0$.
Find the average rate of change of the depth of the water in the tub over the first 10 minutes after the tap is turned on.

6 In the graph the average speed from $t = 0$ to $t = 3$ is

A 2 m/s **B** 1 m/s

C $\frac{2}{3}$ m/s **D** $1\frac{1}{2}$ m/s



7 Two cars start together and travel with constant speed over a 1 kilometre straight track. Car 1 has speed 60 km/h; car 2 travels at $\frac{3}{4}$ of this speed.
Illustrate this situation with distance–time graphs for both cars on the one set of axes.

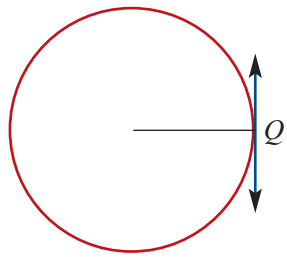
15.4 Finding the gradient of a curve at a given point

In the previous section the average rate of change of a function over a stated interval has been considered. It has also been established that, in general (except for linear functions), the rate of change of a function is different at each different point on the graph of the function. If we are to fully analyse how a function is behaving we need to be able to establish the rate of change of the function at every given point.

Defining tangent

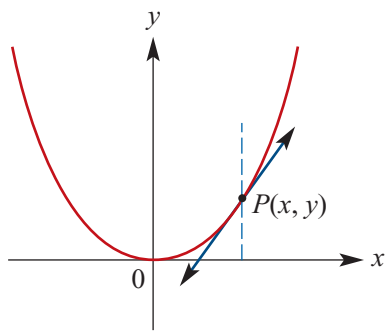
For a circle a tangent is a line which intersects the circle at one and only one point.

There are other curves for which tangency can be defined in this way, e.g. an ellipse. However, for some very simple curves, the tangent cannot be described by the definition we use to describe tangents to circles.



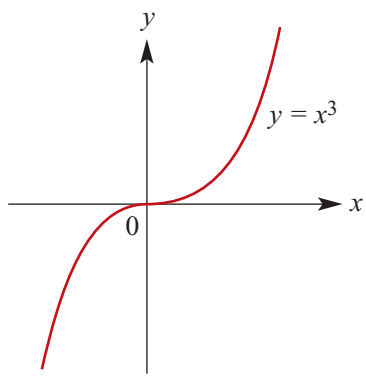
Consider, for example, a parabola.

The tangent at P intersects the graph only at (x, y). But the line parallel to the y-axis through P also has this property. We may try to get around this problem by indicating that the tangent line must only touch the curve, but does not cross it.



For many curves, however, this is not true.

The tangent to this curve at the point (0, 0) is the x-axis and it certainly crosses the curve.

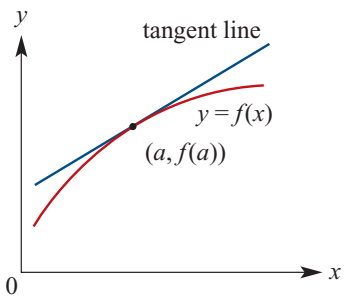


To define tangent adequately for all curves demands mathematics which we will meet in the following chapters. For the present we state that the tangent at a point P of a curve is a line through P which has a gradient that is the same as the gradient of the curve at P.

The rate of change at a given point can be determined by finding the gradient of the tangent at that point.

We refer to this rate of change at a specific point as the **instantaneous rate of change**.

If $y = f(x)$, then the **instantaneous rate of change** of y with respect to x at the point $(a, f(a))$ is the gradient of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



Initially, the instantaneous rate of change at a given point will be estimated by drawing the tangent at the specified point and finding the gradient of this tangent. Any attempt to draw a tangent by hand is unlikely to be very accurate, so this technique will provide us with an approximation only.

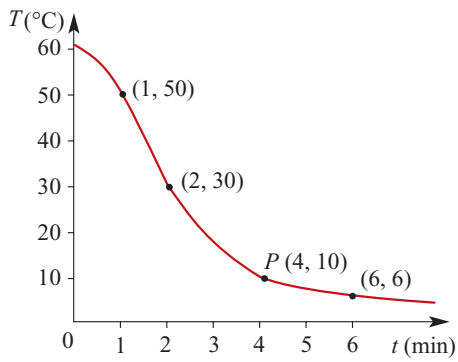
Example 10

This graph represents the temperature T (°C) of a kettle at time t (min) after taking it from the heat and putting it outside.

a What does the gradient represent?

b Find the average rate of change of temperature between $t = 2$ and $t = 6$.

c Draw the tangent at P and hence find an approximate value for the rate of change at $t = 4$.



Solution

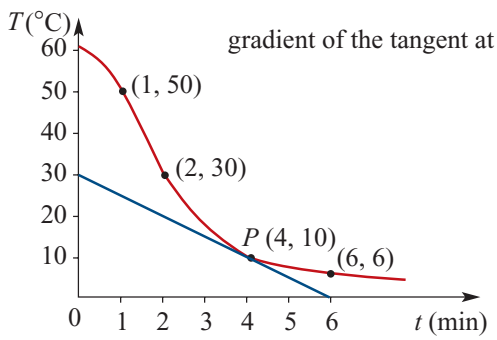
a The gradient represents the rate of increase of temperature with respect to time.

b Average rate of change $= \dfrac{6 - 30}{6 - 2}$

$= \dfrac{-24}{4}$

$= -6$ °C per minute

c

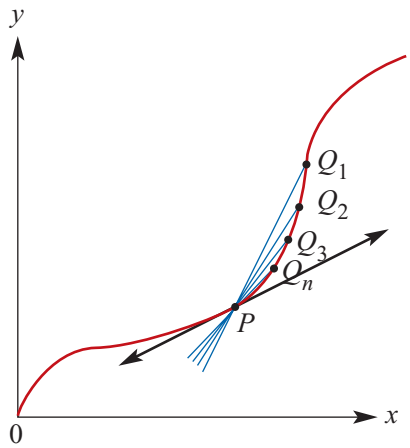


gradient of the tangent at $P = \dfrac{-30}{6}$

$= -5$

Another technique that may be used to find an approximation for the instantaneous rate of change is to use the gradient of a chord from the point in question, P, to another point on the curve, Q, that is very close to P. So the gradient of the chord PQ gives us an approximation to the gradient of the tangent at P, which is of course equal to the gradient of the curve at P.

We can consider chords $PQ_1, PQ_2, \ldots, PQ_n, \ldots$ As the point Q_n is chosen 'closer' to P it can be seen that the gradient of the the chord PQ_n is 'closer' to the gradient of the tangent at P. The gradient of the curve at P is the gradient of the tangent at P.

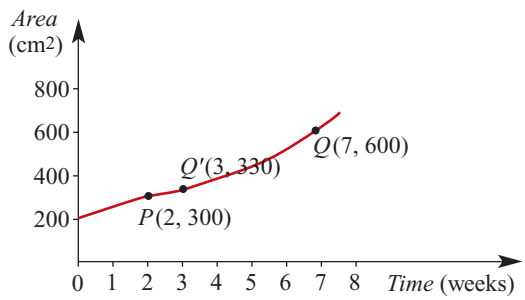


Example 11

The graph represents the area covered by a spreading plant. Area is measured in square cm and time in weeks.

a Find the gradient of the chord PQ.

b Q' has coordinates (3, 330). Find the gradient of PQ' and hence give an approximate value of the gradient of the graph at P.



Solution

a Gradient of chord $PQ = \dfrac{600 - 300}{7 - 2}$

$= \dfrac{300}{5}$

$= 60$

i.e. the average rate of increase of area for the plant from $t = 2$ to $t = 7$ is 60 cm^2 per week.

b Gradient of $PQ' = \dfrac{330 - 300}{1}$

$= 30$

$\therefore$ gradient at P is approximately 30

i.e. the rate of change of area with respect to time is approximately 30 cm^2 per week when $t = 2$.

Example 12

Estimate the gradient of the curve $y = x^3 + 1$ at the point (2, 9).

Solution

Consider the points $P(2, 9)$ and $Q(2.01, (2.01)^3 + 1)$.

$$\text{The gradient of } PQ = \frac{(2.01)^3 + 1 - 9}{2.01 - 2}$$
$$= 12.0601$$

An even better approximation may be made by choosing $P(2, 9)$ and $Q(2.001, (2.001)^3 + 1)$.

Example 13

By considering the chord joining the points where $x = 3$ and $x = 3.1$ estimate, without accurate drawing, the gradient of the curve $y = 2^x$ at $x = 3$.
Repeat for $x = 3.1$ and $x = 3.001$.

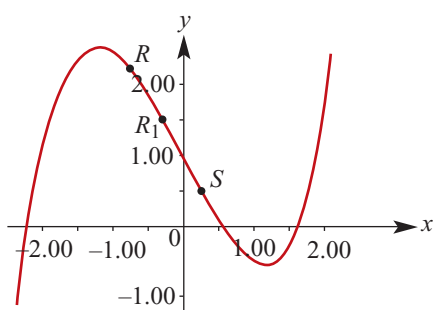
Solution

When $x = 3$, $y = 8$.
When $x = 3.1$, $y = 8.5742$ (correct to 4 decimal places).
The gradient of the line joining (3, 8) to (3.1, 8.5742) = 5.7419.
An estimate of the gradient of $y = 2^x$ at $x = 3$ is 5.742.
When $x = 3.001$, $y = 8.005547$ and the gradient = 5.547.
(Note the true gradient (to 4 decimal places) is 5.5452).

The above method of approximating the rate of change at a single point may be modified slightly.

The gradient of $y = 0.5x^3 - 2x + 1$ at the point (0, 1) will be investigated.

The graph of $y = 0.5x^3 - 2x + 1$ is shown.



First find the gradient of RS where $R = (-0.75, 2.2891)$, $S = (0.25, 0.5078)$.

Gradient of $RS = -1.7813$ (values of R and S were found to 4 decimal places).

Consider a new chord R_1S where $R_1 = (-0.25, 1.4922)$ and $S = (0.25, 0.5078)$.

Gradient of $R_1S = -1.9688$.

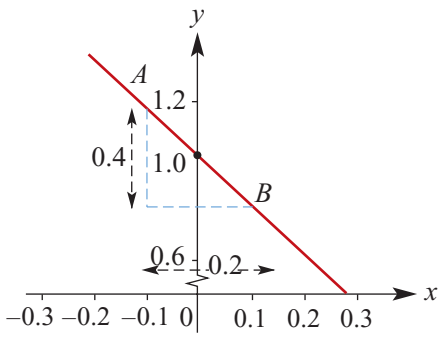
Zoom in on the region of the graph in the vicinity of $x = 0$, and it can be seen that this section of the graph appears increasingly linear. By assuming this section is in fact linear and

finding the gradient of this section of the curve, an approximation to the gradient of the curve at $x = 0$ can be found.

Consider a chord AB where $A = (-0.1, 1.1995)$ and $B = (0.1, 0.8005)$.

A particular section of the graph around (0, 1) can be plotted.

From this section of graph it can be seen that the gradient of the section of the curve being considered is -2, and the final approximation can be made that the gradient of the curve $y = 0.5x^3 - 2x + 1$ is -2 at the point with coordinates (0, 1).

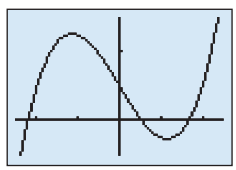


This method may be used without even sketching the graph.

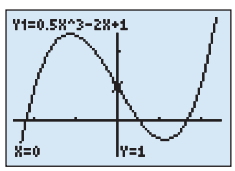
Using a graphics calculator

A graphics calculator may be used in this process as shown.

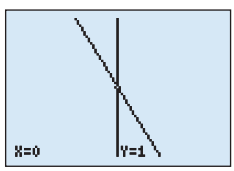
The graph of $y = 0.5x^3 - 2x + 1$ is plotted in a suitable window.



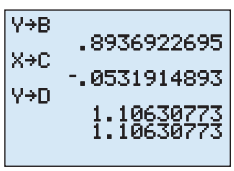
The process is to zoom in around the point (0, 1). Use **TRACE** to take the cursor to the point (0, 1).



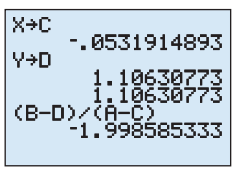
Choose **2:Zoom In** from the **ZOOM** menu and press ENTER. Take the cursor to the point (0, 1) and again choose **2:Zoom In** and press ENTER.



Choose two points on this section of the curve through (0, 1) and save the values as (A, B) and (C, D).



Hence evaluate $\dfrac{B - D}{A - C}$.



Better approximations may be found by taking points closer to (0, 1).

Exercise 15D

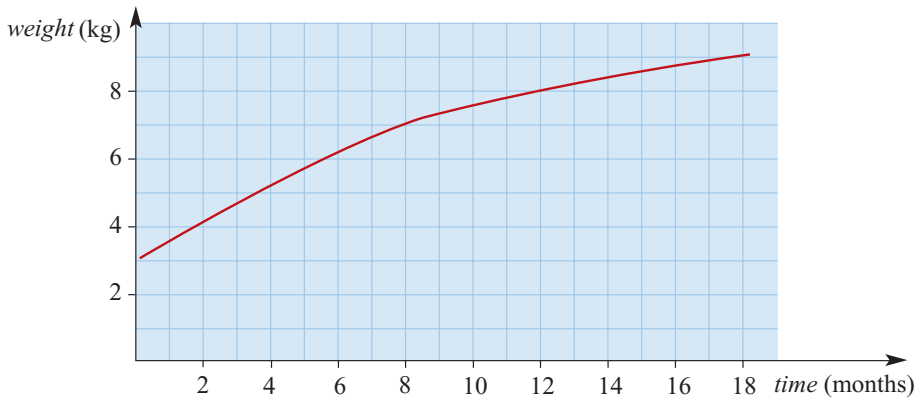
Example 10

1 The graph shown below is the weight–time graph for the growth of a baby boy measured at 2-monthly intervals for the first 18 months of his life. By drawing a tangent at the appropriate points:

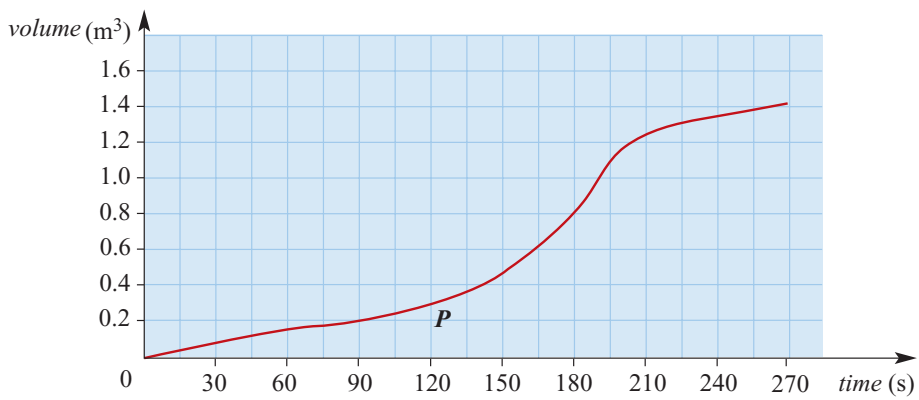
a estimate the average rate of growth of the baby over these 18 months

b estimate the rate of growth of the baby at 6 months

c estimate the rate of growth of the baby at 15 months.



2 A building is heated by gas. The volume–time graph for a 270-second period is shown:



a Place a rule along the tangent at P and hence find the rate of use of gas at time $t = 120$.

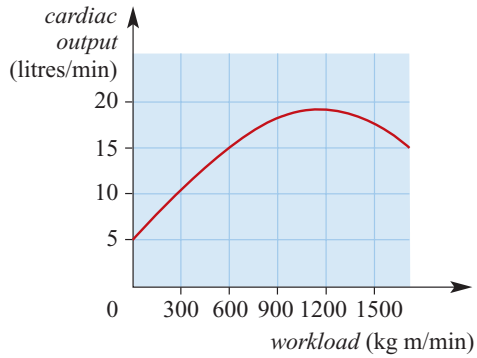
b Find the rate of use at $t = 160$.

c Find the rate of use at $t = 220$.

3 Cardiac output is an important factor in athletic endurance. The graph shows a stress-test graph of cardiac output (measured in litres/min of blood) versus workload (measured in kg m/min).

a Estimate the average rate of change of cardiac output with respect to workload as the workload increases from 0 to 1200 kg m/min.

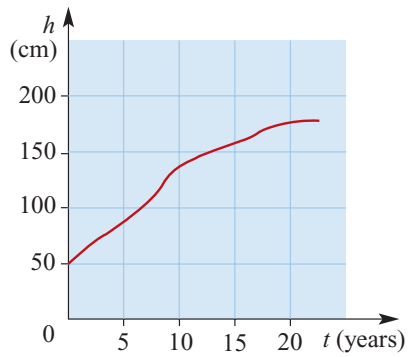
b Estimate the instantaneous rate of change of cardiac output with respect to workload at the point where the workload is 450 kg m/min.



4 The graph of a person's height h (cm) versus t (years) from some time after birth to age 20 is shown.

a When is the growth rate greatest?

b Estimate the growth rate after 5 years.

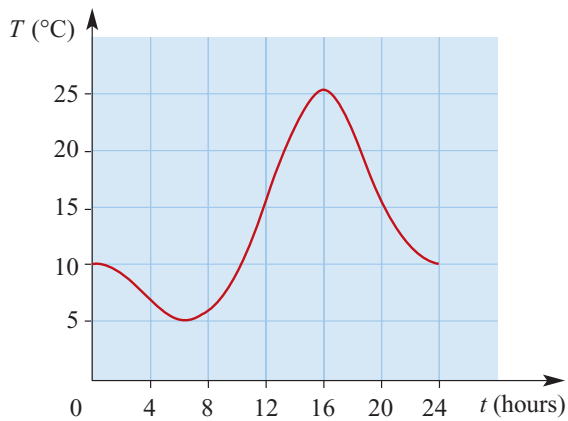


5 Temperature (T°C) varies with time (t hours) over a 24-hour period, as illustrated in the graph.

a Estimate the maximum temperature and the time at which this occurs.

b The temperature rise between 10.00 and 14.00 is approximately linear. Estimate the rate at which the temperature is increasing in this period.

c Estimate the instantaneous rate of change of temperature at $t = 20$.



6 By considering the chord joining the points at which $x = 1.2$ and $x = 1.4$, estimate without accurate drawing the gradient of the curve $y = \frac{1}{x}$ at $x = 1.3$.

Example 12

7 Draw the graph of $y = \sqrt{16 - x^2}$, $-4 \leq x \leq 4$. Use an appropriate technique to find an estimate for the gradient of the curve at the points:

a $x = 0$ **b** $x = 2$ **c** $x = 3$

8 It is known that the straight line $y = 4x - 4$ touches the curve $y = x^2$ at the point (2, 4). Sketch the graphs of both of these functions on the one set of axes.
Find the gradient of $y = x^2$ at (2, 4).

9 Water is being collected in a water tank. The volume, V cubic metres, of water in the tank after time t minutes is given by $V = 3t^2 + 4t + 2$.

a Find the average rate of change of volume between times $t = 1$ and $t = 3$.

b Find an estimate for the rate of change of volume at $t = 1$.

10 A population of bacteria is growing. The population, P million, after time t minutes is given by $P = 3 \times 2^t$.

a Find the average rate of change of population between times $t = 2$ and $t = 4$.

b Find an estimate for the rate of change of population at $t = 2$.

11 Water is flowing out of a water tank. The volume, V cubic metres, of water in the tank after t minutes is given by $V = 5 \times 10^5 - 10^2 \times 2^t$, $0 \leq t \leq 12$.

a Find the average rate of change of volume between times $t = 0$ and $t = 5$.

b Find an estimate for the rate of change of volume when $t = 6$.

c Find an estimate for the rate of change of volume when $t = 12$.

12 By considering the chord joining the points where $x = 1.2$ and $x = 1.3$ estimate the gradient of the curve $y = x^3 + x^2$ at $x = 1.3$.

13 Use the technique of examples 12 and 13 to estimate the gradients of each of the following at the stated point:

a $y = x^3 + 2x^2$, (1, 3) **b** $y = 2x^3 + 3x$, (1, 5)

c $y = -x^3 + 3x^2 + 2x$, (2, 8) **d** $y = 2x^3 - 3x^2 - x + 2$, (3, 26)

14 The volume, V, of a cube with edge length x is given by $V = x^3$.

a Find the average rate at which the volume of the cube changes with respect to x as x increases from $x = 2$ to $x = 4$.

b Find the instantaneous rate at which V changes with respect to x when $x = 2$.

15 Let $y = 2x^2 - 1$.

a Find the average rate at which y changes with respect to x over the interval [1, 4].

b Find the instantaneous rate at which y changes with respect to x when $x = 1$.

16 Let $y = \sin x$:

a Find the average rate at which y changes with respect to x over each of the following intervals:

i $\left[0, \frac{\pi}{2}\right]$ **ii** $\left[0, \frac{\pi}{4}\right]$ **iii** [0, 0.5] **iv** [0, 0.1]

b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

17 Let $y = 10^x$:

a Find the average rate at which y changes with respect to x over each of the following intervals:

i [0, 1] **ii** [0, 0.5] **iii** [0, 0.1] **iv** [0, 0.01]

b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

15.5 Displacement, velocity and acceleration

One of the key applications of rates of change is in the study of the motion of a particle.

In this section only motion in a straight line is considered.

The **displacement** of a particle is a specification of its position.

Consider motion on a straight line with reference point O.

We say that displacement to the right of O is positive and to the left negative.

A particle is moving along the straight line, let x metres denote its displacement relative to O.

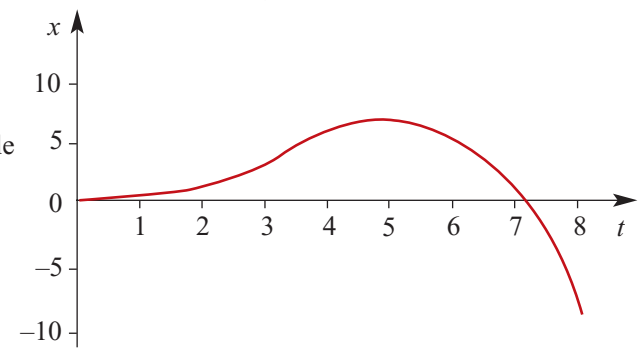
At time $t = 0$, $x = 0$ (time is measured in seconds)

At time $t = 5$, $x = 6.25$

At time $t = 8$, $x = -8.96$.

At $t = 0$ the particle starts from rest and moves to the right. At $t = 5$ the particle stops and moves back in the opposite direction. Its displacement–time graph is shown below.

Note that from $t = 0$ until $t = 7.1$ the displacement is positive; i.e. the position of the particle is to the right of O. For t greater than 7.1 the particle is to the left of O; it has negative displacement.

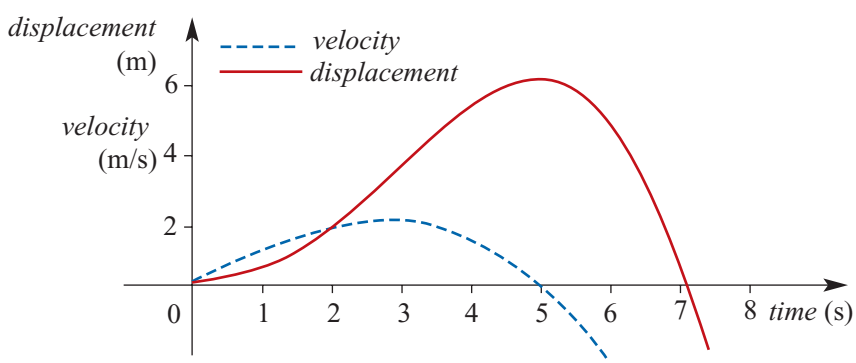


Velocity is the rate of change of displacement with respect to time.

$$\text{The } \textbf{average velocity} = \frac{\text{change in displacement}}{\text{time elapsed}}$$

For example, for the time interval $0 \leq t \leq 5$

$$\text{average velocity} = \frac{6.25}{5} = 1.25 \text{ metres per second (m/s)}$$



For the time interval $5 \leq t \leq 8$

$$\text{average velocity} = \frac{-8.96 - 6.25}{3}$$
$$= -5.07 \text{ metres per second (m/s)}$$

For $0 < t < 5$ the velocity is positive.

For $t > 5$ the velocity is negative.

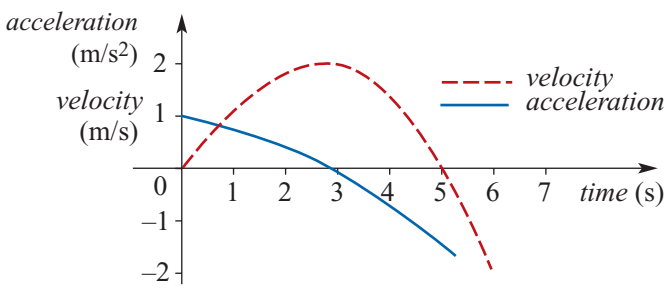
The negative velocity indicates that the particle is travelling from right to left. The relationship between displacement and velocity is illustrated in the preceding graph. The vertical axis is in both metres per second for velocity and metres for displacement.

Note that when $t = 5$ the velocity is 0.

Acceleration is the rate of change of velocity with respect to time. We show both the velocity–time and the acceleration–time graphs on the one set of axes. The units of acceleration are metres per second per second and we abbreviate this to m/s^2.

Note that the acceleration is 0 for $t \approx 2.98$ and this occurs when the velocity changes from increasing to decreasing.

The instantaneous velocity of the particle at time t is given by the gradient of the displacement–time graph for the point corresponding to time t.



Example 14

A particle is moving in a straight line. It was initially at rest at a point O. It moves to the right of O with a constant velocity and reaches a point A, 15 metres from O, after 5 seconds. It then returns to O. The return trip takes 10 seconds. It stops at O.

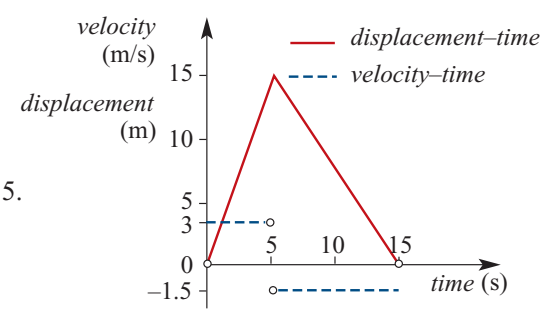
On the one set of axes draw the displacement–time graph and the velocity–time graph for the motion.

Solution

Note: The gradient of the displacement–time graph for $0 < t < 5$ is 3.

The gradient for $5 < t < 15$ is -1.5.

The gradient of the displacement–time graph determines the velocity–time graph.

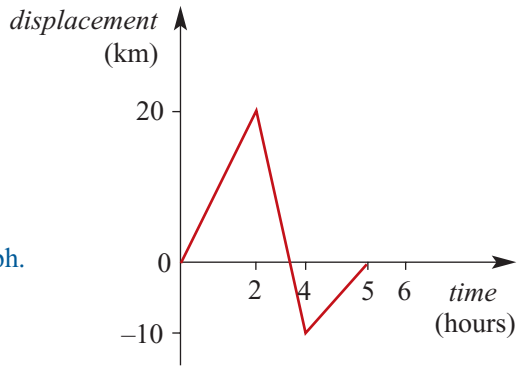


Example 15

The graph shown is the displacement–time graph that corresponds to the cycle trip of a boy who lives at the edge of a long straight road. The road runs north–south and north is chosen to be the positive direction.

a Describe his trip.

b Draw the corresponding velocity–time graph.



Solution

a The boy heads north for 2 hours.

His velocity for this period is $\frac{20-0}{2} = 10$ km/h.

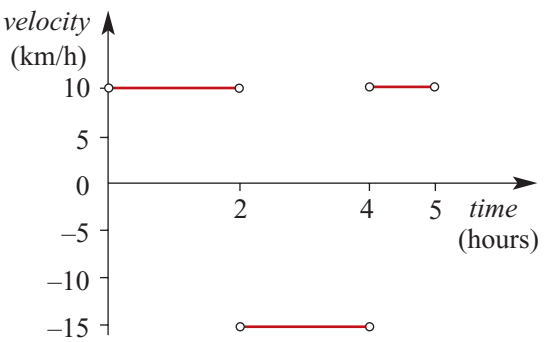
He turns and rides south for 2 hours.

His velocity for this period is $\frac{-10-20}{2} = -15$ km/h.

He turns and rides north until he reaches home.

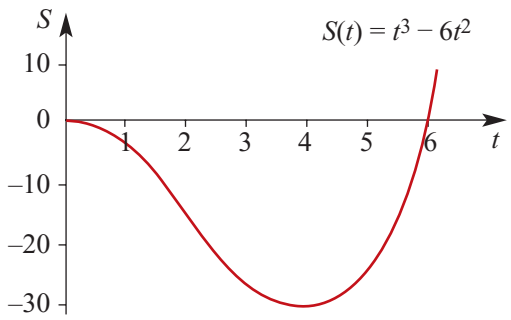
His velocity for this period is $\frac{0-(-10)}{1} = 10$ km/h.

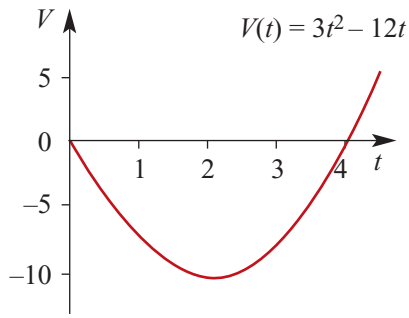
b



Example 16

The displacement of a particle moving in a straight line is given by the function $S(t) = t^3 - 6t^2$, $t \geq 0$. The graph of S against t is shown. The corresponding velocity–time graph is also shown. The function describing the velocity is $V(t) = 3t^2 - 12t$.





a Find the average velocity of the particle for the intervals:
i [3.5, 4.5] **ii** [3.9, 4.1] **iii** [3.99, 4.01]
b From part a what is the instantaneous velocity when $t = 4$?
c **i** For what values of t is the velocity positive?
ii For what values of t is the velocity negative?

Solution

a **i** $\text{Average velocity} = \dfrac{S(4.5) - S(3.5)}{1} = \dfrac{-30.375 + 30.625}{1} = 0.25$

ii $\text{Average velocity} = \dfrac{S(4.1) - S(3.9)}{0.2} = \dfrac{-31.939 + 31.941}{0.2} = 0.01$

iii $\text{Average velocity} = S(4.01) - S(3.99) = \dfrac{-31.999399 + 31.999401}{0.02}$
$= 0.0001$

b The results of part a give the instantaneous velocity as zero when $t = 4$ and this is consistent with the result suggested by both graphs.

c **i** From the displacement–time graph it can be seen that the velocity is positive for $t > 4$.

ii From the displacement–time graph it can be seen that the velocity is negative for $0 \leq t < 4$.

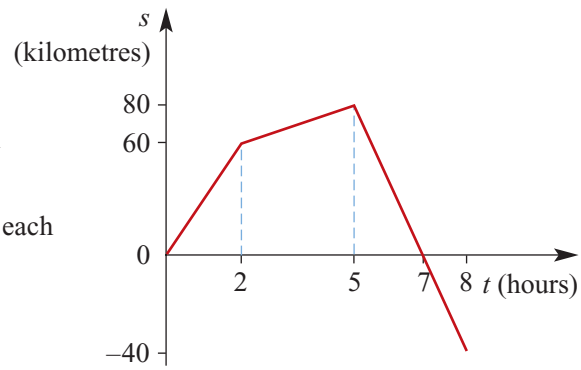
Exercise 15E

1 Let $s(t) = 6t - 2t^3$ be the displacement function of a particle moving in a straight line, where t is in seconds and s is in metres.

a Find the average velocity for the time interval [0, 1].
b Find the average velocity for the time interval [0.8, 1].
c Find an estimate for the instantaneous velocity for $t = 1$.
d State the values of t for which the displacement, s, is positive.
e State the values of t for which the velocity is positive.

Example 15

2 The following is the displacement–time graph for a train travelling on a straight track. Displacement is measured from the door of the station master's office at Jimbara station.



a What was the train's velocity over each of the following time intervals:

i [0, 2] **ii** [2, 5]

iii [5, 8]

b Describe the train journey.

c Draw a velocity–time graph for the train's motion for the interval [0, 8].

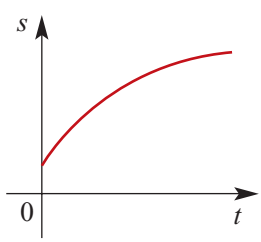
3 The motion of a particle moving in a straight line is described by the following information:

- For the time period (0, 2), velocity $= -3$.
- For the time period (2, 5), velocity $= 3$.
- For the time period (5, 7), velocity $= 4$.

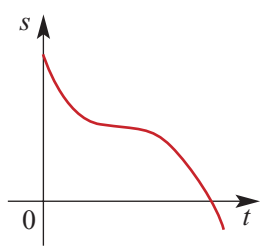
Draw the displacement–time graph for the interval [0, 7].

4 Match each displacement–time graph shown with its velocity–time graph.

a



b



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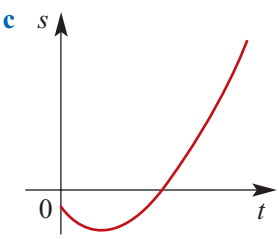
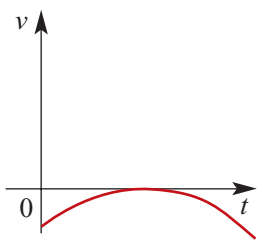
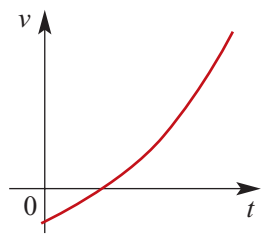


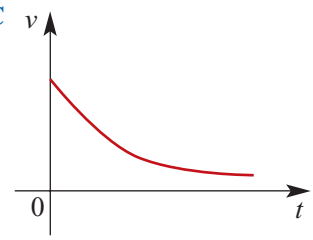
A



B

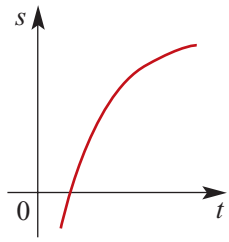


C

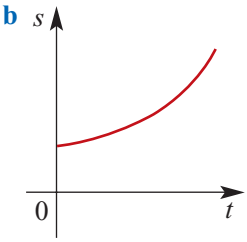


5 The graphs of four displacement–time graphs are shown. In each case determine the sign of the velocity and whether the particle is speeding up or slowing down.

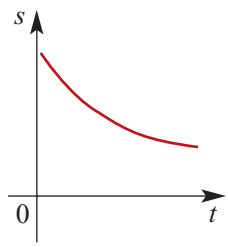
a



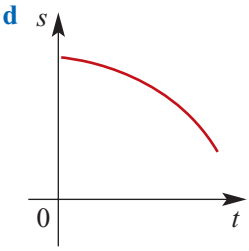
b



c



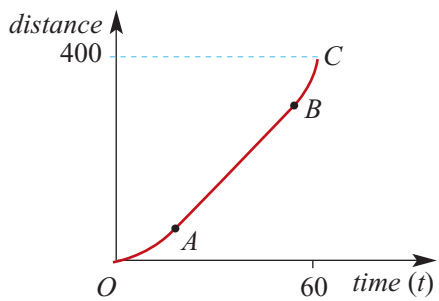
d



6 The distance–time graph for a 400-metre runner is shown. Explain what is happening between:

a O and A **b** A and B

c B and C



Example 16

7 The table shows the distance, d metres, of a ball from its starting position t seconds after being thrown into the air.

t	0	1	2	3	4	5	6
d	0	25	40	45	40	25	0

Use scales 2 cm = 1 second and 1 cm = 5 m and draw the graph of d against t.
From your graph find:

a when the ball returns to the starting point

b the average velocity of the ball from $t = 1$ to $t = 2$

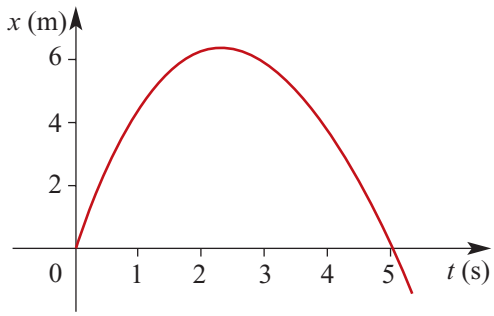
c the average velocity of the ball from $t = 1$ to $t = 1.5$

d an estimate of the velocity of the ball when $t = 1$

e an estimate of the velocity of the ball when $t = 4$

f an estimate of the velocity of the ball when $t = 5$.

8 A particle moves along a horizontal straight line. It starts from rest at a point O. The graph is the displacement–time graph for this motion.



a At what time is the velocity zero?

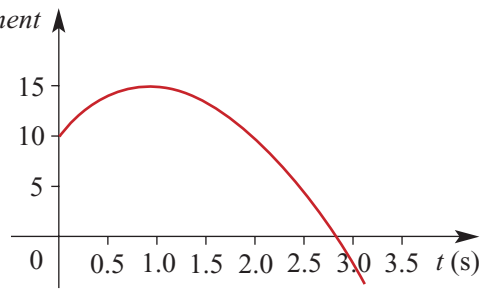
b For which values of t is the velocity positive?

c If to the right of O is taken as positive displacement, how far does the particle go to the right?

d How long does it take to return to O?

e Estimate the velocity of the particle at $t = 1$.

9 A stone is thrown vertically upwards from the edge of a platform which is 10 m above the ground. The displacement–time graph for the motion of the stone is shown. The motion of the stone is in a straight line and the reference point for displacement is taken as a point at ground level, directly below where the stone was thrown.



a From the graph estimate the velocity with which the stone is thrown.

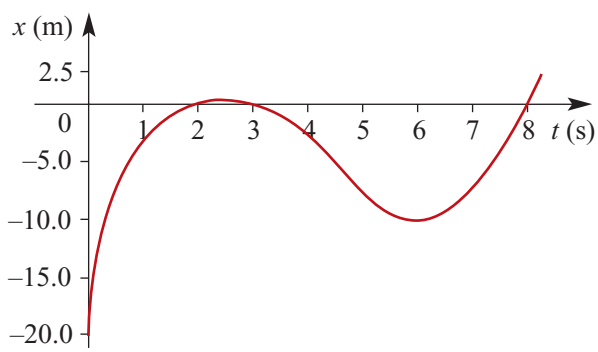
b What is the maximum height reached by the stone?

c At what time does the stone reach its maximum height?

d At what time does the stone hit the ground?

e From the graph estimate the speed at which the stone hits the ground.

10 A particle is moving in a horizontal straight line. Displacement is measured from a point O. The particle starts at a point 20 m to the left of O. The displacement–time graph for the motion of the particle is as shown.



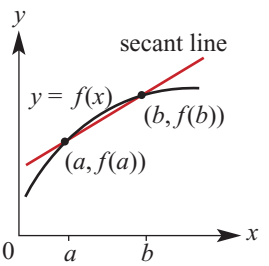
a At which times is the particle at O?

b For which values of t is the particle moving to the right?

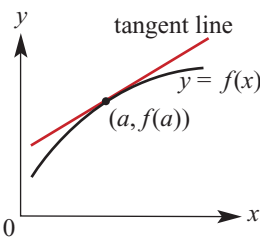
c For which values of t is the particle stationary?

Chapter summary

- Where a real-life situation is modelled by a straight line graph, the gradient represents a rate of change of one quantity with respect to a second.
- Average speed $= \dfrac{\text{total distance travelled}}{\text{total time taken}}$
- If $y = f(x)$, then the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the secant line joining $(a, f(a))$ to $(b, f(b))$.



- If $y = f(x)$, then the **instantaneous rate of change** of y with respect to x at the point $(a, f(a))$ is the gradient of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



Multiple-choice questions

1 A bushwalker walks 12 km in 2 hours, stops for 45 minutes and then walks a further 8 km in another 1.25 hours. The average walking speed of the bushwalker over the entire walk is

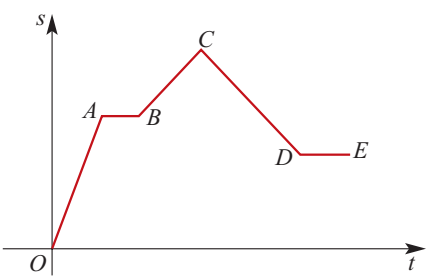
A 10 km/h **B** 9 km/h **C** 5 km/h
D 4 km/h **E** 7.2 km/h

2 Postal workers sort 12 000 letters during the normal day shift of 8 hours and, with a reduced workforce during the 2 hours overtime shift, they sort a further 2500 letters. The average rate of letter sorting per hour is

A 1375 letters per hour **B** 1450 letters per hour **C** 1300 letters per hour
D 1400 letters per hour **E** 1500 letters per hour

Questions 3 to 5 refer to the following information:

The graph shows the movement of a vehicle over a period of time. It represents the distance(s) from a fixed point at a given time (t).



Review

3 The line segment OA represents a stage of the movement during which the vehicle is

A speeding up **B** slowing down **C** travelling north
D travelling at a constant speed **E** stationary

4 The line segment AB represents a stage of the movement during which the vehicle is

A speeding up **B** slowing down **C** travelling east
D travelling at a constant speed greater than zero **E** stationary

5 The section(s) of the graph which represent(s) the vehicle when it is stationary is/are

A at O **B** at A and C **C** between C and D
D between A and B and D and E **E** at no time

6 The average rate of change of the function $y = 3 \times 2^x$ over the interval $[0, 2]$ is

A 9 **B** 4.5 **C** 12 **D** 6 **E** 5

7 The population of trout in a trout pond is growing. If the population P, after t weeks is given by $P = 10 \times 1.1^t$, the average rate of growth of the population during the 5^{th} week is closest to

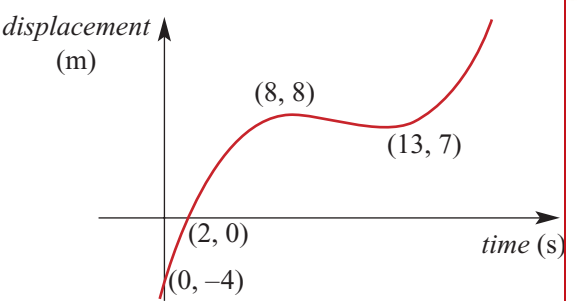
A 16 trout per week **B** 15 trout per week **C** 1.5 trout per week
D 4 trout per week **E** 15.35 trout per week

8 Given $f(x) = 2x^3 + 3x$, the average rate of change of $f(x)$ with respect to x for the interval $[-2, 2]$ is

A 0 **B** -22 **C** -11 **D** 22 **E** 11

Questions 9 to 10 refer to the following information:

A particle moves along a horizontal line. The graph of the particle's displacement relative to the origin over time is shown.



9 The particle has a velocity of zero at

A 8 s and 13 s **B** 2 s **C** 0 s **D** 8 s and 7 s **E** -4 s

10 The time interval(s) during which the particle has a negative velocity are

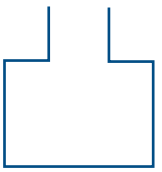
A $8 < t < 13$ **B** $0 < t < 2$ and $8 < t < 13$ **C** $0 < t < 2$
D $7 < t < 8$ **E** $(0, \infty)$

Short-answer questions (technology-free)

1 A liquid is poured at a constant rate into each of the containers shown below. For each container, draw a graph to show how the depth of the water varies with time.

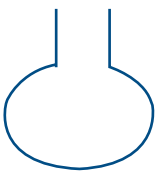
a

b



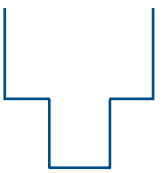
c

d



e

f



2 **a** A car travels from New Delhi to Agra, a distance of 200 km, in 3 hours. If the car travels at a constant speed, draw a distance–time graph and calculate this constant speed. For the distance–time graph, use minutes as the unit of time.

b A girl walks at a constant speed of 1.5 m/s for 20 seconds. Draw a distance–time graph to illustrate this.

c A car travels along a road for $6\frac{1}{2}$ hours. The trip can be described in three sections.

Section A Car travels at a constant speed of 40 km/h for 1 hour.

Section B Car travels at a constant speed of 80 km/h for $3\frac{1}{2}$ hours.

Section C Car travels at a constant speed of 90 km/h for 2 hours.

Draw a distance–time graph which illustrates this motion.

3 The surface area s cm^2 of a cube is given by the expression $s = 6x^2$, where x cm is the length of each edge of the cube. Find the average rate at which the surface area changes with respect to x, as x increases from $x = 2$ to $x = 4$.

4 Let $y = x^3$. Find the average rate at which y changes with respect to x over each of the following intervals:

a $[0, 1]$ **b** $[1, 3]$

5 Let $s(t) = 4t - 6t^3$ be the displacement function of a particle moving in a straight line, where t is in seconds and s is in metres.

a Find the average velocity for the time interval $[0, 1]$.

b Find the average velocity for the time interval $[0.9, 1]$.

c Estimate the instantaneous velocity for $t = 1$.

Review

Extended-response questions

1 The resistance of a copper wire is measured at various temperatures with the following results:

Temperature (°C)	10	20	30	40	50	60
Resistance (ohms)	23.4	23.9	24.4	24.9	25.5	26.0

a Plot the data to see whether the resistance rises approximately linearly with temperature.

b If it does, find the rate of increase in ohms per °C.

2 A rock is allowed to fall from the top of a high cliff. It falls y metres in t seconds, where $y = 4.9t^2$.

a Find the average speed of the rock between:

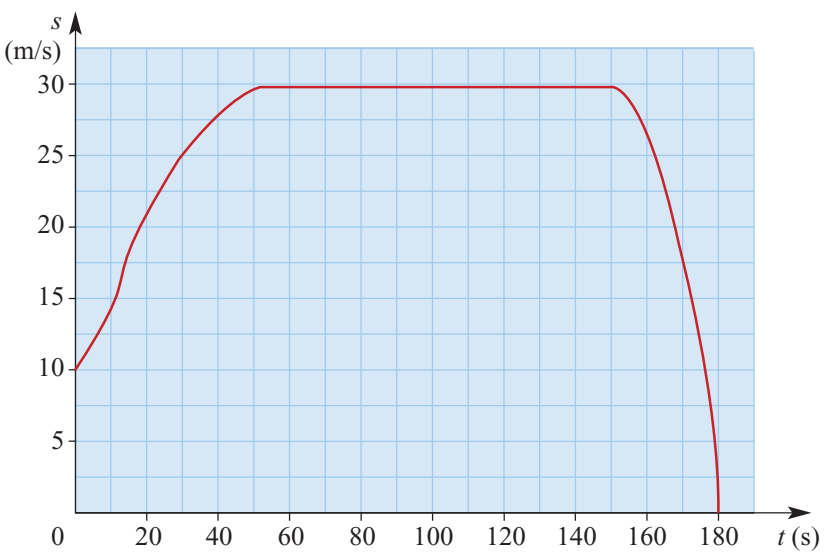
i $t = 0$ and $t = 2$ **ii** $t = 2$ and $t = 4$

b **i** How far has the rock fallen between $t = 4 - h$ and $t = 4$?

ii What is the average speed between $t = 4 - h$ and $t = 4$?

iii If it hits the bottom when $t = 4$, find the speed of impact by using the result of **ii** and taking $h = 0.2, 0.1, 0.05, 0.01, 0.001$.

3 The graph shows the speed s (in metres per second) of a car travelling along a straight highway at time t seconds measured from the time it crosses an intersection.



a Use the graph to show how you find:

i the average rate at which the car's speed is increasing between times $t = 30$ and $t = 50$

ii an estimate of the rate at which the car's speed is increasing when $t = 30$.

b Sketch a graph showing the rate of increase of speed against time for $t = 0$ to $t = 180$.

4 The table shows the girth (w cm) of a watermelon, n days after being fed with a fertiliser.

n (days)	1	2	3	4	5	6	7
w (cm)	15	17	20	24	29	35	41

a Use a scale of 2 cm for 1 day and 2 cm for 10 cm of girth to draw the graph illustrating the information.

b Find the gradient of the chord joining the points where $n = 3$ and $n = 7$ and interpret the result.

c From the same graph estimate the gradient of the curve at $n = 4$.

5 A vending machine in a bus terminus contains cans of soft drinks. On a typical day:

- the machine starts $\frac{1}{4}$ full
- no drinks are sold between 1.00 am and 6.00 am
- the machine is filled at 2.00 pm.

Sketch a graph to show how the number of cans in the machine may vary from 6.00 am to 12 midnight.

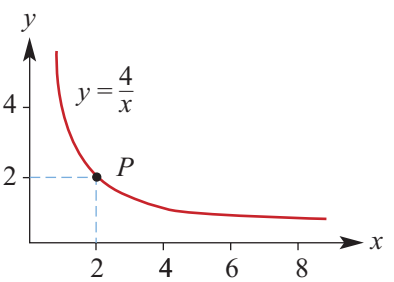
6 **a** $P(a, a^2)$ and $Q(b, b^2)$ are two points on the curve with equation $y = x^2$. Find the gradient of the line joining the points. (Answer in terms of a and b.)

b Use this result to find the gradient of the line for points with $a = 1$ and $b = 2$.

c Use this result to find the gradient of the line if $a = 2$ and $b = 2.01$.

7 The figure shows part of the curve with equation $y = \frac{4}{x}$ and P is the point at which $x = 2$.

a A_1 and A_2 are points on the curve whose x-coordinates are 1.5 and 2.5 respectively. Use your calculator to find their y-coordinates and hence the gradient of A_1A_2.



b Repeat for B_1 and B_2 whose x-coordinates are 1.9 and 2.1 respectively.

c Repeat for C_1 and C_2 whose x-coordinates are 1.99 and 2.01 respectively.

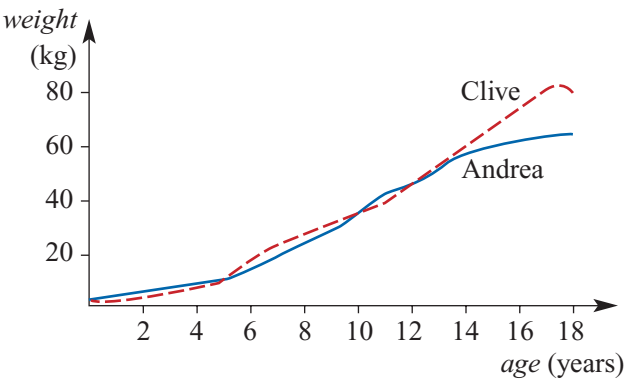
d Repeat for D_1 and D_2 whose x-coordinates are 1.999 and 2.001 respectively.

e What do the results of parts a–d suggest for the value of the gradient at $x = 2$?

8 Use the method of Question 7 to find the gradient of $y = (2x - 1)^3$ at $x = 1$.

Review

9 The following graphs compare the weights of two people over the first 18 years of their lives.



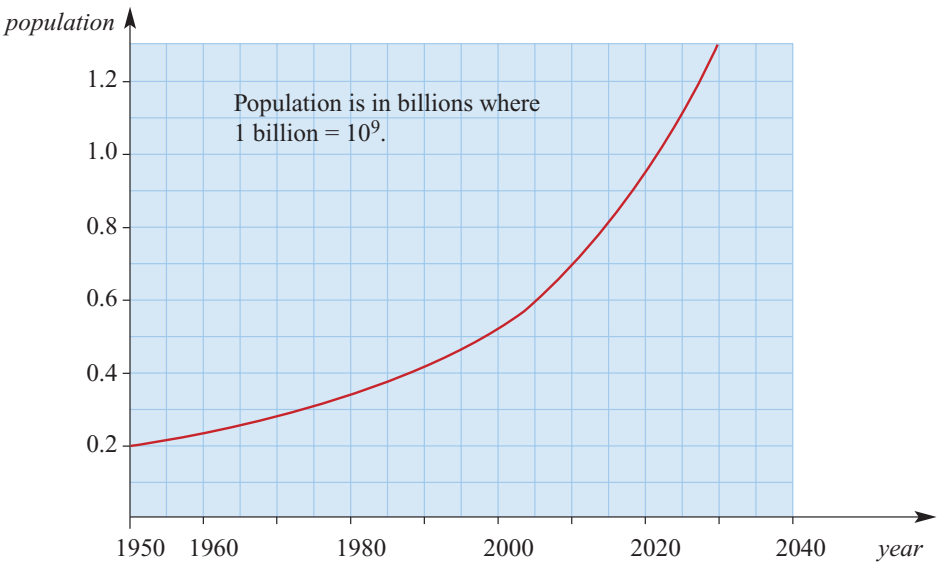
a What was the average rate of change of weight with respect to time for Andrea between the ages of 0 and 18?

b What was the average rate of change of weight with respect to time for Clive between the ages of 0 and 18?

c During which periods did Andrea weigh more than Clive?

d During which periods of time was Clive growing more rapidly than Andrea?

10 The graph below shows an exponential model for Acubaland's population growth. In exponential growth the rate of increase of the population at any time is proportional to the size of the population at that time.



a From the graph find the population of Acubaland in:

i 1960 **ii** 2000

b Calculate the average annual rate of population increase (in billions per year) over the years from 1960 to 2000.

c From the graph estimate the rate of population increase in:

i 1960 **ii** 2000

d How many years do you expect that it will take to double the 2020 population? Explain your reasoning.

11 **a** Draw the graph of $y = 10^x$ and find the gradient of the chord joining the points:

i $x = 2.5$ and $x = 2.8$ **ii** $x = 2.6$ and $x = 2.8$

iii $x = 2.7$ and $x = 2.8$ **iv** $x = 2.75$ and $x = 2.8$

b Comment on your result and investigate further.

12 **a** Use the result that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ to find an expression for the gradient of the line joining points $P(a, a^3)$ and $Q(b, b^3)$ on the curve with equation $y = x^3$.

b Find the gradient of the line for $a = 1$, $b = 2$.

c Find the gradient of the line for $a = 2$, and $b = 2.01$.

d For your expression for the gradient in terms of a and b (in part **a**) let $a = b$ and write your new expression in simplest terms. Interpret this result.

13 The rough sketch graph below shows what happens when three swimmers compete in a 100-metre race. (The vertical axis shows distance travelled by a swimmer.)

a Who wins the race?

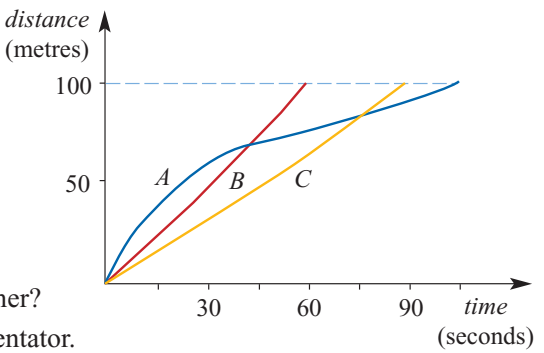
b Who is in front at the 50 m mark?

c What is the approximate distance separating first and third place when the winner finishes?

d What is the approximate time difference between first and third place?

e What is the average speed of each swimmer?

f Describe the race as if you were a commentator.



14 In the following, $f(x)$ is the rule for a well-behaved function f.

a If for $y = f(x)$ the average rate of change of y with respect to x is m, in the interval $[a, b]$, then for the same interval find the rate of change of y with respect to x, where:

i $y = f(x) + c$ **ii** $y = cf(x)$ **iii** $y = -f(x)$

b Give similar statements for the instantaneous rates of change.

CHAPTER

16

Differentiation of Polynomials

Objectives

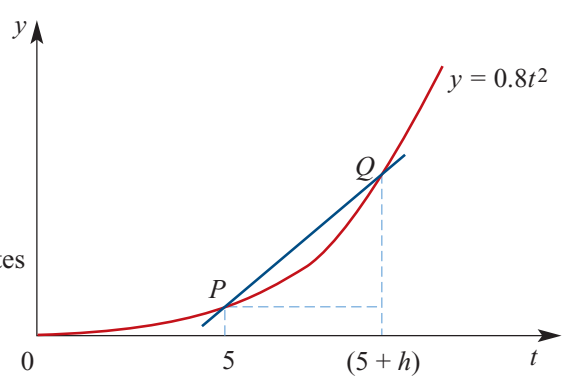
- To understand the concept of **limit**.
- To understand the **definition of differentiation**.
- To understand and use the notation for the **derivative** of a polynomial function.
- To be able to find the **gradient** of a curve of a polynomial function by calculating its derivative.
- To **apply** the rules for differentiating polynomials to solving problems.

In the previous chapter the rate of change of one quantity with respect to a second quantity has been considered. In this chapter a technique will be developed for calculating the rate of change for polynomial functions. To illustrate this an introductory example will be considered.

On planet X, an object falls a distance of y metres in t seconds, where $y = 0.8t^2$. (**Note:** On Earth the commonly used model is $y = 4.9t^2$.)

Can a general expression for the speed of such an object after t seconds be found? In the previous chapter it was found that the gradient of the curve at a given point P can be approximated by finding the gradient of a chord PQ, where Q is a point on the curve as close as possible to P.

The gradient of chord PQ approximates the speed of the object at P. The 'shorter' this chord is made the better the approximation.



Let P be the point on the curve where $t = 5$. Let Q be the point on the curve corresponding to h seconds after $t = 5$, i.e. Q is the point on the curve where $t = 5 + h$.

$$\text{The gradient of chord } PQ = \frac{0.8\left((5+h)^2 - 5^2\right)}{(5+h) - 5}$$
$$= \frac{0.8\left((5+h)^2 - 5^2\right)}{h}$$
$$= 0.8(10 + h)$$

The table gives the gradient for different values of h. Use your calculator to check these.

h	Gradient of PQ
0.7	8.56
0.6	8.48
0.5	8.40
0.4	8.32
0.3	8.24
0.2	8.16
0.1	8.08

If values of h of smaller and smaller magnitude are taken, it is found that the gradient of chord PQ gets closer and closer to 8. At the point where $t = 5$ the gradient is 8. Thus the speed of the object at the moment $t = 5$ is 8 metres per second.

The speed of the object at the moment $t = 5$ is the limiting value of the gradients of PQ as Q approaches P.

A formula for the speed of the object at any time t is required. Let P be the point with coordinates $(t, 0.8t^2)$ on the curve and Q be the point with coordinates $(t + h, 0.8(t + h)^2)$.

$$\text{The gradient of chord } PQ = \frac{0.8\left((t+h)^2 - t^2\right)}{(t+h) - t}$$
$$= 0.8(2t + h)$$

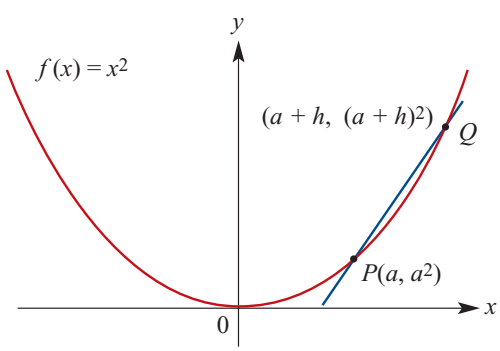
From this an expression for the speed can be found. Consider the limit as h approaches 0, that is, the value of $0.8(2t + h)$ as h becomes arbitrarily small.

The speed at time t is $1.6t$ metres per second. (The gradient of the curve at the point corresponding to time t is $1.6t$.) This technique can be used to investigate the gradient of similar functions.

16.1 The gradient of a curve at a point, and the gradient function

Consider the function $f: R \to R, f(x) = x^2$.

The gradient of the chord PQ in the adjacent graph



$$= \frac{(a+h)^2 - a^2}{a + h - a}$$
$$= \frac{a^2 + 2ah + h^2 - a^2}{a + h - a}$$
$$= 2a + h$$

and the gradient at P can be seen to be $2a$. The limit of $(2a + h)$ as h approaches 0 is $2a$.

It can be seen that there is nothing special about a. So if x is a real number a similar formula holds.

The gradient of the function $y = x^2$ at any point x is equal to $2x$.

It is said that $2x$ is **the derivative of x^2 with respect to x** or, more briefly, **the derivative of x^2 is $2x$**.

This itself is clearly the rule for a function of x and we refer to this function as the **gradient** (or **derived**) function.

The straight line that passes through P and has gradient $2a$ is called the **tangent** to the curve at P.

From the discussion at the beginning of the chapter it can be seen that the derivative of $0.8t^2$ is $1.6t$.

Example 1

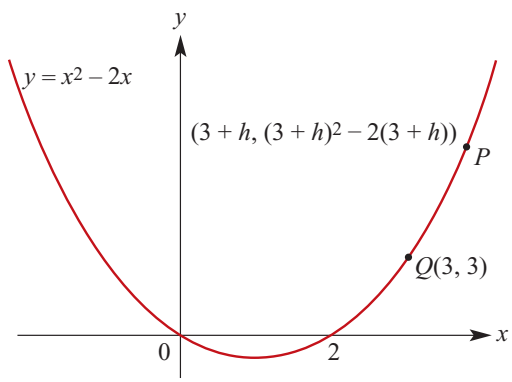
By first considering the gradient of the chord QP, find the gradient of $y = x^2 - 2x$ at the point Q with coordinates (3, 3).

Solution

Consider chord PQ:

Gradient of PQ

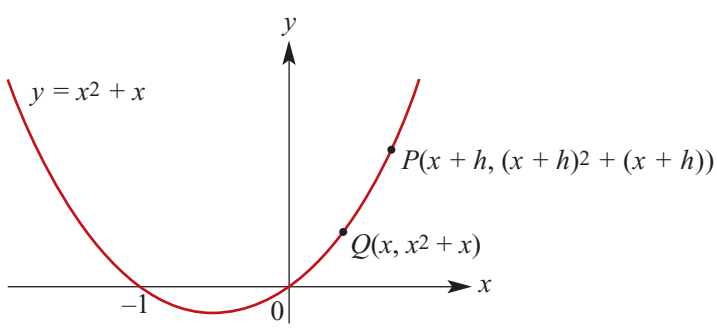
$$= \frac{(3+h)^2 - 2(3+h) - 3}{3+h-3}$$
$$= \frac{9 + 6h + h^2 - 6 - 2h - 3}{3+h-3}$$
$$= \frac{4h + h^2}{h}$$
$$= 4 + h$$



Consider h as it approaches zero. The gradient at the point (3, 3) is 4.

Example 2

Find the gradient of chord PQ and hence the derivative of $x^2 + x$.



Solution

$$\begin{aligned}\text{The gradient of chord } PQ &= \frac{(x+h)^2+(x+h)-(x^2+x)}{x+h-x}\\ &= \frac{x^2+2xh+h^2+x+h-x^2-x}{h}\\ &= \frac{2xh+h^2+h}{h}\\ &= 2x+h+1\end{aligned}$$

From this it is seen that the derivative of x^2+x is $2x+1$.

The notation for limit of $2x+h+1$ as h approaches 0 is $\lim_{h\to 0} 2x+h+1$.

The derivative of a function with rule $f(x)$ may be found by first finding an expression for the gradient of the chord from $Q(x, f(x))$ to $P(x+h, f(x+h))$ and then finding the limit of this expression as h approaches 0.

Example 3

By first considering the gradient of the chord from $Q(x, f(x))$ to $P(x+h, f(x+h))$ for the curve $f(x)=x^3$, find the derivative of x^3.

Solution

$$\begin{aligned}f(x) &= x^3\\ f(x+h) &= (x+h)^3\end{aligned}$$

$$\begin{aligned}\text{The gradient of chord } PQ &= \frac{f(x+h)-f(x)}{(x+h)-x}\\ &= \frac{(x+h)^3-x^3}{(x+h)-x}\\ \text{The derivative of } f(x) &= \lim_{h\to 0}\frac{(x+h)^3-x^3}{(x+h)-x}\\ &= \lim_{h\to 0}\frac{x^3+3x^2h+3h^2x+h^3-x^3}{h}\\ &= \lim_{h\to 0}\frac{3x^2h+3h^2x+h^3}{h}\\ &= \lim_{h\to 0} 3x^2+3hx+h^2\\ &= 3x^2\end{aligned}$$

We have found that the derivative of x^3 is $3x^2$. The following example provides practice in determining limits.

CAS APPENDIX

Example 4

Find:

a $\lim_{h \to 0} 22x^2 + 20xh + h$

b $\lim_{h \to 0} \dfrac{3x^2h + 2h^2}{h}$

c $\lim_{h \to 0} 3x$

d $\lim_{h \to 0} 4$

Solution

a $\lim_{h \to 0} 22x^2 + 20xh + h = 22x^2$

b $\lim_{h \to 0} \dfrac{3x^2h + 2h^2}{h} = \lim_{h \to 0} 3x^2 + 2h$
$= 3x^2$

c $\lim_{h \to 0} 3x = 3x$

d $\lim_{h \to 0} 4 = 4$

Exercise 16A

1 A space vehicle moves so that the distance travelled over its first minute of motion is given by $y = 4t^4$, where y is the distance travelled in metres and t the time in seconds. By finding the gradient of the chord between the points where $t = 4$ and $t = 5$, estimate the speed of the space vehicle when $t = 5$.

2 A population of insects grows so that the population, P, at time t (days) is given by $P = 1000 + t^2 + t$, where $t > 0$. By finding the gradient of the chord between the points where $t = 3$ and $t = 3 + h$, find an estimate for the rate of growth of the insect population at time $t = 3$.

Example 4

3 Find:

a $\lim_{h \to 0} \dfrac{2x^2h^3 + xh^2 + h}{h}$

b $\lim_{h \to 0} \dfrac{3x^2h - 2xh^2 + h}{h}$

c $\lim_{h \to 0} 20 - 10h$

d $\lim_{h \to 0} \dfrac{30hx^2 + 2h^2 + h}{h}$

e $\lim_{h \to 0} 5$

Example 3

4 Find:

a $\lim_{h \to 0} \dfrac{(x + h)^2 + 2(x + h) - (x^2 + 2x)}{h}$ i.e. the derivative of $y = x^2 + 2x$

b $\lim_{h \to 0} \dfrac{(5 + h)^2 + 3(5 + h) - 40}{h}$ i.e. the gradient of $y = x^2 + 3x$, where $x = 5$

c $\lim_{h \to 0} \dfrac{(x + h)^3 + 2(x + h)^2 - (x^3 + 2x^2)}{h}$ i.e. the derivative of $y = x^3 + 2x^2$

Example 1

5 For a curve with equation $y = 3x^2 - x$:

a find the gradient of chord PQ, where P is the point $(1, 2)$ and Q is the point $((1 + h), 3(1 + h)^2 - (1 + h))$

b find the gradient of PQ when $h = 0.1$

c find the gradient of the curve at P.

6 For a curve with equation $y = \frac{2}{x}$:

a find the gradient of chord AB where A is the point (2, 1) and B is the point $\left((2+h), \frac{2}{2+h}\right)$

b find the gradient of AB when $h = 0.1$

c find the gradient of the curve at A.

7 For a curve with equation $y = x^2 + 2x - 3$:

a find the gradient of chord PQ, where P is the point (2, 5) and Q is the point $((2 + h), (2 + h)^2 + 2(2 + h) - 3)$

b find the gradient of PQ when $h = 0.1$

c find the gradient of the curve at P.

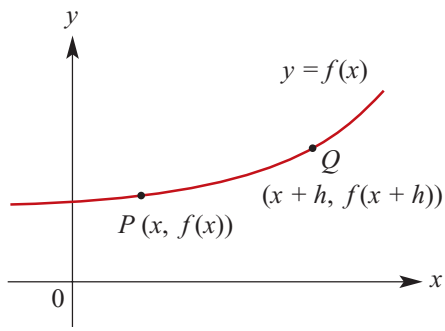
16.2 The derived function

In Section 16.1 we saw how the gradient function of a function with rule $f(x)$ could be derived by considering the gradient of a chord PQ on the curve of $y = f(x)$.

Consider the graph $y = f(x)$ of the function $f: R \to R$.

The gradient of the chord $PQ = \dfrac{f(x+h) - f(x)}{x + h - x}$

$= \dfrac{f(x+h) - f(x)}{h}$



Therefore the gradient of the graph at P is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The reader is referred to Section 16.4 for a further discussion of limits.

The gradient or derived function is denoted by f',

where $f': R \to R$ and $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

In this chapter only polynomial functions are considered. For a polynomial function the derived function always exists and is defined for every number in the domain of f.

Determining the derivative of an expression or the derived function by evaluating the limit is called **differentiation by first principles**.

Example 5

For $f(x) = x^2 + 2x$ find $f'(x)$ by first principles.

Solution

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h} \\
&= \lim_{h \to 0} 2x + h + 2 \\
&= 2x + 2 \\
\therefore \quad f'(x) &= 2x + 2
\end{aligned}
$$

Example 6

For $f(x) = 2 - x^3$ find $f'(x)$ by first principles.

Solution

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{2 - (x+h)^3 - (2 - x^3)}{h} \\
&= \lim_{h \to 0} \frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - (2 - x^3)}{h} \\
&= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3}{h} \\
&= \lim_{h \to 0} -3x^2 - 3xh - h^2 \\
&= -3x^2
\end{aligned}
$$

The following results have been obtained:

For $f(x) = x$, $f'(x) = 1$.

For $f(x) = x^2$, $f'(x) = 2x$.

For $f(x) = x^3$, $f'(x) = 3x^2$.

For $f(x) = x^4$, $f'(x) = 4x^3$.

For $f(x) = 1$, $f'(x) = 0$.

This suggests the following general result:

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, $n = 1, 2, 3, \ldots$

and for $f(x) = 1$, $f'(x) = 0$

From the previous section it can be seen that for k, a constant:

> If $f(x) = kx^n$, the derivative function f' has rule $f'(x) = knx^{n-1}$.

It is worth making a special note of the results:

> If $g(x) = kf(x)$, where k is a constant, then $g'(x) = kf'(x)$.

That is, **the derivative of a number multiple is the multiple of the derivative**.
For example, for $g(x) = 5x^2$, the derived function $g'(x) = 5(2x) = 10x$.

Another important rule for differentiation is:

> If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

That is, **the derivative of the sum is the sum of the derivatives**.

For example, for $f(x) = x^2 + 2x$, the derived function $f'(x) = 2x + 2$.

The process of finding the derivative function is called **differentiation**.

Example 7

Find the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

Solution

$$\begin{aligned} f(x) &= x^5 - 2x^3 + 2 \\ \text{then } f'(x) &= 5x^4 - 2(3x^2) + 2(0) \\ &= 5x^4 - 6x^2 \end{aligned}$$

Example 8

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and $f'(1)$.

Solution

$$\begin{aligned} f(x) &= 3x^3 - 6x^2 + 1 \\ \text{then } f'(x) &= 3(3x^2) - 6(2x) + 1(0) \\ &= 9x^2 - 12x \\ f'(1) &= 9 - 12 \\ &= -3 \end{aligned}$$

Example 9

Find the gradient of the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$ at the point $(1, -2)$.

Solution

Now $f'(x) = 9x^2 - 12x$ and $f'(1) = 9 - 12 = -3$.

The gradient of the curve is -3 at the point $(1, -2)$.

An alternative notation for the derivative is the following:

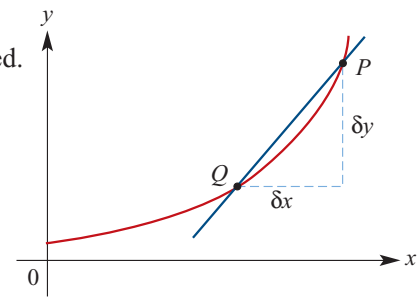
If $y = x^3$, then the derivative can be denoted by $\frac{dy}{dx}$, so that $\frac{dy}{dx} = 3x^2$.

In general, if y is a function of x, the derivative of y with respect to x is denoted by $\frac{dy}{dx}$ and with the use of different symbols z, where z is a function of t. The derivative of z with respect to t is $\frac{dz}{dt}$.

In this notation d is not a factor and cannot be cancelled. This came about because in the eighteenth century the standard diagram for finding the limiting gradient was labelled as in the figure. (δ is the lower case Greek letter for 'd', and is pronounced *delta*.)

δx means a difference in x.

δy means a difference in y.



Example 10

a If $y = t^2$, find $\frac{dy}{dt}$. **b** If $x = t^3 + t$, find $\frac{dx}{dt}$. **c** If $z = \frac{1}{3}x^3 + x^2$, find $\frac{dz}{dx}$.

Solution

a $y = t^2$

then $\frac{dy}{dt} = 2t$

b $x = t^3 + t$

then $\frac{dx}{dt} = 3t^2 + 1$

c $z = \frac{1}{3}x^3 + x^2$

then $\frac{dz}{dx} = x^2 + 2x$

Example 11

a For $y = (x + 3)^2$, find $\frac{dy}{dx}$. **b** For $z = (2t - 1)^2(t + 2)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^2 + 3x}{x}$, find $\frac{dy}{dx}$. **d** Differentiate $y = 2x^3 - 1$ with respect to x.

Solution

a It is first necessary to write $y = (x + 3)^2$ in expanded form.

$\therefore \quad y = x^2 + 6x + 9$

and $\frac{dy}{dx} = 2x + 6$

b Expanding:

$z = (4t^2 - 4t + 1)(t + 2)$

$= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2$

$= 4t^3 + 4t^2 - 7t + 2$

and $\frac{dz}{dt} = 12t^2 + 8t - 7$

c First divide by x:

$y = x + 3$

$\therefore \quad \frac{dy}{dx} = 1$

d $y = 2x^3 - 1$

$\therefore \quad \frac{dy}{dx} = 6x^2$

Operator notation

'Find the derivative of $2x^2 - 4x$ with respect to x' can also be written as $\frac{d}{dx}(2x^2 - 4x)$, and, in general, $\frac{d}{dx}(f(x)) = f'(x)$.

Example 12

Find:

a $\frac{d}{dx}(5x - 4x^3)$ **b** $\frac{d}{dz}(5z^2 - 4z)$ **c** $\frac{d}{dz}(6z^3 - 4z^2)$

Solution

a $\frac{d}{dx}(5x - 4x^3)$
$= 5 - 12x^2$

b $\frac{d}{dz}(5z^2 - 4z)$
$= 10z - 4$

c $\frac{d}{dz}(6z^3 - 4z^2)$
$= 18z^2 - 8z$

Example 13

Find the coordinates of the points on curves determined by each of the following equations at which the gradient has the given value:

a $y = x^3$; gradient $= 8$ **b** $y = x^2 - 4x + 2$; gradient $= 0$

c $y = 4 - x^3$; gradient $= -6$

Solution

a $y = x^3$ implies $\frac{dy}{dx} = 3x^2$
$\therefore 3x^2 = 8$
$\therefore \quad x = \pm\sqrt{\frac{8}{3}} = \frac{\pm 2\sqrt{6}}{3}$
$\therefore$ coordinates are
$\left(\frac{2\sqrt{6}}{3}, \frac{16\sqrt{6}}{9}\right)$ and $\left(\frac{-2\sqrt{6}}{3}, \frac{-16\sqrt{6}}{9}\right)$

b $y = x^2 - 4x + 2$ implies $\frac{dy}{dx} = 2x - 4$
$\therefore 2x - 4 = 0$
$\therefore \quad x = 2$
$\therefore$ coordinates are $(2, -2)$

c $y = 4 - x^3$ implies $\frac{dy}{dx} = -3x^2$
$\therefore -3x^2 = -6$
$\therefore \quad x^2 = 2$
$\therefore \quad x = \pm\sqrt{2}.$
$\therefore$ coordinates are $\left(2^{\frac{1}{2}}, 4 - 2^{\frac{3}{2}}\right)$ and $\left(-2^{\frac{1}{2}}, 4 + 2^{\frac{3}{2}}\right)$

Exercise 16B

Examples 5, 6

1 For each of the following, find $f'(x)$ by finding $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$:

a $f(x) = 3x^2$ **b** $f(x) = 4x$ **c** $f(x) = 3$ **d** $f(x) = 3x^2 + 4x + 3$

e $f(x) = 2x^3 - 4$ **f** $f(x) = 4x^2 - 5x$ **g** $f(x) = 3 - 2x + x^2$

Example 7

2 Find the derivative of each of the following with respect to x.

a $x^2 + 4x$ **b** $2x + 1$ **c** $x^3 - x$

d $\dfrac{1}{2}x^2 - 3x + 4$ **e** $5x^3 + 3x^2$ **f** $-x^3 + 2x^2$

3 For each of the following find $f'(x)$:

a $f(x) = x^{12}$ **b** $f(x) = 3x^7$ **c** $f(x) = 5x$

d $f(x) = 5x + 3$ **e** $f(x) = 3$ **f** $f(x) = 5x^2 - 3x$

g $f(x) = 10x^5 + 3x^4$ **h** $f(x) = 2x^4 - \dfrac{1}{3}x^3 - \dfrac{1}{4}x^2 + 2$

Example 10

4 For each of the following, find $\dfrac{dy}{dx}$:

a $y = -x$ **b** $y = 10$ **c** $y = 4x^3 - 3x + 2$

d $y = \dfrac{1}{3}(x^3 - 3x + 6)$ **e** $y = (x + 1)(x + 2)$ **f** $y = 2x(3x^2 - 4)$

g $y = \dfrac{10x^5 + 3x^4}{2x^2}, x \neq 0$

Example 9

5 **a** For the curve with equation $y = x^3 + 1$ find the gradient at points:

i $(1, 2)$ **ii** $(a, a^3 + 1)$

b Find the derivative of $x^3 + 1$ with respect to x.

6 **a** Given that $y = x^3 - 3x^2 + 3x$, find $\dfrac{dy}{dx}$. Hence show that $\dfrac{dy}{dx} \geq 0$ for all x, and interpret this in terms of the graph of $y = x^3 - 3x^2 + 3x$.

b Given that $y = \dfrac{x^2 + 2x}{x}$, for $x \neq 0$, find $\dfrac{dy}{dx}$.

c Differentiate $y = (3x + 1)^2$ with respect to x.

7 At the points on the following curves corresponding to the given values of x, find the y-coordinate and the gradient.

a $y = x^2 - 2x + 1, x = 2$ **b** $y = x^2 + x + 1, x = 0$

c $y = x^2 - 2x, x = -1$ **d** $y = (x + 2)(x - 4), x = 3$

e $y = 3x^2 - 2x^3, x = -2$ **f** $y = (4x - 5)^2, x = \dfrac{1}{2}$

8 **a** For each of the following, find $f'(x)$ and $f'(1)$, if $y = f(x)$ then find the $\{(x, y): f'(x) = 1\}$ i.e. the coordinates of the points where the gradient is 1.

i $y = 2x^2 - x$ **ii** $y = 1 + \frac{1}{2}x + \frac{1}{3}x^2$

iii $y = x^3 + x$ **iv** $y = x^4 - 31x$

b What is the interpretation of $\{(x, y): f'(x) = 1\}$ in terms of the graphs?

Example 12

9 Find:

a $\frac{d}{dt}(3t^2 - 4t)$ **b** $\frac{d}{dx}(4 - x^2 + x^3)$ **c** $\frac{d}{dz}(5 - 2z^2 - z^4)$

d $\frac{d}{dy}(3y^2 - y^3)$ **e** $\frac{d}{dx}(2x^3 - 4x^2)$ **f** $\frac{d}{dt}(9.8t^2 - 2t)$

Example 13

10 Find the coordinates of the points on the curves given by the following equations at which the gradient has the given values:

a $y = x^2$; gradient $= 8$ **b** $y = x^3$; gradient $= 12$

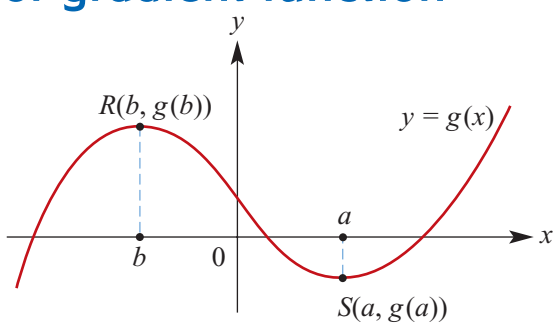
c $y = x(2 - x)$; gradient $= 2$ **d** $y = x^2 - 3x + 1$; gradient $= 0$

e $y = x^3 - 6x^2 + 4$; gradient $= -12$ **f** $y = x^2 - x^3$; gradient $= -1$

16.3 Graphs of the derived or gradient function

Consider the gradient for different intervals of the graph of $y = g(x)$ shown opposite.

At a point $(a, g(a))$ of the graph $y = g(x)$ the gradient is $g'(a)$.



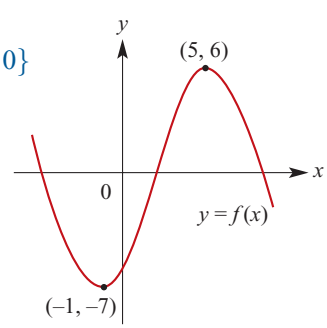
Some of the features of the graph are:

- $x < b$ the gradient is positive, i.e. $g'(x) > 0$
- $x = b$ the gradient is zero, i.e. $g'(b) = 0$
- $b < x < a$ the gradient is negative, i.e. $g'(x) < 0$
- $x = a$ the gradient is zero, i.e. $g'(a) = 0$
- $x > a$ the gradient is positive, i.e. $g'(x) > 0$

Example 14

For the graph of $f: R \to R$, find:

a $\{x: f'(x) > 0\}$ **b** $\{x: f'(x) < 0\}$ **c** $\{x: f'(x) = 0\}$



Solution

a $\{x: f'(x) > 0\} = \{x: -1 < x < 5\} = (-1, 5)$

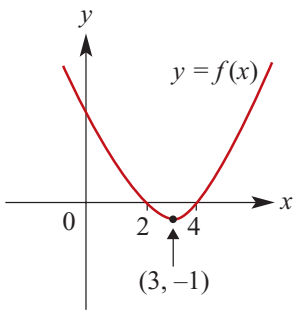
b $\{x: f'(x) < 0\} = \{x: x < -1\} \cup \{x: x > 5\}$
$= (-\infty, -1) \cup (5, \infty)$

c $\{x: f'(x) = 0\} = \{-1, 5\}$

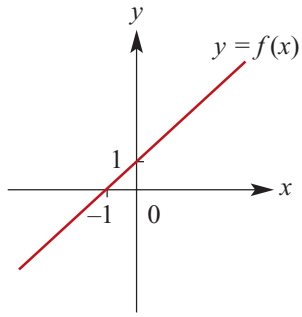
Example 15

Sketch the graph of $y = f'(x)$ for each of the following. (It is impossible to determine all features.)

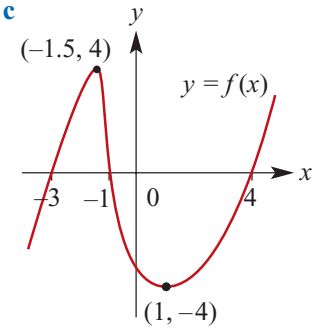
a



b

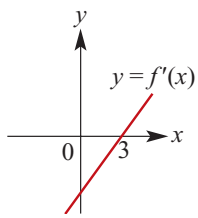


c

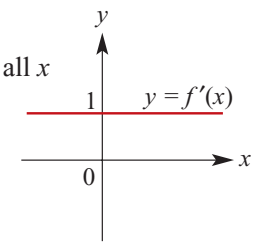


Solution

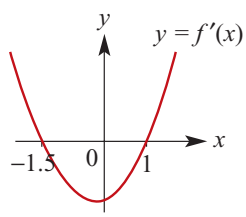
a $f'(x) > 0$ for $x > 3$
$f'(x) < 0$ for $x < 3$
$f'(x) = 0$ for $x = 3$



b $f'(x) = 1$ for all x

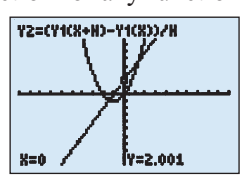


c $f'(x) > 0$ for $x > 1$
$f'(x) < 0$ for $-1.5 < x < 1$
$f'(x) > 0$ for $x < -1.5$
$f'(-1.5) = 0$ and $f(1) = 0$



Using a graphics calculator

The graphics calculator may be used to generate the gradient function for any function that has previously been entered into the calculator.



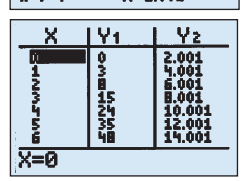
In the **Y=** screen enter:

Y1 = X² + 2X

Y2 = (Y1(X + H) − Y1(X))/H

In the Home screen store a small value in memory location **H**.

For example, **0.001 → H**.



Observe both the graph screen and the table screen.

Finally enter **Y3 = 2X + 2** to check the result.

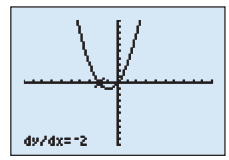
Change the function for **Y1** and observe the result.

An alternative method is to use the **8:nDeriv** function from the **MATH** menu of the calculator.

To generate the gradient function for a previously entered function $\mathbf{Y_1}$, enter, **Y2 = nDeriv(Y1, X, X, 0.001)**.

The gradient at a point can be found at a point of the graph on the **GRAPH** window.

For $\mathbf{Y1 = X^2 + 2X}$, press ZOOM and select **6:Zstandard**. The graph appears. Now select **6:dy/dx** from the **CALC** menu and enter **−2**.



An angle associated with the gradient of a curve at a point

The gradient of a curve at a point is the gradient of the tangent at that point. A straight line, the tangent, is associated with each point on the curve.

If α is the angle a straight line makes with the positive direction of the x-axis, then the gradient, m, of the straight line is equal to $\tan\alpha$, i.e., $\tan\alpha = m$.

If $\alpha = 45°$ then $\tan\alpha = 1$ and the gradient is 1.

If $\alpha = 20°$ then the gradient of the straight line is $\tan 20°$.

If $\alpha = 135°$ then $\tan\alpha = -1$ and the gradient is -1.

Example 16

Find the coordinates of the points on the curve with equation $y = x^2 - 7x + 8$ at which the tangent:

a makes an angle of $45°$ with the positive direction of the x-axis

b is parallel to the line $y = -2x + 6$.

Solution

a $\dfrac{dy}{dx} = 2x - 7$

$2x - 7 = 1$ $(\tan 45° = 1)$

and $2x = 8$

$\therefore \quad x = 4$

$y = 4^2 - 7 \times 4 + 8 = -4$

coordinates are $(4, -4)$

b **Note:** $y = -2x + 6$ has gradient -2

$\therefore \quad 2x - 7 = -2$

and $2x = 5$

$\therefore \quad x = \dfrac{5}{2}$

coordinates are $\left(\dfrac{5}{2}, -\dfrac{13}{4}\right)$

Example 17

The planned path for a flying saucer leaving a planet is defined by the equation $y = \frac{1}{4}x^4 + \frac{2}{3}x^3$ for $x > 0$.

The units are kilometres. (The x-axis is horizontal and the y-axis vertical.)

a Find the direction of motion when the x-value is:

i 2 **ii** 3

b Find a point on the flying saucer's path where the path is inclined at 45° to the positive x-axis. (i.e. where the gradient of the path is 1).

c Are there any other points on the path which satisfy the situation described in part b?

Solution

a $\frac{dy}{dx} = x^3 + 2x^2$

i When $x = 2$, $\frac{dy}{dx} = 8 + 8 = 16$

$\tan^{-1}16 = 86.42°$ (to the x-axis)

ii When $x = 3$, $\frac{dy}{dx} = 27 + 18 = 45$

$\tan^{-1}45 = 88.73°$ (to the x-axis)

b, c When the flying saucer is flying at 45° to the direction of the x-axis, the gradient of the curve of its path is given by tan 45°.

Thus to find the point at which this happens we consider equation $\frac{dy}{dx} = \tan 45°$.

$\therefore \quad x^3 + 2x^2 = 1$

$\therefore \quad x^3 + 2x^2 - 1 = 0$

$\therefore \quad (x + 1)(x^2 + x - 1) = 0$

$\therefore \quad x = -1$ or $x = \frac{-1 \pm \sqrt{5}}{2}$

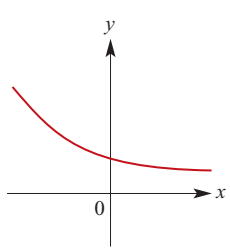
The only acceptable solution is $x = \frac{-1 + \sqrt{5}}{2}$ $(x \approx 0.62)$ as the other two possibilities give negative values for x and we are only considering positive values for x.

Exercise 16C

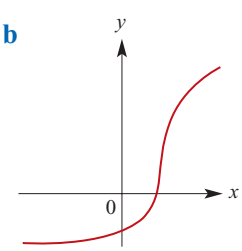
Skillsheet TEACHER

1 On which of the following curves is $\frac{dy}{dx}$ positive for all values of x?

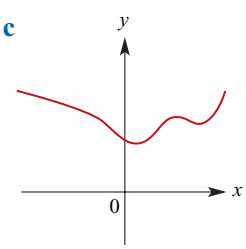
a



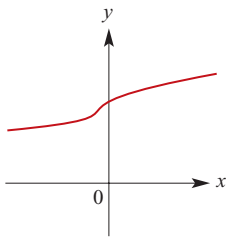
b



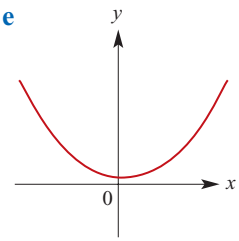
c



d

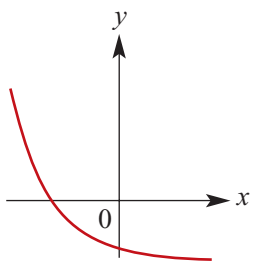


e

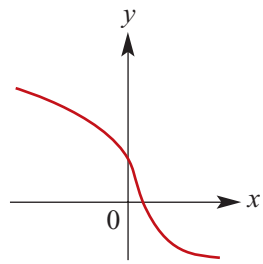


2 On which of the following curves is $\frac{dy}{dx}$ negative for all values of x?

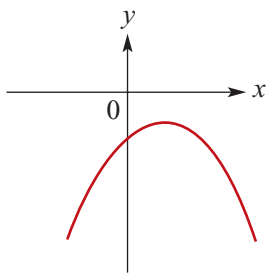
a



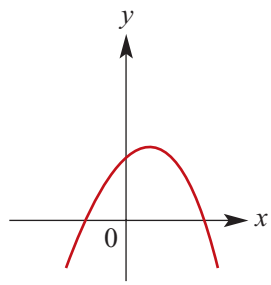
b



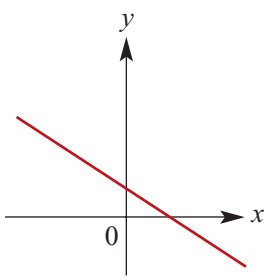
c



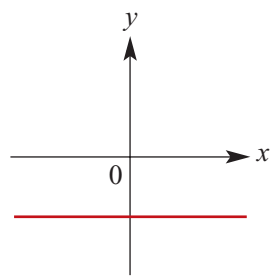
d



e



f



3 For the function $f(x) = 2(x-1)^2$ find the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) > 0$

d $f'(x) < 0$ **e** $f'(x) = -2$

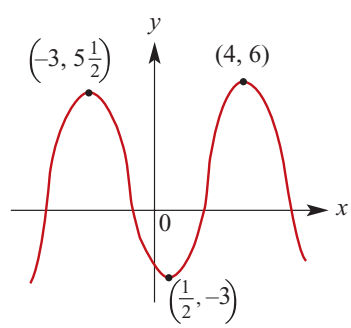
Example 15

4 For the graph of $y = h(x)$ shown here find:

a $\{x: h'(x) > 0\}$

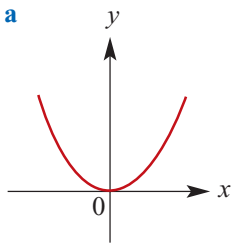
b $\{x: h'(x) < 0\}$

c $\{x: h'(x) = 0\}$

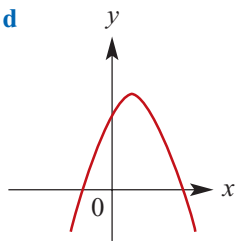


5 Which of the graphs labelled **A**–**F** correspond to each of the graphs labelled **a**–**f**?

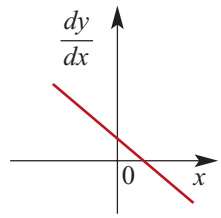
a



d



A



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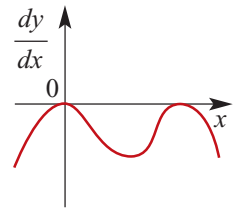

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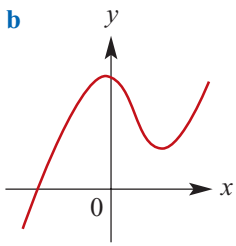

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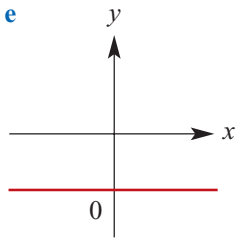

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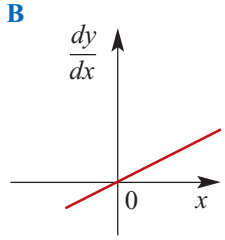

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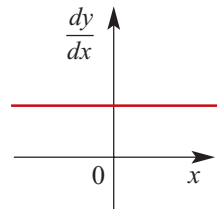

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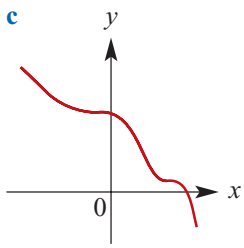
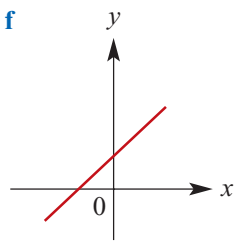
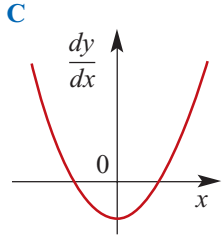


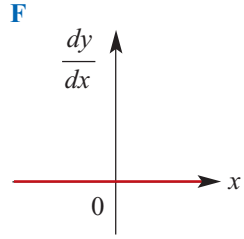
f



C



F



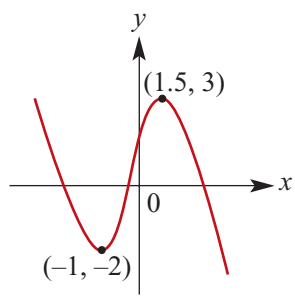
Example 15

6 For the graph of $y = f(x)$ find:

a $\{x: f'(x) > 0\}$

b $\{x: f'(x) < 0\}$

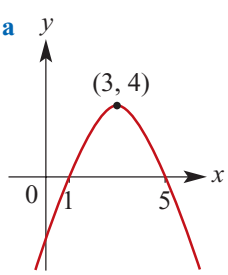
c $\{x: f'(x) = 0\}$



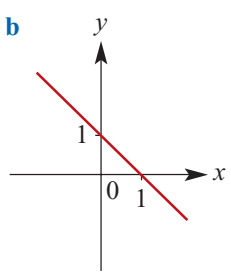
Example 16

7 Sketch the graph of $y = f'(x)$ for each of the following

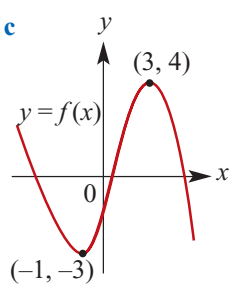
a



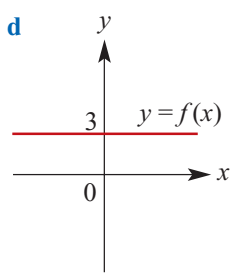
b



c



d



Example 17

8 Find the coordinates of the points on the curve $y = x^2 - 5x + 6$ at which the tangent:

a makes an angle of 45° with the positive direction of the x-axis, i.e. where the gradient is 1

b is parallel to the line $y = 3x + 4$.

Example 17

9 Find the coordinates of the points on the parabola $y = x^2 - x - 6$ at which:

a the gradient is zero

b the tangent is parallel to the line $x + y = 6$.

10 Use a graphics calculator to plot the graph of $y = f'(x) =$ where:

a $f(x) = \sin x$ **b** $f(x) = \cos x$ **c** $f(x) = 2^x$

11 A car moves away from a set of traffic lights so that the distance $S(t)$ metres covered after t seconds is modelled by $S(t) = (0.2)t^3$.

a Find its speed after t seconds. **b** What will its speed be when $t = 1, 3, 5$?

12 A rocket is launched from Cape York Peninsula so that after t seconds its height $h(t)$ metres is given by $h(t) = 20t^2$, $0 \leq t \leq 150$. After $2\frac{1}{2}$ minutes this model is no longer appropriate.

a Find the height and the speed of the rocket when $t = 150$.

b After how long will its speed be 1000 m/s?

13 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point $(2, -2)$.

a Find the values of a and b.

b Find the coordinates of the point where the gradient is 0.

16.4 Limits and continuity

Limits

We consider the limit of a function $f(x)$ to be the value that $f(x)$ approaches as x approaches a given value. $\lim\limits_{x \to a} f(x) = p$ means that, as x approaches a, $f(x)$ approaches p. It is important to understand that it is possible to get as close as desired to p as x approaches a. This idea of prescribed closeness is discussed in Section 16.6. Note that $f(x)$ may or may not be defined at $x = a$.

For many functions $f(a)$ is defined, so to evaluate the limit we simply substitute the value a into the rule for the function.

Example 18

If $f(x) = 3x^2$ find $\lim\limits_{x \to 2} 3x^2$.

Solution

Since $f(x) = 3x^2$ is defined at $x = 2$

$$\lim_{x \to 2} 3x^2 = 3(2)^2$$
$$= 12$$

If the function is not defined at the value for which the limit is to be found, a different procedure is used.

Example 19

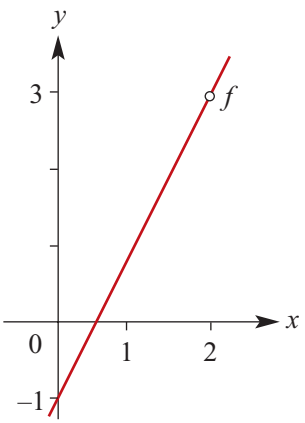
For $f(x) = \dfrac{2x^2 - 5x + 2}{x - 2}$, $x \neq 2$, find $\lim\limits_{x \to 2} f(x)$.

Solution

Observe that $f(x)$ is defined for $x \in R \setminus \{2\}$.

Examine the behaviour of $f(x)$ for values of x 'near' 2.

$x < 2$	$x > 2$
$f(1.7) = 2.4$	$f(2.3) = 3.6$
$f(1.8) = 2.6$	$f(2.2) = 3.4$
$f(1.9) = 2.8$	$f(2.1) = 3.2$
$f(1.99) = 2.98$	$f(2.01) = 3.02$
$f(1.999) = 2.998$	$f(2.001) = 3.002$



From the table it is apparent that as x takes values closer and closer to 2, regardless of whether x approaches 2 from the left or from the right, the values of $f(x)$ become closer and closer to 3.

i.e. $\lim\limits_{x \to 2} f(x) = 3$

This may also be seen by observing that:

$$f(x) = \frac{(2x - 1)(x - 2)}{x - 2}, n \neq 2$$
$$= 2x - 1, x \neq 2,$$

The graph of $f: R \setminus \{2\} \to R, f(x) = 2x - 1$ is shown.

The following important results are useful for the evaluation of limits:

- $\lim\limits_{x \to c} (f(x) + g(x)) = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$

  i.e. the limit of the sum is the sum of the limits.
- $\lim\limits_{x \to c} kf(x) = k \lim\limits_{x \to c} f(x)$, k being a given number
- $\lim\limits_{x \to c} (f(x)g(x)) = \lim\limits_{x \to c} f(x) \lim\limits_{x \to c} g(x)$

  i.e. the limit of the product is the product of the limits.
- $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$, provided $\lim\limits_{x \to c} g(x) \neq 0$

  i.e. the limit of the quotient is the quotient of the limits.

Example 20

Find:

a $\lim_{x \to 0}(x^2 + 2)$ **b** $\lim_{x \to 3} \dfrac{x^2 - 3x}{x - 3}$ **c** $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2}$

d $\lim_{x \to 3}(2x + 1)(3x - 2)$ **e** $\lim_{x \to 3} \dfrac{x^2 - 7x + 10}{x^2 - 25}$

Solution

a $\lim_{x \to 0}(x^2 + 2) = \lim_{x \to 0} x^2 + \lim_{x \to 0} 2 = 0 + 2 = 2$

b $\lim_{x \to 3} \dfrac{x^2 - 3x}{x - 3} = \lim_{x \to 3} \dfrac{x(x - 3)}{x - 3} = \lim_{x \to 3} x = 3$

c $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \dfrac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2}(x + 1) = 3$

d $\lim_{x \to 3}(2x + 1)(3x - 2) = \lim_{x \to 3}(2x + 1) \lim_{x \to 3}(3x - 2) = 7 \times 7 = 49$

e $\lim_{x \to 3} \dfrac{x^2 - 7x + 10}{x^2 - 25} = \lim_{x \to 3} \dfrac{(x - 2)(x - 5)}{(x + 5)(x - 5)} = \dfrac{\lim_{x \to 3}(x - 2)}{\lim_{x \to 3}(x + 5)} = \dfrac{1}{8}$

Example 21

Find:

a $\lim_{h \to 0}(3h + 4)$ **b** $\lim_{x \to 2} 4x(x + 2)$ **c** $\lim_{x \to 3} \dfrac{5x + 2}{x - 2}$

Solution

a $\begin{aligned} \lim_{h \to 0}(3h + 4) &= \lim_{h \to 0}(3h) + \lim_{h \to 0}(4) \\ &= 0 + 4 \\ &= 4 \end{aligned}$

b $\begin{aligned} \lim_{x \to 2} 4x(x + 2) &= \lim_{x \to 2}(4x) \lim_{x \to 2}(x + 2) \\ &= 8 \times 4 \\ &= 32 \end{aligned}$

c $\begin{aligned} \lim_{x \to 3} \dfrac{5x + 2}{x - 2} &= \lim_{x \to 3}(5x + 2) \div \lim_{x \to 3}(x - 2) \\ &= 17 \div 1 \\ &= 17 \end{aligned}$

The notation of limits is used to describe the behaviour of graphs, and a similar notation has been used previously in the book.

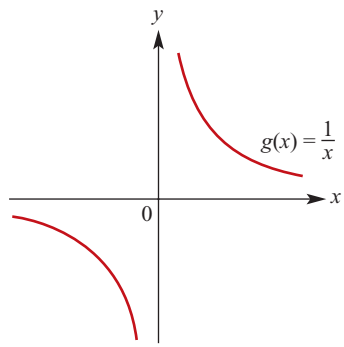
Consider $f: R \setminus \{0\} \to R$, $f(x) = \dfrac{1}{x^2}$. Observe that as $x \to 0$, both from the left and from the right, $f(x)$ increases without bound. The limit notation for this is $\lim_{x \to 0} f(x) = \infty$.

For $g: R \setminus \{0\} \to R$, $g(x) = \dfrac{1}{x}$, the behaviour of $g(x)$ as x approaches 0 from the left is different from the behaviour as x approaches 0 from the right.

With limit notation this is written as:

$$\lim_{x \to 0^-} g(x) = -\infty \quad \text{and} \quad \lim_{x \to 0^+} f(x) = \infty$$

Now examine this function as the magnitude of x becomes very large. It can be seen that as x increases without bound through positive values, the corresponding values of $g(x)$ approach zero. Likewise as x decreases without bound through negative values, the corresponding values of $g(x)$ also approach zero.



Symbolically this is written as:

$$\lim_{x \to \infty} g(x) = 0 \quad \text{and} \quad \lim_{x \to -\infty} g(x) = 0$$

Many functions approach a limiting value or limit as x approaches $\pm\infty$.

Left and right limits

An idea which is useful in the following discussion is the existence of limits from the left and from the right.

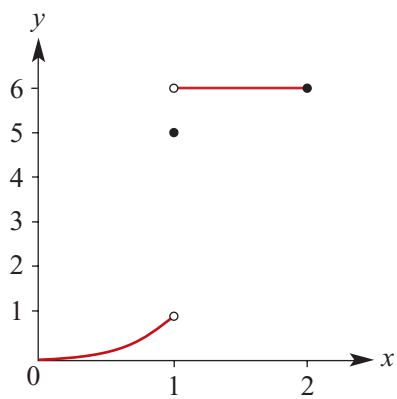
If the value of $f(x)$ approaches the number p as x approaches a from the right-hand side, then it is written as $\lim_{x \to a^+} f(x) = p$. If the value of $f(x)$ approaches the number p as x approaches a from the left-hand side, then it is written as $\lim_{x \to a^-} f(x) = p$.

The limit as x approaches a exists only if the limits from the left and the right both exist and are equal. Then $\lim_{x \to a} f(x) = p$.

The following is an example of when the limit does not exist for a particular value.

$$\text{Let } f(x) = \begin{cases} x^3 & \text{for } 0 \leq x < 1 \\ 5 & \text{for } x = 1 \\ 6 & \text{for } 1 < x \leq 2 \end{cases}$$

It is clear from the graph of f that the $\lim_{x \to 1} f(x)$ does not exist. However, if x is allowed to approach 1 from the left, then $f(x)$ approaches 1. On the other hand if x is allowed to approach 1 from the right, then $f(x)$ approaches 6. Also note that $f(1) = 5$.



Continuity at a point: informal definition

A function with rule $f(x)$ is said to be continuous when $x = a$ if the graph of $y = f(x)$ can be drawn through the point with coordinates $(a, f(a))$ without a break. Otherwise there is said to be a discontinuity at $x = a$.

Most of the functions considered in this course are continuous for their domains.

A more formal definition of continuity follows.

A function f is continuous at a point a if $f(a)$, $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a^-} f(x)$ all exist and are equal. Or equivalently:

A function f is continuous at the point $x = a$ if the following three conditions are met:

1 $f(x)$ is defined at $x = a$ **2** $\lim\limits_{x \to a} f(x)$ exists **3** $\lim\limits_{x \to a} f(x) = f(a)$

The function is said to be discontinuous at a point if it is not continuous at that point. A function is said to be continuous everywhere if it is continuous for all real numbers.

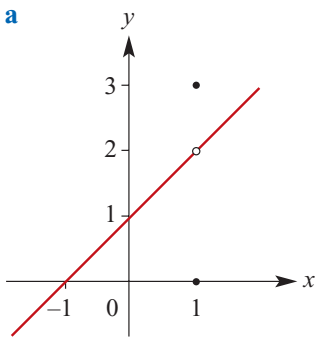
The polynomial functions are all continuous for R. The function with rule $f(x) = \dfrac{1}{x}$ has a discontinuity at $x = 0$, as $f(0)$ is not defined. It is continuous elsewhere in its domain.

Hybrid functions, as introduced in Chapter 5, provide examples of functions which have points of discontinuity where the function is defined.

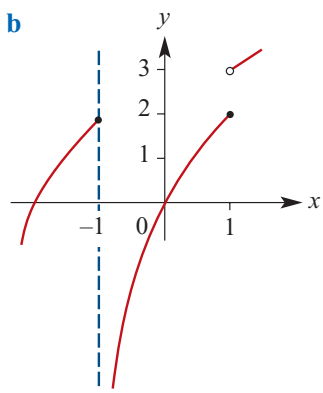
Example 22

State the values for x for which the functions whose graphs are shown below have a discontinuity.

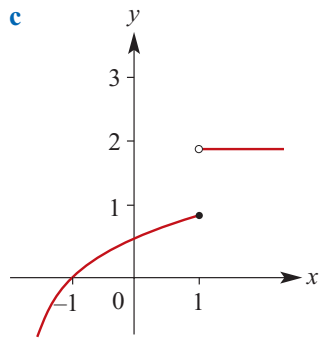
a



b



c



Solution

a There is a discontinuity at $x = 1$, as $f(1) = 3$ but $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x) = 2$.

b There is a discontinuity at $x = -1$, as $f(-1) = 2$ and $\lim\limits_{x \to -1^-} f(x) = 2$ but $\lim\limits_{x \to -1^+} f(x) = -\infty$ and a discontinuity at $x = 1$ as $f(1) = 2$ and $\lim\limits_{x \to 1^-} f(x) = 2$ but $\lim\limits_{x \to 1^+} f(x) = 3$.

c There is a discontinuity at $x = 1$, as $f(1) = 1$ and $\lim\limits_{x \to 1^-} f(x) = 1$ but $\lim\limits_{x \to 1^+} f(x) = 2$.

Example 23

For each of the following functions state the values of x for which there is a discontinuity and use the definition of continuity in terms of $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why each is discontinuous:

a $f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x + 1 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -2x + 1 & \text{if } x < 0 \end{cases}$

c $f(x) = \begin{cases} x & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 0 \\ -2x + 1 & \text{if } x \geq 0 \end{cases}$

d $f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 0 \\ -2x + 1 & \text{if } x < 0 \end{cases}$

e $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$

Solution

a $f(0) = 0$ but $\lim_{x \to 0^-} f(x) = 1$, therefore there is a discontinuity at $x = 0$

b $f(0) = 0$ but $\lim_{x \to 0^-} f(x) = 1$, therefore there is a discontinuity at $x = 0$

c $f(-1) = -1$ but $\lim_{x \to -1^+} f(x) = 1$, therefore there is a discontinuity at $x = 1$
$f(0) = 1$ but $\lim_{x \to 0^-} f(x) = 0$, therefore there is a discontinuity at $x = 0$

d No discontinuity

e No discontinuity

Skillsheet TEACHER

Exercise 16D

Examples 18–21

1 Find the following limits:

a $\lim_{x \to 3} 15$

b $\lim_{x \to 6} (x - 5)$

c $\lim_{x \to \frac{1}{2}} (3x - 5)$

d $\lim_{t \to -3} \dfrac{(t-2)}{(t+5)}$

e $\lim_{t \to -1} \dfrac{t^2 + 2t + 1}{t + 1}$

f $\lim_{x \to 0} \dfrac{(x+2)^2 - 4}{x}$

g $\lim_{t \to 1} \dfrac{t^2 - 1}{t - 1}$

h $\lim_{x \to 9} \sqrt{x + 3}$

i $\lim_{x \to 0} \dfrac{x^2 - 2x}{x}$

j $\lim_{x \to 2} \dfrac{x^3 - 8}{x - 2}$

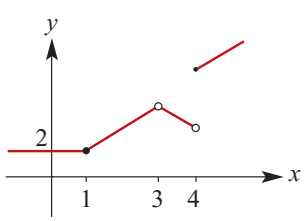
k $\lim_{x \to 2} \dfrac{3x^2 - x - 10}{x^2 + 5x - 14}$

l $\lim_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 - 6x + 5}$

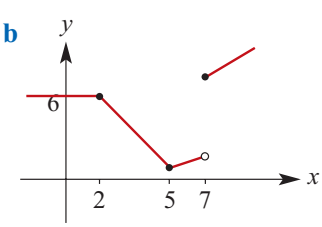
Example 22

2 For each of the following graphs give the values of x at which a discontinuity occurs. Give reasons.

a



b



Example 23

3 For each of the following functions state the values of x at which there is a discontinuity and use the definition of continuity in terms of $f(a)$, $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a^-} f(x)$ to explain why each stated value of x corresponds to a discontinuity.

a $f(x) = \begin{cases} 3x & \text{if } x \geq 0 \\ -2x + 2 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 + 2 & \text{if } x \geq 1 \\ -2x + 1 & \text{if } x < 1 \end{cases}$

c $f(x) = \begin{cases} -x & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 0 \\ -3x + 1 & \text{if } x \geq 0 \end{cases}$

4 The rule of a particular function is given below. For what values of x is the graph of this function discontinuous?

$$y = \begin{cases} 2, & x < 1 \\ (x-4)^2 - 9, & 1 \leq x < 7 \\ x - 7, & x \geq 7 \end{cases}$$

16.5 When is a function differentiable?

A function f is said to be differentiable at x if $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists.

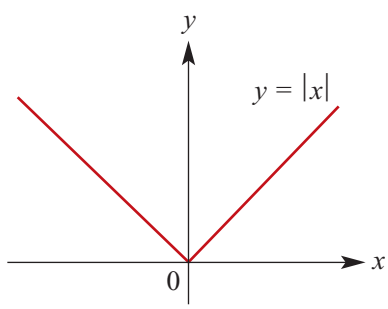
The polynomial functions considered in this chapter are differentiable for all x. However this is not true for all functions.

$$\text{Let } f: R \to R,\ f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This function is called the modulus function or absolute value function and it is denoted by $f(x) = |x|$, f is not differentiable at $x = 0$:

$$\frac{f(0+h) - f(0)}{h} = \begin{cases} \dfrac{h}{h} & h > 0 \\ \dfrac{-h}{h} & h < 0 \end{cases}$$

$$= \begin{cases} 1 & h > 0 \\ -1 & h < 0 \end{cases}$$



So $\lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist.

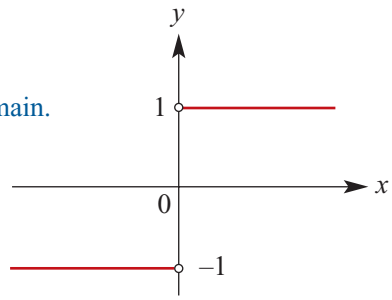
i.e. f is not differentiable at $x = 0$.

Note: The gradient to the left of 0 is -1 and to the right of 0 the gradient is 1.

Example 24

Let $f: R \to R$, $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ $(= |x|)$

Sketch the graph of the derivative for a suitable domain.



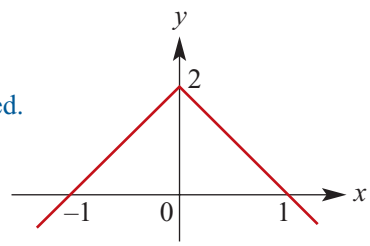
Solution

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$f'(x)$ is not defined at $x = 0$

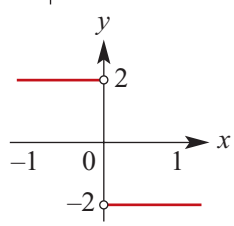
Example 25

Draw a sketch graph of f' where the graph of f is as illustrated. Indicate where f' is not defined.



Solution

The derivative does not exist at $x = 0$;
i.e. the function is not differentiable at $x = 0$.



It was shown in the previous section that some hybrid functions are continuous for R. There are hybrid functions which are differentiable for R. The smoothness of the 'joins' determines if this is the case.

Example 26

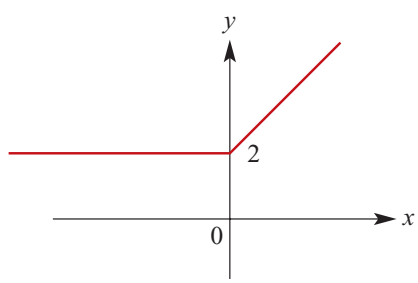
For the function with following rule find $f'(x)$ and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ 2x + 1 & \text{if } x < 0 \end{cases}$$

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$$

In particular $f'(0)$ is defined and is equal to 2. Also $f(0) = 1$. The two sections of the graph of $y = f(x)$ join smoothly at (0, 1).



Example 27

For the function with rule

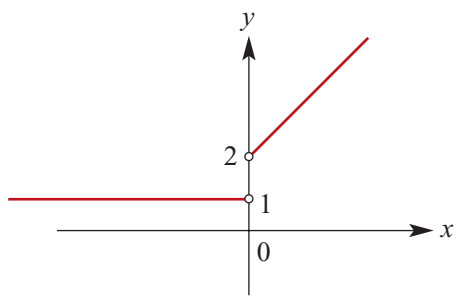
$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases}$$

state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$.

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x > 0 \\ 1 & \text{if } x < 0 \end{cases}$$

$f'(0)$ is not defined as the limits from the left and right are not equal. The function is differentiable for $R\backslash\{0\}$.

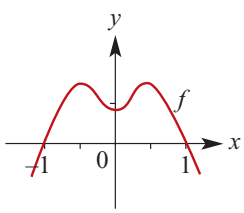


Exercise 16E

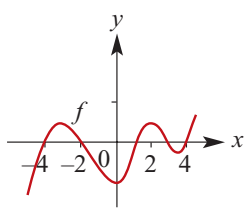
Examples 24–26

1 In each of the figures below a function graph f is given. Sketch the graph of f'. Obviously your sketch of f' cannot be exact, but $f'(x)$ should be 0 at values of x for which the gradient of f is zero; $f'(x)$ should be < 0 where the original graph slopes downward, and so on.

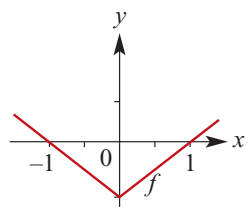
a



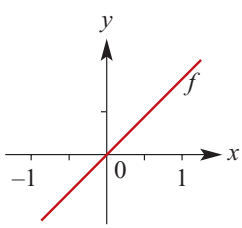
b



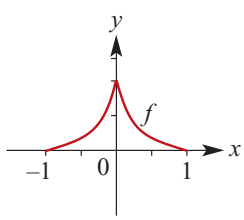
c



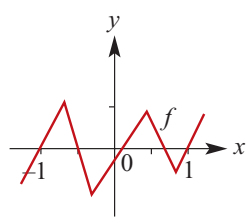
d



e



f



Example 26

2 For the function with following rule find $f'(x)$ and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} -x^2 + 3x + 1 & \text{if } x \geq 0 \\ 3x + 1 & \text{if } x < 0 \end{cases}$$

Example 27

3 For the function with following rule state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 1 \\ -2x + 3 & \text{if } x < 1 \end{cases}$$

Example 27

4 For the function with following rule state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} -x^2 - 3x + 1 & \text{if } x \geq -1 \\ -2x + 3 & \text{if } x < -1 \end{cases}$$

16.6 Limits – a formal approach

Note: This section is optional and has been included for students who wish to consider the study of limits from a more formal perspective. It is *not* essential for Mathematical Methods Units 1 & 2.

If a is a number, then we define the *absolute value* of a to be

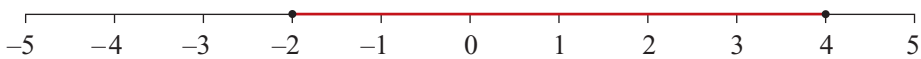
$$\begin{array}{ll} a \text{ itself} & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{array}$$

Thus the absolute value of a number is always a positive number or 0.

For example, the absolute value of 3 is itself and the absolute value of -3 is $-(-3) = 3$.

We denote the absolute value of a number a by $|a|$. Hence $|3| = 3$ and $|-3| = 3$.

In the following example absolute value is used as a measure of distance on the number line. For example, $\{x: |x - 1| \leq 3\}$ is illustrated on the number line below.



The interval is centred at 1 and the extremities are 3 units from 1.

Consider $\{x: |x| < 1\}$. This may be described as the set of numbers on the number line which are less than 1 unit from 0.

$$\begin{aligned} \text{We note } \{x: |x| < 1\} &= \{x: 0 - 1 < x < 0 + 1\} \\ &= \{x: -1 < x < 1\} \\ \text{and } \{x: |x - 1| \leq 3\} &= \{x: 1 - 3 \leq x \leq 3 + 1\} \\ &= \{x: -2 \leq x \leq 4\} \\ \text{In a similar way } \{x: |x - 2| < 3\} &= \{x: 2 - 3 < x < 3 + 2\} \\ &= \{x: -1 < x < 5\} \end{aligned}$$

Definition

Let f be a function defined on a set S with the possible exception of the number c in S.

We say $\lim\limits_{x \to c} f(x) = L$ if, given any number $\varepsilon > 0$ (however small), we can find a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ when $0 < |x - c| < \delta$, x in S.

The condition $0 < |x - c|$ in this definition is important. It shows that $x \neq c$.

It is possible to have two functions, f and g, such that

$$f(x) = g(x),\ x \neq c,$$

and $$f(c) \neq g(c)$$

and $$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

For example:

let $c = 1$

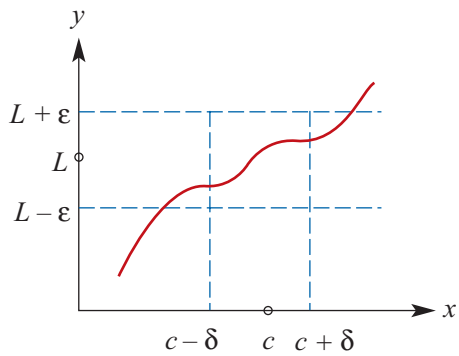
and let $f: R \to R,\ f(x) = \begin{cases} x & x \neq 1 \\ 2 & x = 1 \end{cases}$

and $g: R \to R,\ g(x) = \begin{cases} x & x \neq 1 \\ 3 & x = 1 \end{cases}$

The definition given above may be paraphrased as follows.

Given any $\varepsilon > 0$ we can find a $\delta > 0$ so that, for all $x \neq c$ in the interval $(c - \delta, c + \delta)$, $f(x)$ is in $(L - \varepsilon, L + \varepsilon)$.

To see what this means graphically, draw the horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ (see graph). Then, however close together these lines may be, we can determine a vertical strip bounded by the lines $x = c - \delta$, $x = c + \delta$ such that every point on the graph of f lying in this strip, except perhaps $(c, f(c))$, is between the two horizontal lines.



Example 28

Show that $\lim_{x \to 0} x^2 = 0$.

Solution

The definition of limit given above tells us that for the limit to exist, given $\varepsilon > 0$, there exists δ such that $0 < |x - 0| < \delta$ implies $|x^2 - 0| < \varepsilon$. It is important to note that the ε is arbitrary (and in particular may be chosen to be as small as you like).

In this type of example it often helps to use equivalent statements. We use the symbol '$\Leftrightarrow$' for 'equivalence of' statements. This is not a convention used in the rest of this book but in this formal context will aid presentation.

$$|x^2 - 0| < \varepsilon \Leftrightarrow |x^2| < \varepsilon \Leftrightarrow |x| < \sqrt{\varepsilon}$$

Thus for a given ε choose $\delta = \sqrt{\varepsilon}$

Thus for $0 < |x - 0| < \sqrt{\varepsilon}$ we have $|x^2 - 0| < \varepsilon$

Example 29

Show that $\lim_{h \to 0} \dfrac{2h + h^2}{h} = 2$.

Solution

Consider $f: R \setminus \{0\} \to R,\ f(h) = 2 + h$

Then $|f(h) - 2| < \varepsilon \Leftrightarrow |2 + h - 2| < \varepsilon \Leftrightarrow |h| < \varepsilon$

Choose $\delta = \varepsilon$

Thus for $0 < |h - 0| < \varepsilon$ we have $|f(h) - 2| < \varepsilon$

Example 30

Show that $\lim\limits_{x \to 1} \dfrac{5x^2 - 5}{x - 1} = 10$

Solution

Let $f: R \setminus \{1\} \to R,\ f(x) = \dfrac{5x^2 - 5}{x - 1}$

$$|f(x) - 10| < \varepsilon \Leftrightarrow |5x + 5 - 10| < \varepsilon \Leftrightarrow 5\,|x - 1| < \varepsilon \Leftrightarrow |x - 1| < \frac{\varepsilon}{5}$$

Choose $\delta = \dfrac{\varepsilon}{5}$

For $0 < |x - 1| < \dfrac{\varepsilon}{5}$ we have $|f(x) - 10| < \varepsilon$

These examples illustrate that we may achieve a prescribed closeness and the formal language used above describes this idea precisely.

MC TEST D&D TEST

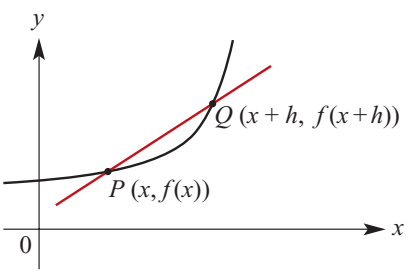
Chapter summary

- The notation for limit as h approaches 0 is written as $\lim_{h \to 0}$.
- The following are important results for limits:
  - $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
    i.e. the limit of the sum is the sum of the limits.
  - $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$, k being a given number
  - $\lim_{x \to c} (f(x)g(x)) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$
    i.e. the limit of the product is the product of the limits.
  - $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided $\lim_{x \to c} g(x) \neq 0$
    i.e. the limit of the quotient is the quotient of the limits.
- A function f is defined as continuous at the point $x = a$ if three conditions are met:
  1 $f(x)$ is defined at $x = a$
  2 $\lim_{x \to a} f(x)$ exists
  3 $\lim_{x \to a} f(x) = f(a)$
- The function is said to be discontinuous at a point if it is not continuous at that point. We say that a function is continuous everywhere if it is continuous for all real numbers.
- For the graph of $y = f(x)$ of the function $f : R \to R$:

  The gradient of the chord $PQ = \frac{f(x+h) - f(x)}{h}$.

  The gradient of the graph at P is given by

  $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.



  This limit gives a rule for the derived function denoted by f', where $f' : R \to R$ and

  $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- The general rule of the derived function of $f(x) = x^n$, $n = 1, 2, 3, \ldots$

  For $f(x) = x^n$, $f'x = nx^{n-1}$, $n = 1, 2, 3, \ldots$

  For $f(x) = 1$, $f'(x) = 0$

  For example:

  For $f(x) = x^2$, $f'(x) = 2x$. For $f(x) = x^3$, $f'(x) = 3x^2$.

  For $f(x) = x^4$, $f'(x) = 4x^3$. For $f(x) = 1$, $f'(x) = 0$.
- The derivative of a number multiple is the multiple of the derivative:

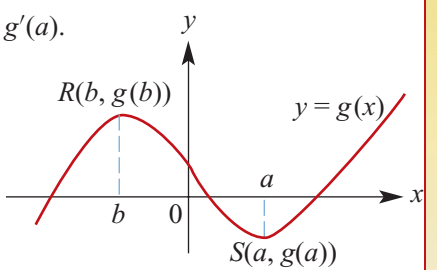
  For $g(x) = kf(x)$, where k is a constant

  $g'(x) = kf'(x)$

  For example: $g(x) = 3x^2$, $g'(x) = 3(2x) = 6x$

Review

- The derivative of a constant is always zero:
  For $g(x) = a$, $g'(x) = 0$
  For example: $f(x) = 27.3$, $f'(x) = 0$
- The derivative of the sum is the sum of the derivatives:
  If $f(x) = g(x) + h(x)$ then
  $f'(x) = g'(x) + h'(x)$.
  For example: $f(x) = x^2 + x^3$, $f'(x) = 2x + 3x^2$
  $g(x) = 3x^2 + 4^3$, $g'(x) = 3(2x) + 4(3x^2) = 6x + 12x^2$
- At a point $(a, g(a))$ on the curve $y = g(x)$ the gradient is $g'(a)$.
  For $x < b$ the gradient is positive, i.e. $g'(x) > 0$
  $x = b$ the gradient is zero, i.e. $g'(b) = 0$
  $b < x < a$ the gradient is negative, i.e. $g'(a) < 0$
  $x = a$ the gradient is zero, i.e. $g'(a) = 0$
  $x > a$ the gradient is positive, i.e. $g'(x) > 0$



Multiple-choice questions

1 The gradient of the curve $y = x^3 + 4x$ at the point where $x = 2$ is
A 12 **B** 4 **C** 10 **D** 16 **E** 8

2 The gradient of the chord of the curve $y = 2x^2$ between the points where $x = 1$ and $x = 1 + h$ is given by
A $2(x + h)^2 - 2x^2$ **B** $4 + 2h$ **C** 4 **D** $4x$ **E** $4 + h$

3 If $y = 2x^4 - 5x^3 + 2$, then $\frac{dy}{dx}$ equals
A $8x^3 - 5x^2 + 2$ **B** $4x^4 - 15x^2 + 2$ **C** $4x^4 - 10x^2$
D $8x^3 - 15x + 2$ **E** $8x^3 - 15x^2$

4 If $f(x) = x^2(x + 1)$ then $f'(-1)$ equals
A -1 **B** 1 **C** 2 **D** -2 **E** 5

5 If $f(x) = (x - 3)^2$, then $f'(x)$ equals
A $x - 3$ **B** $x - 6$ **C** $2x - 6$ **D** $2x + 9$ **E** $2x$

6 If $y = \frac{2x^4 + 9x^2}{3x}$, then $\frac{dy}{dx}$ equals
A $\frac{2x^4}{3} + 6x$ **B** $2x + 3$ **C** $2x^2 + 3$ **D** $\frac{8x^3 + 18x}{3}$ **E** $8x^3 + 18x$

7 Given that $y = x^2 - 6x + 9$, the values of x for which $\frac{dy}{dx} \geq 0$ are
A $x \geq 3$ **B** $x > 3$ **C** $x \geq -3$ **D** $x \leq -3$ **E** $x < 3$

8 If $y = 2x^4 - 36x^2$, the points at which the tangent to the curve is parallel to the x-axis are
A 1, 0 and 3 **B** 0 and 3 **C** -3 and 3 **D** 0 and -3 **E** -3, 0 and 3

9 The coordinates of the point on the graph of $y = x^2 + 6x - 5$ at which the tangent is parallel to the line $y = 4x$ are

A $(-1, -10)$ **B** $(-1, -2)$ **C** $(1, 2)$ **D** $(-1, 4)$ **E** $(-1, 10)$

10 If $y = -2x^3 + 3x^2 - x + 1$, then $\frac{dy}{dx}$ is equals

A $6x^2 + 6x - 1$ **B** $-6x^2 + 6x$ **C** $-6x^2 + 3x - 1$
D $-6x^2 + 6x - 1$ **E** $6x^2 - 6x - 1$

Short-answer questions (technology-free)

1 Find $\frac{dy}{dx}$ when:

a $y = 3x^2 - 2x + 6$ **b** $y = 5$ **c** $y = 2x(2 - x)$
d $y = 4(2x - 1)(5x + 2)$ **e** $y = (x + 1)(3x - 2)$ **f** $y = (x + 1)(2 - 3x)$

2 Find $\frac{dy}{dx}$ when:

a $y = -x$ **b** $y = 10$ **c** $y = \frac{(x + 3)(2x + 1)}{4}$

d $y = \frac{2x^3 - x^2}{3x}$ **e** $y = \frac{x^4 + 3x^2}{2x^2}$

3 For each of the following functions find the y-coordinates and the gradient at the point on the curve for the given value of x:

a $y = x^2 - 2x + 1, x = 2$ **b** $y = x^2 - 2x,\ x = -1$
c $y = (x + 2)(x - 4), x = 3$ **d** $y = 3x^2 - 2x^3,\ x = -2$

4 Find the coordinates of the points on the curves given by the following equations at which the gradient has the given value:

a $y = x^2 - 3x + 1;\ \frac{dy}{dx} = 0$ **b** $y = x^3 - 6x^2 + 4;\ \frac{dy}{dx} = -12$
c $y = x^2 - x^3;\ \frac{dy}{dx} = -1$ **d** $y = x^3 - 2x + 7;\ \frac{dy}{dx} = 1$
e $y = x^4 - 2x^3 + 1;\ \frac{dy}{dx} = 0$ **f** $y = x(x - 3)^2;\ \frac{dy}{dx} = 0$

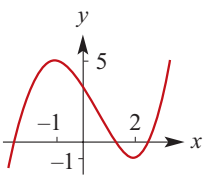
5 For the function with rule $f(x) = 3(2x - 1)^2$ find the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) > 0$
d $f'(x) < 0$ **e** $f(x) > 0$ **f** $f'(x) = 3$

6 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point (1, 1). Find:

a the values of a and b **b** the coordinates of the points where the gradient is 0.

7 Sketch the graph of $y = f'(x)$. (All details cannot be determined but the axis intercepts and shape of graph can be determined.)



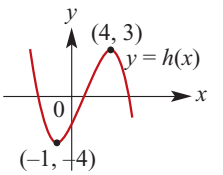
Review

8 For the graph of $y = h(x)$ find:

a $\{x: h'(x) > 0\}$

b $\{x: h'(x) < 0\}$

c $\{x: h'(x) = 0\}$

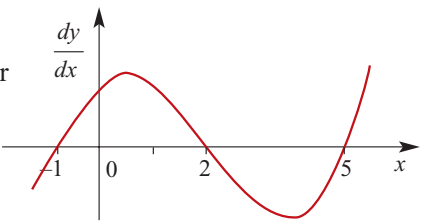


Extended-response questions

1 The diagram to the right shows part of the graph of $\frac{dy}{dx}$ against x.

Sketch a possible shape of y against x over the same interval if:

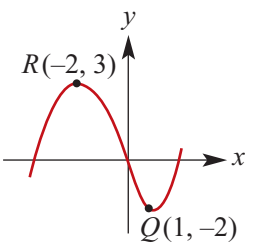
- $y = -1$ when $x = -1$
- $y = 0$ when $x = 0$
- $y = 1$ when $x = 2$.



2 The graph shown is that of a polynomial of the form $P(x) = ax^3 + bx^2 + cx + d$.

Find the values of a, b, c and d.

Note: $Q(1, -2)$ is not a turning point.



3 A body moves in a path described by the equation $y = \frac{1}{5}x^5 + \frac{1}{2}x^4$, $x \geq 0$.

Units are in kilometres and x and y are the horizontal and vertical axes respectively.

a What will be the direction of motion (give the answer as angle between direction of motion and the x-axis) when the x-value is

i 1 km? **ii** 3 km?

b Find a value of x for which the gradient of the path is 32.

4 A trail over a mountain pass can be modelled by the curve with equation $y = 2 + 0.12x - 0.01x^3$, where x and y are, respectively, the horizontal and vertical distances measured in kilometres, $0 \leq x \leq 3$.

a Find the gradients at the beginning and the end of the trail.

b Calculate the point where the gradient is zero, and calculate also the height of the pass.

5 A tadpole begins to swim vertically upwards in a pond and after t seconds it is $25 - 0.1t^3$ cm below the surface.

a How long does the tadpole take to reach the surface, and what is its speed then?

b What is the average speed over this time?

6 **a** Show that the gradients of the curve $y = x(x - 2)$ at the points (0, 0) and (2, 0) only differ in sign. What is the geometrical interpretation for this?

b If the gradient of the curve $y = x(x - 2)(x - 5)$ at the points (0, 0), (2, 0) and (5, 0) are l, m and n respectively, show that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 0$.

CHAPTER

17

Applications of Differentiation of Polynomials

Objectives

- To be able to find the equation of the **tangent** and the **normal** at a given point of a polynomial curve.
- To use the derivative of a polynomial in **rates of change** problems.
- To be able to find the **stationary points** on the curves of certain polynomial functions and to state the nature of such points.
- To use differential calculus for **sketching the graphs** of polynomial functions.
- To be able to apply differential calculus to the solution of **maximum and minimum problems**.

17.1 Tangents and normals

The derivative of a function is a new function which gives the measure of the gradient at each point of the curve. If the gradient is known, it is possible to find the equation of the tangent for a given point on the curve.

Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then if f is differentiable for $x = x_1$, the equation of the tangent at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.

Example 1

Find the equation of the tangent to the curve $y = x^3 + \frac{1}{2}x^2$ at the point $x = 1$.

Solution

When $x = 1$, $y = \frac{3}{2}$ so $\left(1, \frac{3}{2}\right)$ is a point on the tangent.

Further $\frac{dy}{dx} = 3x^2 + x$.

Thus the gradient of the tangent to the curve at $x = 1$ is 4 and the equation of the tangent is $y - \frac{3}{2} = 4(x - 1)$

which becomes $y = 4x - \frac{5}{2}$.

The **normal** to a curve at a point on the curve is the line which passes through the point and is perpendicular to the tangent at that point.

Recall from Chapter 2 that lines with gradients m_1 and m_2 are perpendicular if, and only if, $m_1 m_2 = -1$.

Thus if a tangent has a gradient m, the normal has gradient $-\frac{1}{m}$.

Example 2

CAS APPENDIX

Find the equation of the normal to the curve with equation $y = x^3 - 2x^2$ at the point $(1, -1)$.

Solution

The point $(1, -1)$ is on the normal.

Further $\frac{dy}{dx} = 3x^2 - 4x$.

Gradient of tangent $= -1$

Thus the gradient of the normal at $x = 1$ is $\frac{-1}{-1} = +1$

Hence the equation of the normal is $y + 1 = 1(x - 1)$

i.e. the equation of the normal is $y = x - 2$.

Exercise 17A

Examples 1, 2

1 Find the equation of the tangent and the normal at the given point for:

a $f(x) = x^2$, $(2, 4)$ **b** $f(x) = (2x - 1)^2$, $(2, 9)$

c $f(x) = 3x - x^2$, $(2, 2)$ **d** $f(x) = 9x - x^3$, $(1, 8)$

2 Find the equation of the tangent to the curve with equation $y = 3x^3 - 4x^2 + 2x - 10$ at the point of intersection with the y-axis.

3 Find the equation of the tangent to $y = x^2$ at the point (1, 1) and the equation of the tangent to $y = \frac{1}{6}x^3$ at the point $\left(2, \frac{4}{3}\right)$.
Show that these tangents are parallel and find the perpendicular distance between them.

4 Find the equations of the tangents to the curve $y = x^3 - 6x^2 + 12x + 2$ which are parallel to the line $y = 3x$.

5 The curve with the equation $y = (x - 2)(x - 3)(x - 4)$ cuts the x-axis at the points $P = (2, 0)$, $Q = (3, 0)$, $R = (4, 0)$.

a Prove that the tangents at P and R are parallel.

b At what point does the normal to the curve at Q cut the y-axis?

6 For the curve with equation $y = x^2 + 3$ show that $y = 2ax - a^2 + 3$ is the equation of the tangent at the point $(a, a^2 + 3)$.
Hence find the coordinates of the two points on the curve, the tangents of which pass through the point (2, 6).

7 **a** Find the equation of the tangent at the point (2, 4) to the curve $y = x^3 - 2x$.

b Find the coordinates of the point where the tangent meets the curve again.

8 **a** Find the equation of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at the point (1, 4).

b At what points of the curve is the tangent parallel to the line $4x + y - 3 = 0$?

17.2 Rates of change

We have used the derivative of a function to find the gradient of the corresponding curve. It is clear that the process of differentiation may be used to tackle many kinds of problems involving rates of change.

For the function with rule $f(x)$ the average rate of change for $x \in [a, b]$ is given by $\frac{f(b) - f(a)}{b - a}$. The instantaneous rate of change of f with respect to x when $x = a$ is $f'(a)$. Average rate of change has been discussed in earlier chapters.

The derivative of y with respect to x, $\frac{dy}{dx}$, gives the instantaneous rate of change of y with respect to x.

If $\frac{dy}{dx} > 0$ the change is an increase in the value of y corresponding to an increase in x and if $\frac{dy}{dx} < 0$ the change is a decrease in the value of y corresponding to an increase in x.

Example 3

For the function with rule $f(x) = x^2 + 2x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

Solution

a The average rate of change $= \dfrac{f(3) - f(2)}{3 - 2} = 15 - 8 = 7$

b For the interval the average rate of change $= \dfrac{f(2+h) - f(2)}{2 + h - 2}$

$$= \frac{(2+h)^2 + 2(2+h) - 8}{h}$$

$$= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h}$$

$$= \frac{6h + h^2}{h}$$

$$= 6 + h$$

c When $x = 2$, the instantaneous rate of change $= f'(2) = 6$. This can also be seen from the result of part b.

Example 4

A balloon develops a microscopic leak and decreases in volume. Its volume $V(\text{cm}^3)$ at time t (seconds) is $V = 600 - 10t - \dfrac{1}{100}t^2$, $t > 0$.

a Find the rate of change of volume after

i 10 seconds **ii** 20 seconds

b For how long could the model be valid?

Solution

a $\dfrac{dV}{dt} = -10 - \dfrac{t}{50}$

i When $t = 10$

$$\frac{dV}{dt} = -10 - \frac{1}{5}$$

$$= -10\frac{1}{5}$$

i.e. the volume is decreasing at a rate of $10\frac{1}{5}$ cm^3 per second.

ii When $t = 20$

$$\frac{dV}{dt} = -10 - \frac{2}{5}$$

$$= -10\frac{2}{5}$$

i.e. the volume is decreasing at a rate of $10\frac{2}{5}$ cm^3 per second.

b The model will not be meaningful when $V < 0$. Consider $V = 0$.

$600 - 10t - \dfrac{1}{100}t^2 = 0$

$\therefore \quad t = \dfrac{10 \pm \sqrt{100 + \frac{1}{100} \times 600 \times 4}}{-0.02}$

$\therefore \quad t = -1056.78$ or $t = 56.78$ (to 2 decimal places)

$\therefore$ the model may be suitable for $0 < t < 56.78$

Displacement, velocity and acceleration for a body moving in a straight line were introduced in Section 15.5. Displacement was specified with respect to a reference point O on that line.

- Velocity (v m/s): $v = \dfrac{ds}{dt}$
- Acceleration (a m/s^2): $a = \dfrac{dv}{dt}$

Example 5

A car starts from rest and moves a distance s metres in t seconds, where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$. What is the initial acceleration and the acceleration when $t = 2$?

Solution

$$s = \frac{1}{6}t^3 + \frac{1}{4}t^2$$

$$\text{Velocity} = \frac{ds}{dt}$$

$$= \frac{1}{2}t^2 + \frac{1}{2}t$$

Velocity is given in metres per second or m/s.

Let $v = \dfrac{ds}{dt}$

Then $\dfrac{dv}{dt} = t + \dfrac{1}{2}$

Let $a = \dfrac{dv}{dt}$

When $t = 0, a = \dfrac{1}{2}$

When $t = 2, a = 2\dfrac{1}{2}$

Hence the required accelerations are $\frac{1}{2}$ m/s^2 and $2\frac{1}{2}$ m/s^2.

Example 6

A point moves along a straight line so that its distance x cm from a point O at time t seconds is given by the formula $x = t^3 - 6t^2 + 9t$.

a Find at what times and in what positions the point will have zero velocity.

b Find its acceleration at those instants.

c Find its velocity when its acceleration is zero.

Solution

Velocity $= v$

$$= \frac{dx}{dt}$$

$$= 3t^2 - 12t + 9$$

a When $v = 0$

$3(t^2 - 4t + 3) = 0$

$(t - 1)(t - 3) = 0$

$t = 1 \quad \text{or} \quad t = 3$

i.e. the velocity is zero when $t = 1$ and $t = 3$ and where $x = 4$ and $x = 0$.

b Acceleration $= \frac{dv}{dt}$

$= 6t - 12$

$\therefore$ acceleration $= -6$ m/s^2 when $t = 1$ and acceleration $= 6$ m/s^2 when $t = 3$.

c Acceleration is zero when $6t - 12 = 0$, i.e. when $t = 2$

When $t = 2$

velocity $v = 3 \times 4 - 24 + 9$

$= -3$ m/s

Exercise 17B

Example 3

1 Let $y = 35 + 12x^2$.

a Find the change in y as x changes from 1 to 2. What is the average rate of change of y with respect to x in this interval?

b Find the change in y as x changes from $2 - h$ to 2. What is the average rate of change of y with respect to x in this interval?

c Find the rate of change of y with respect to x when $x = 2$.

Example 4

2 According to a business magazine the expected assets, $\$M$, of a proposed new company will be given by $M = 200\,000 + 600t^2 - \frac{200}{3}t^3$, where t is the number of months after the business is set up.

a Find the rate of growth of assets at time t months.

b Find the rate of growth of assets at time $t = 3$ months.

c Will the rate of growth of assets be 0 at any time?

Examples 5, 6

3 The position of a body moving in a straight line, x cm from the origin, at time t seconds ($t \geq 0$) is given by $x = \frac{1}{3}t^3 - 12t + 6$.

a Find the rate of change of position with respect to time at $t = 3$.

b Find the time at which the velocity is zero.

4 Let $s = 10 + 15t - 4.9t^2$ be the height (in metres) of an object at time t (in seconds).

a Find the velocity at time t. **b** Find the acceleration at time t.

5 As a result of a survey, the marketing director of a company found that the revenue, $\$R$, from selling n produced items at $\$P$ is given by the rule $R = 30P - 2P^2$.

a Find $\frac{dR}{dP}$ and explain what it means.

b Calculate $\frac{dR}{dP}$ when $P = 5$ and $P = 10$.

c For what selling prices is revenue rising?

6 The population, P, of a new housing estate t years after 30 January 1997 is given by the rule $P = 100(5 + t - 0.25t^2)$.

a Find the rate of change of the population after:

i 1 year **ii** 2 years **iii** 3 years

7 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by $V(t) = \frac{5}{8}\left(10t^2 - \frac{t^3}{3}\right)$, $0 \leq t \leq 20$.

a Find the volume of water in the flask at time:

i $t = 0$ **ii** $t = 20$

b Find the rate of flow of water into the flask.

c Sketch the graph of $V'(t)$ against t for $0 \leq t \leq 20$.

8 A model aeroplane flying level at 250 m above the ground suddenly dives. Its height h (m) above the ground at time (t seconds) after beginning to dive is given by

$$h(t) = 8t^2 - 80t + 250 \qquad t \in [0, 10].$$

Find the rate at which the plane is losing height at:

a $t = 1$ **b** $t = 3$ **c** $t = 5$

9 A particle moves along a straight line so that after t seconds its distance from O, a fixed point on the line, is s m, where $s = t^3 - 3t^2 + 2t$.

a When is the particle at O?

b What is its velocity and acceleration at these times?

c What is the average velocity during the first second?

10 A car starts from rest and moves a distance s (m) in t (s), where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$.

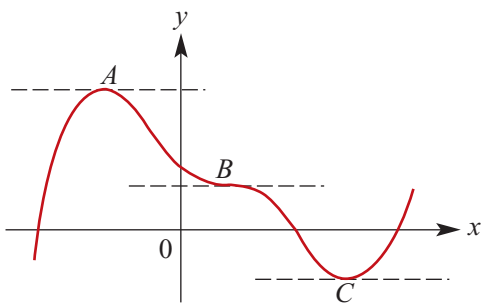
a What is the acceleration when $t = 0$?

b What is the acceleration when $t = 2$?

17.3 Sketch graphs of polynomials

Stationary points

In the previous chapter we have seen that the gradient at a point $(a, g(a))$ of the curve with rule $y = g(x)$ is given by $g'(a)$. A point $(a, g(a))$ on a curve $y = g(x)$ is said to be a stationary point if $g'(a) = 0$.



(Equivalently: for $y = g(x)$, $\frac{dy}{dx} = 0$ when $x = a$ implies that $(a, g(a))$ is a stationary point.)

For example, in the graph above there are stationary points at A, B and C. At such points the tangents are parallel to the x-axis (illustrated as dotted lines).

The reason for the name **stationary points** becomes clear if we look at an application to motion of a particle.

Example 7

The displacement x metres of a particle moving in a straight line is given by $x = 9t - \frac{1}{3}t^3$ for $0 \leq t \leq 3\sqrt{3}$, where t seconds is the time taken.
Find the maximum displacement.

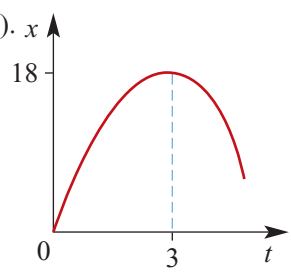
Solution

$\frac{dx}{dt} = 9 - t^2$, and maximum displacement occurs when $\frac{dx}{dt} = 0$.

So $t = 3$ or $t = -3$ (but $t = -3$ lies outside the domain).
At $t = 3$, $x = 18$.

Thus the stationary point is (3, 18) and the maximum displacement is 18 metres.

Note that the stationary point occurs when the rate of change of displacement with respect to time (i.e. velocity) is zero. The particle stopped moving forward at $t = 3$.



Example 8

Find the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $p = 2t^3 - 5t^2 - 4t + 13$ for $t > 0$ **c** $y = 4 + 3x - x^3$

Solution

a $y = 9 + 12x - 2x^2$

$\frac{dy}{dx} = 12 - 4x$

Stationary point occurs when $\frac{dy}{dx} = 0$, i.e. when $12 - 4x = 0$

i.e. at $x = 3$

When $x = 3$, $y = 9 + 12 \times 3 - 2 \times 3^2 = 27$

Thus the stationary point is at (3, 27).

b $p = 2t^3 - 5t^2 - 4t + 13$

$\frac{dp}{dt} = 6t^2 - 10t - 4, t > 0$

$\frac{dp}{dt} = 0$ implies $2(3t^2 - 5t - 2) = 0$

$\therefore \quad (3t + 1)(t - 2) = 0$

$\therefore t = -\frac{1}{3}$ or $t = 2$

But $t > 0$, therefore the only acceptable solution is $t = 2$.

When $t = 2$, $p = 16 - 20 - 8 + 13 = 1$

So the corresponding stationary point is (2, 1).

c $y = 4 + 3x - x^3$

$\frac{dy}{dx} = 3 - 3x^2$

$\frac{dy}{dx} = 0$ implies $3(1 - x^2) = 0$

$\therefore \quad x = \pm 1$

$\therefore$ stationary points occur at (1, 6) and $(-1, 2)$.

Example 9

The curve with equation $y = x^3 + ax^2 + bx + c$ passes through (0, 5) with stationary point (2, 7). Find a, b, c.

Solution

When $x = 0$, $y = 5$

Thus $5 = c$

$\frac{dy}{dx} = 3x^2 + 2ax + b$ and at $x = 2$, $\frac{dy}{dx} = 0$

Therefore $0 = 12 + 4a + b$ (1)

The point (2, 7) is on the curve and therefore

$7 = 2^3 + 2^2a + 2b + 5$

$\therefore \quad 2 = 8 + 4a + 2b$

$\therefore \quad 4a + 2b + 6 = 0$ (2)

Subtract (2) from (1)

$$-b + 6 = 0$$
$$\therefore \quad b = 6$$

Substitute in (1)

$$0 = 12 + 4a + 6$$
$$-18 = 4a$$
$$-\frac{9}{2} = a$$
$$\therefore \quad a = -\frac{9}{2}, b = 6, c = 5$$

Exercise 17C

Example 8

1 Find the coordinates of the stationary points of each of the following functions:

a $f(x) = x^2 - 6x + 3$ **b** $y = x^3 - 4x^2 - 3x + 20, x > 0$

c $z = x^4 - 32x + 50$ **d** $q = 8t + 5t^2 - t^3$ for $t > 0$

e $y = 2x^2(x - 3)$ **f** $y = 3x^4 - 16x^3 + 24x^2 - 10$

Example 9

2 The curve with equation $y = ax^2 + bx + c$ passes through $(0, -1)$ and has a stationary point at $(2, -9)$. Find a, b, c.

3 The curve with equation $y = ax^2 + bx + c$ has a stationary point at $(1, 2)$. When $x = 0$, the slope of the curve is 45°. Find a, b, c.

4 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point $(2, -2)$.

a Find the values of a and b. **b** Find the coordinates of the turning point.

5 The curve of the equation $y = x^2 + ax + 3$ has a stationary point when $x = 4$. Find a.

6 The curve with equation $y = x^2 - ax + 4$ has a stationary point when $x = 3$. Find a.

7 Find the coordinates of the stationary points of each of the following:

a $y = x^2 - 5x - 6$ **b** $y = (3x - 2)(8x + 3)$

c $y = 2x^3 - 9x^2 + 27$ **d** $y = x^3 - 3x^2 - 24x + 20$

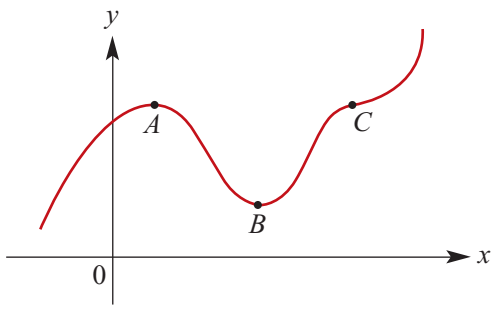
e $y = (x + 1)^2(x + 4)$ **f** $y = (x + 1)^2 + (x + 2)^2$

8 The curve with equation $y = ax^2 + bx + 12$ has a stationary point at $(1, 13)$. Find a and b.

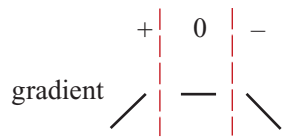
9 The curve with equation $y = ax^3 + bx^2 + cx + d$ has a gradient of -3 at $\left(0, 7\frac{1}{2}\right)$ and a turning point at $(3, 6)$. Find a, b, c and d.

17.4 Types of stationary points

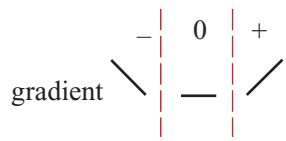
The graph below has three stationary points A, B, C.



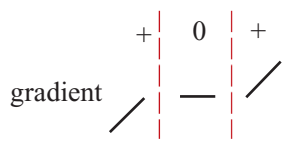
A Point A is called a **local maximum** point. Notice that immediately to the left of A the gradient is positive and immediately to the right the gradient is negative, as shown in the diagram on the right.



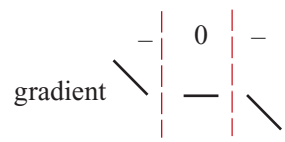
B Point B is called a **local minimum** point. Notice that immediately to the left of B the gradient is negative and immediately to the right the gradient is positive, as shown in the diagram on the right.



C The point C is called a **stationary point of inflexion**, as shown in the diagram on the right.



Clearly it is also possible to have stationary points of inflexion for which the diagram would be like this:



Stationary points of type A and B are referred to as **turning points**.

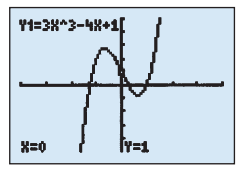
Example 10

For the function $f: R \to R$, $f(x) = 3x^3 - 4x + 1$:

a find the stationary points and state their nature **b** sketch the graph

Solution

It is useful to use the graphics calculator to plot the graph before commencing the sketch process. The graphics calculator does not give exact values.



a $y = f(x)$ has stationary points where $f'(x) = 0$.

$f'(x) = 9x^2 - 4 = 0$

$\therefore\ x = \pm\frac{2}{3}$

There are stationary points at $\left(-\frac{2}{3}, f\left(-\frac{2}{3}\right)\right)$, and $\left(\frac{2}{3}, f\left(\frac{2}{3}\right)\right)$,

that is at $\left(-\frac{2}{3}, 2\frac{7}{9}\right)$, and $\left(\frac{2}{3}, -\frac{7}{9}\right)$.

$f'(x)$ is of constant sign for each of

$$\left\{x: x < -\frac{2}{3}\right\}, \left\{x: -\frac{2}{3} < x < \frac{2}{3}\right\} \quad \text{and} \quad \left\{x: x > \frac{2}{3}\right\}$$

To calculate the sign of $f'(x)$ for each of these sets, simply choose a representative number in the set.

Thus $f'(-1) = 9 - 4 = 5 > 0$

$f'(0) = 0 - 4 = -4 < 0$

$f'(1) = 9 - 4 = 5 > 0$

We can thus put together the following table:

x		$-\frac{2}{3}$		$\frac{2}{3}$	
$f'(x)$	+	0	−	0	+
shape of f	↗	—	↘	—	↗

$\therefore$ there is a local maximum at $\left(-\frac{2}{3}, 2\frac{7}{9}\right)$ and a local minimum at $\left(\frac{2}{3}, -\frac{7}{9}\right)$.

b To sketch the graph of this function we need to find the axis intercepts and investigate the behaviour of the graph for $x > \frac{2}{3}$ and $x < -\frac{2}{3}$.

First, $f(0) = 1$ $\therefore$ the y-intercept is 1.

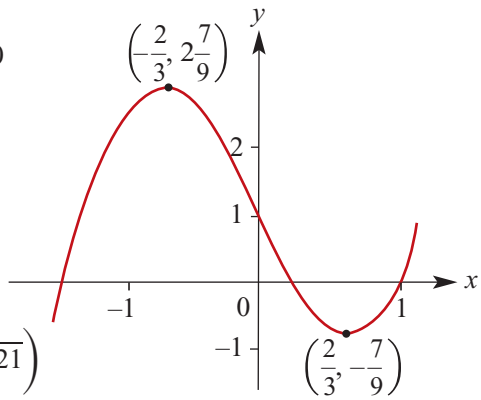
Consider $f(x) = 0$ which implies $3x^3 - 4x + 1 = 0$.

By inspection (factor theorem), $(x - 1)$ is a factor and by division

$$3x^3 - 4x + 1 = (x - 1)(3x^2 + 3x - 1)$$

Now $(x-1)(3x^2+3x-1)=0$
implies that $x=1$ or $3x^2+3x-1=0$

$$3x^2+3x-1$$
$$=3\left[\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-\frac{1}{3}\right]$$
$$=3\left[\left(x+\frac{1}{2}\right)^2-\frac{21}{36}\right]$$
$$=3\left(x+\frac{1}{2}-\frac{1}{6}\sqrt{21}\right)\left(x+\frac{1}{2}+\frac{1}{6}\sqrt{21}\right)$$



Thus the x-intercepts are at

$$x=-\frac{1}{2}+\frac{1}{6}\sqrt{21},\ x=-\frac{1}{2}-\frac{1}{6}\sqrt{21},\ x=1$$

For $x>\frac{2}{3}$, $f(x)$ becomes larger.

For $x<-\frac{2}{3}$, $f(x)$ becomes smaller.

Exercise 17D

Example 10

1 For each of the following find all stationary points and state their nature. Sketch the graph of each function.

a $y=9x^2-x^3$ **b** $y=x^3-3x^2-9x$ **c** $y=x^4-4x^3$

2 Find the stationary points (and state their type) for each of the following functions:

a $y=x^2(x-4)$ **b** $y=x^2(3-x)$ **c** $y=x^4$
d $y=x^5(x-4)$ **e** $y=x^3-5x^2+3x+2$ **f** $y=x(x-8)(x-3)$

Example 10

3 Sketch the graphs of each of the following functions:

a $y=2+3x-x^3$ **b** $y=2x^2(x-3)$ **c** $y=x^3-3x^2-9x+11$

4 The graphs corresponding to each of the following equations have a stationary point at $(-2, 10)$. For each graph find the nature of the stationary point at $(-2, 10)$.

a $y=2x^3+3x^2-12x-10$ **b** $y=3x^4+16x^3+24x^2-6$

5 For the function $y=x^3-6x^2+9x+10$:

a find the intervals where y is increasing, i.e. $\left\{x: \frac{dy}{dx}>0\right\}$
b find the stationary points on the curve corresponding to $y=x^3-6x^2+9x+10$
c sketch the curve carefully between $x=0$ and $x=4$.

6 Find the maximum value of the product of two numbers x and $10-x$ which add up to 10.

7 For the function $f: R \to R,\ f(x) = 1 + 12x - x^3$, determine the values of x for which the function is increasing.

8 Let $f: R \to R$, where $f(x) = 3 + 6x - 2x^3$.

a Determine the values of x for which the graph of f has positive gradient.

b Find the values of x for which the graph of f has increasing gradient.

9 Let $f(x) = x(x + 3)(x - 5)$.

a Find the values of x for which $f'(x) = 0$.

b Sketch the graph of $f(x)$ for $-5 \leq x \leq 6$, giving the coordinates of the intersections with the axes and the coordinates of the turning points.

10 Sketch the graph of $y = x^3 - 6x^2 + 9x - 4$. State the coordinates of the axes intercepts and of the turning points.

11 Find the coordinates of the points on the curve $y = x^3 - 3x^2 - 45x + 2$ where the tangent is parallel to the x-axis.

12 Let $f(x) = x^3 - 3x^2$.

a Find:

i $\{x: f'(x) < 0\}$ **ii** $\{x: f'(x) > 0\}$ **iii** $\{x: f'(x) = 0\}$

b Sketch the graph of $y = f(x)$.

13 Sketch the graph of $y = x^3 - 9x^2 + 27x - 19$ and state the coordinates of stationary points.

14 Sketch the graph of $y = x^4 - 8x^2 + 7$.
All axis intercepts and all turning points should be identified and their coordinates given.

17.5 Finding the tangent to a curve at a given point

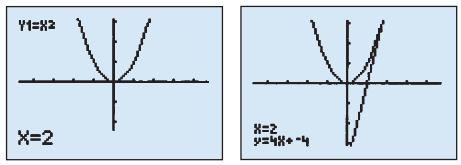
Using a graphics calculator

When the graph of a function is displayed on the screen it is possible to draw the tangent to the graph at a given point. This is done by choosing **5:Tangent(** from the **DRAW** menu, which will cause the graph window to be displayed. If more than one graph is plotted use the up and down arrow keys to choose the graph of the function to which you wish to draw the tangent. Then enter the x-coordinate of the point through which the tangent is to pass.

Press ENTER and the tangent will be drawn. In the bottom left corner of the screen the x-coordinate of the point at which the tangent intersects the graph of the function is displayed, as is the equation of the tangent.

For example, enter $y = x^2$ in the **Y=** window. From the **DRAW** menu select **5:Tangent(** .

Enter $x = 2$. The result is shown below.



Example 11

Find the equations of the tangents to the curve $y = x^3 - 4x^2 + 5x - 12$ at the points where the gradient of the tangents is 1.

Solution

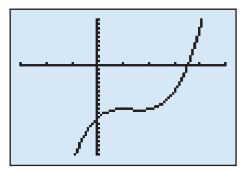
The derivative gives the gradient function, so $\frac{dy}{dx} = 3x^2 - 8x + 5$ is needed to find the x-coordinates of the points where the gradient of the tangents is 1.

Solve $3x^2 - 8x + 5 = 1$

$3x^2 - 8x + 4 = 0$

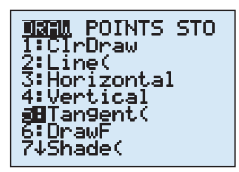
$(3x - 2)(x - 2) = 0$

$x = \frac{2}{3}, 2$



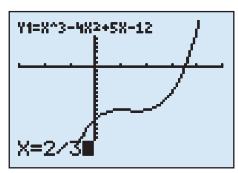
Display the graph of $y = x^3 - 4x^2 + 5x - 12$.

Select **5:Tangent(** from the **DRAW** menu.

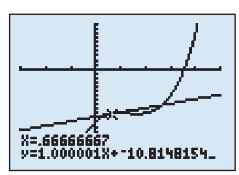


The graph will appear with a cursor, the coordinates of the cursor are displayed at the bottom of the screen and the equation of the curve at the top of the screen.

Enter the x-coordinate of one of the points, e.g. $x = \frac{2}{3}$, and press ENTER.



The tangent to the curve $y = x^3 - 4x^2 + 5x - 12$ at the point where $x = \frac{2}{3}$ is drawn and the equation of the tangent is displayed. The equation is $y = x - 10.8148154$. This is an approximation, with the actual equation being $y = x - \frac{292}{27}$.



By repeating the process the second tangent at the point where $x = 2$ is found to have equation $y = x - 12$.

17.6 Determining stationary points

Using a graphics calculator

Values for the coordinates of stationary points can be found.

CAS APPENDIX

Example 12

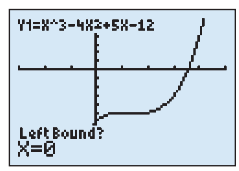
Find the coordinates of the stationary points of the function with rule $y = x^3 - 4x^2 + 5x - 12$.

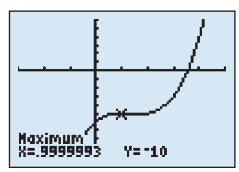
Solution

Enter the rule in the **Y=** screen.

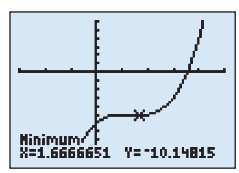
Select **4:maximum** from the **CALC** menu and enter 0 for the left bound. Press ENTER and on the prompt enter 1.5 for the right bound.

Press ENTER twice to achieve the result shown.





Now select **3:minimum** from the CALC menu and enter the left bound as 1.5 and the right bound as 2. The result is as shown.



The stationary points can be found using calculus and occur when $x = 1$ and $x = \frac{5}{3}$. The corresponding y values are -10 and $\frac{-274}{27}$ respectively. Care must be taken if exact values are required.

In the following exercise a graphics calculator is to be used.

Exercise 17E

Example 11 **1** Find the equation of the tangent at the given point:

a $y = x^3 - 2x^2 + 1, (2, 1)$ **b** $y = x^3 - 2x + 1, (2, 5)$

c $y = x^3 - 2x^2 + 2x + 1, (2, 5)$ **d** $y = -x^3 - 2x^2 + 2x + 1, (2, -11)$

Example 12 **2** Use a graphics calculator to find the coordinates of the stationary points of each of the following cubic functions. (Values correct to 2 decimal places where necessary.)

a $y = x^3 - 2x^2 + 1$ **b** $y = x^3 - 2x^2 + 2x + 1$

c $y = 2x^3 - 2x^2 + 1$ **d** $y = -x^3 + 2x^2 + x$

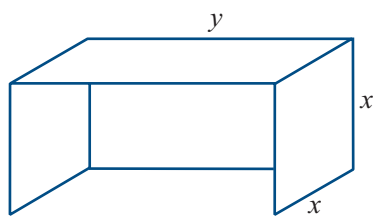
e $y = -x^3 + 3x^2$ **f** $y = x^3 + 2x$

3 A body moves along a straight line so that its distance x cm from a point O at time t seconds is given by $x = t^3 - 6t^2 + 9$. On the same screen plot the graphs of x against t, $\frac{dx}{dt}$ against t, and acceleration against t.

17.7 Applications to maximum and minimum and rate problems

Example 13

A canvas shelter is made up with a back, two square sides and a top. The area of canvas available is 24 m^2.



a Find the dimensions of the shelter that will create the largest possible enclosed volume.

b Sketch the graph of V against x for a suitable domain.

c Find the values of x and y for which $V = 10$ m^3.

Solution

a The volume $V = x^2y$. One of the variables must be eliminated.
We know that area $= 24$ m^2.

$\therefore \quad 2x^2 + 2xy = 24$

Rearranging gives $y = \dfrac{24 - 2x^2}{2x}$

i.e. $y = \dfrac{12}{x} - x$

Substituting in the formula for volume gives

$$V = 12x - x^3$$

Differentiation now gives

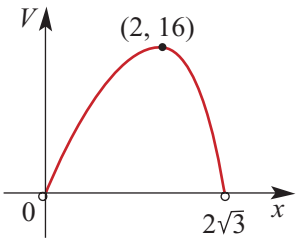
$$\frac{dV}{dx} = 12 - 3x^2$$

Stationary points occur when $\frac{dV}{dx} = 0$, which implies $12 - 3x^2 = 0$.

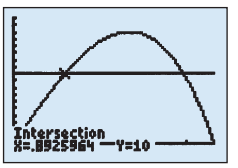
So stationary points occur when $x^2 = 4$, i.e. when $x = \pm 2$, but negative values have no meaning in this problem, so the only solution is $x = 2$.
$\therefore$ maximum at $x = 2$
Dimensions are 2 m, 2 m, 4 m.

x		2	
$\frac{dV}{dx}$	+	0	−
shape of V	/	—	\

b Note that $V > 0$, $x > 0$ and $y > 0$.
This implies $12 - x^2 > 0$ and $x > 0$,
i.e. $0 < x < 2\sqrt{3}$



c Enter $y = 10$ and $y = 12x - x^3$ in the **Y=** window and use **5:intersect** from the **CALC** window to find the solutions of the equation $12x - x^3 = 10$.



The solutions are

$x = 2.9304\ldots$ and $x = 0.8925\ldots$

Possible dimensions to the nearest centimetre are 2.93 m, 2.93 m, 1.16 m and 0.89 m, 0.89 m, 12.55 m.

Example 14

Given that $x + 2y = 4$, calculate the minimum value of $x^2 + xy - y^2$.

Solution

Rearranging $x + 2y = 4$ we have $x = 4 - 2y$.
Let $P = x^2 + xy - y^2$.
Substituting for x: $P = (4 - 2y)^2 + (4 - 2y)y - y^2$

$$= 16 - 16y + 4y^2 + 4y - 2y^2 - y^2$$
$$= 16 - 12y + y^2$$
$$\frac{dP}{dy} = -12 + 2y$$

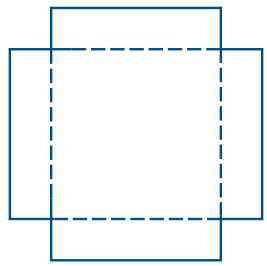
Stationary values occur when $\frac{dP}{dy} = 0$
i.e. when $-12 + 2y = 0$
which implies $y = 6$
$\therefore$ a minimum when $y = 6$

y		6	
$\frac{dP}{dy}$	−	0	+
shape of P	↘	—	↗

When $y = 6$, $x = -8$, the minimum value of $x^2 + xy - y^2$ is -20.

Example 15

From a square piece of metal of side length 2 m, four squares are removed as shown in the figure opposite.
The metal is then folded along the dotted lines to give an open box with sides of height x m.



a Show that the volume, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.

b Find the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.

c Sketch the graph of V against x for a suitable domain.

d Find the value(s) of x for which $V = 0.5$ m^3.

Solution

a Length of box $= (2 - 2x)$ metres, height $= x$ metres

$$\begin{aligned}\therefore \text{Volume} &= (2-2x)^2 x \\ &= (4 - 8x + 4x^2)x \\ &= 4x^3 - 8x^2 + 4x \text{ m}^3\end{aligned}$$

b Let $V = 4x^3 - 8x^2 + 4x$.
Maximum point will occur when $\dfrac{dV}{dx} = 0$.

$\dfrac{dV}{dx} = 12x^2 - 16x + 4$ and $\dfrac{dV}{dx} = 0$ implies that

$$12x^2 - 16x + 4 = 0$$
$$\therefore \quad 3x^2 - 4x + 1 = 0$$
$$\therefore \quad (3x - 1)(x - 1) = 0$$
$$\therefore x = \frac{1}{3} \text{ or } x = 1$$

Note: When $x = 1$, the length of box $= 2 - 2x$, which is zero, $\therefore$ only value to be considered is $x = \dfrac{1}{3}$.

We show the entire gradient chart for completeness.

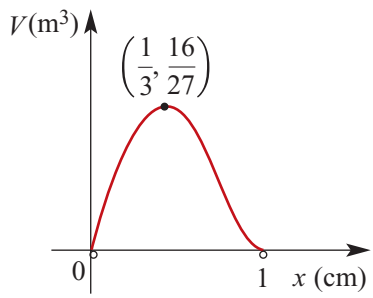
x		$\frac{1}{3}$		1
$\frac{dV}{dx}$	+	0	−	0
shape	↗	—	↘	—

$\therefore$ a maximum occurs when $x = \dfrac{1}{3}$

and maximum volume $= \left(2 - 2 \times \dfrac{1}{3}\right)^2 \times \dfrac{1}{3}$

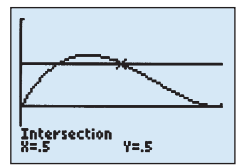
$$= \frac{16}{9} \times \frac{1}{3}$$
$$= \frac{16}{27} \text{ m}^3$$

c



d This is achieved by solving the equation $V = 0.5$, i.e. $4x^3 - 8x^2 + 4x = 0.5$.

The solution is to be found numerically using a graphics calculator.



5:intersect from the **CALC** menu can be used.

One solution is $x = 0.5$ and the other $x = 0.19098\ldots$

It is clear from this that $x - \frac{1}{2}$ is a factor of $4x^3 - 8x^2 + 4x - \frac{1}{2}$ and this can be verified by using the factor theorem. By division:

$$4x^3 - 8x^2 + 4x - \frac{1}{2} = \left(x - \frac{1}{2}\right)(4x^2 - 6x + 1)$$

$$\therefore V = 0.5 \text{ when } x = \frac{1}{2} \text{ or } 4x^2 - 6x + 1 = 0$$

Using the quadratic formula:

$4x^2 - 6x + 1 = 0$ is equivalent to $x = \dfrac{6 \pm \sqrt{20}}{8}$

i.e. $x = \dfrac{1}{2}$ or $\dfrac{3 \pm \sqrt{5}}{4}$

The domain for V is $(0, 1)$

$$\therefore x = \frac{1}{2} \text{ or } \frac{3 - \sqrt{5}}{4}$$

Exercise 17F

1 The area, A km^2, of an oil slick is growing so that t hours after a leak $A = \dfrac{t}{2} + \dfrac{1}{10}t^2$.

a Find the area covered at the end of 1 hour.

b Find the rate of increase of the area after 1 hour.

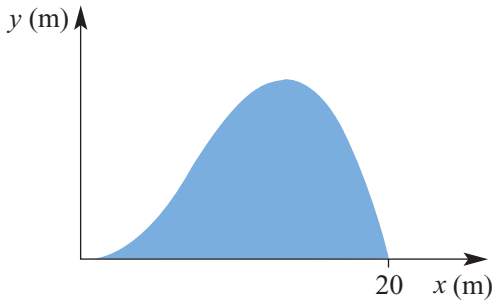
2 A particle moves in a straight line so that after t seconds it is s metres from a fixed point O on the line, and $s = t^4 + 3t^2$.

a Find the acceleration when $t = 1$, $t = 2$, $t = 3$.

b Find the average acceleration between $t = 1$ and $t = 3$.

3 A bank of earth has cross-section as shown in the diagram. All dimensions are in metres.

The curve defining the bank has equation $y = \frac{x^2}{400}(20 - x)$ for $x \in [0, 20]$.



a Find the height of the bank where:

i $x = 5$ **ii** $x = 10$ **iii** $x = 15$

b Find the value of x for which the height is a maximum and state the maximum height of the mound.

c Find the values of x for which:

i $\frac{dy}{dx} = \frac{1}{8}$ **ii** $\frac{dy}{dx} = -\frac{1}{8}$

4 A cuboid has a total surface area of 150 cm^2 and a square base of side x cm.

a Show that the height, h cm, of the cuboid is given by $h = \frac{75 - x^2}{2x}$.

b Express the volume of the cuboid in terms of x.

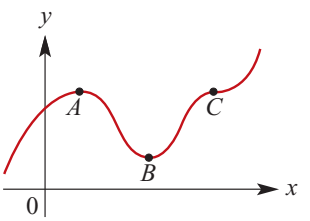
c Hence determine its maximum volume as x varies.

MC TEST D&D TEST

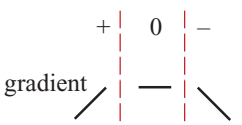
Review

Chapter summary

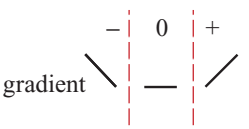
- Let P be the point on the curve $y = f(x)$ with coordinates (x_1, y_1). If f is differentiable for $x = x_1$, the equation of the tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$.
  The equation of the normal at (x_1, y_1) is $y - y_1 = -\dfrac{1}{f'(x_1)}(x - x_1)$.
- For a body moving in a straight line with displacement s metres from a point O at time t seconds:
  velocity $v = \dfrac{ds}{dt}$ acceleration $a = \dfrac{dv}{dt}$
- A point with coordinates $(a, g(a))$ on a curve $y = g(x)$ is said to be a stationary point if $g'(a) = 0$.
- **Types of stationary points**
  The graph shown here has three stationary points, A, B, C.



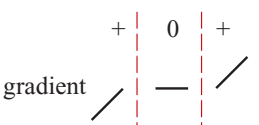
A The point A is called **a local maximum** point. Notice that immediately to the left of A the gradient is positive and immediately to the right the gradient is negative. A diagram to represent this is shown.



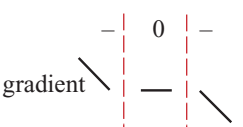
B The point B is called **a local minimum** point. Notice that immediately to the left of B the gradient is negative and immediately to the right the gradient is positive. A diagram to represent this is shown.



C The point C is called a **stationary point of inflexion**. A diagram for this is shown.



Clearly it is also possible to have stationary points of inflexion for which our diagram would be as shown.



Stationary points of type A and B are referred to as **turning points**.

Multiple-choice questions

1 The equation of the tangent to the curve $y = x^3 + 2x$ at the point $(1, 3)$ is

A $y = x$ **B** $y = 5x$ **C** $y = 5x + 2$
D $y = 5x - 2$ **E** $y = x - 2$

2 The equation of the normal to the curve $y = x^3 + 2x$ at the point $(1, 3)$ is

A $y = -5x$ **B** $y = -5x + 2$ **C** $y = \frac{1}{5}x + 2\frac{2}{5}$
D $y = -\frac{1}{5}x + 2\frac{2}{5}$ **E** $y = -\frac{1}{5}x + 3\frac{1}{5}$

3 The equation of the tangent to the curve $y = 2x - 3x^3$ at the origin is

A $y = 2$ **B** $y = -2x$ **C** $y = x$ **D** $y = -x$ **E** $y = 2x$

4 The average rate of change of the function $f(x) = 4x - x^2$ between $x = 0$ and $x = 1$ is

A 3 **B** -3 **C** 4 **D** -4 **E** 0

5 A particle moves in a straight line so that its position S m relative to O at a time t seconds ($t > 0$) is given by $S = 4t^3 + 3t - 7$. The initial velocity of the particle is

A 0 m/s **B** -7 m/s **C** 3 m/s **D** -4 m/s **E** 15 m/s

6 The function $y = x^3 - 12x$ has stationary points at $x =$

A 0 and 12 **B** -4 and 4 **C** -2 and 4 **D** -2 and 2 **E** 2 only

7 The curve $y = 2x^3 - 6x$ has a gradient of 6 at $x =$

A 2 **B** $\sqrt{2}$ **C** -2 and 2 **D** $-\sqrt{2}$ and $\sqrt{2}$ **E** 0, $\sqrt{2}$

8 The rate of change of the curve $f(x) = 2x^3 - 5x^2 + x$ at $x = 2$ is

A 5 **B** -2 **C** 2 **D** -5 **E** 6

9 The average rate of change of the function $y = \frac{1}{2}x^4 + 2x^2 - 5$ between $x = -2$ and $x = 2$ is

A 0 **B** 5.5 **C** 11 **D** 22 **E** 2.75

10 The minimum value of the function $y = x^2 - 8x + 1$ is

A 1 **B** 4 **C** -15 **D** 0 **E** -11

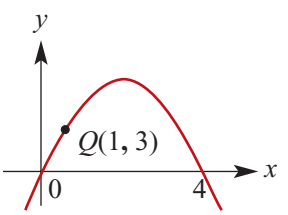
Short-answer questions (technology-free)

1 The graph of $y = 4x - x^2$ is shown.

a Find $\frac{dy}{dx}$.

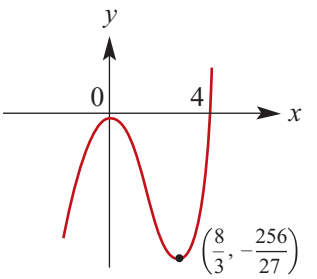
b Find the gradient of the curve at $Q(1, 3)$.

c Find the equation of the tangent at Q.



Review

2 The graph of $y = x^3 - 4x^2$ is as shown.



a Find $\frac{dy}{dx}$.

b Find the gradient of the tangent of the curve at the point $(2, -8)$.

c Find the equation of the tangent at the point $(2, -8)$.

d Find the coordinates of the point Q where the tangent crosses the curve again.

3 Let $y = x^3 - 12x + 2$.

a Find $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$.

b State the nature of each of these stationary points.

c Find the corresponding y-value for each of these.

4 Write down the values of x for which each of the following derived functions are zero, and in each case prepare a gradient chart to determine whether the corresponding points are maxima, minima or stationary points of inflexion.

a $\frac{dy}{dx} = 3x^2$ **b** $\frac{dy}{dx} = -3x^3$

c $f'(x) = (x - 2)(x - 3)$ **d** $f'(x) = (x - 2)(x + 2)$

e $f'(x) = (2 - x)(x + 2)$ **f** $f'(x) = -(x - 1)(x - 3)$

g $\frac{dy}{dx} = -x^2 + x + 12$ **h** $\frac{dy}{dx} = 15 - 2x - x^2$

5 For each of the following find all stationary points and state the nature of each.

a $y = 4x - 3x^3$ **b** $y = 2x^3 - 3x^2 - 12x - 7$ **c** $y = x(2x - 3)(x - 4)$

6 Sketch the graphs of each of the following. Give the coordinates of their stationary points and of their axes intercepts.

a $y = 3x^2 - x^3$ **b** $y = x^3 - 6x^2$

c $y = (x + 1)^2(2 - x)$ **d** $y = 4x^3 - 3x$

e $y = x^3 - 12x^2$

Extended-response questions

Questions 1–8 involve rates of change.

1 The height, in metres, of a stone thrown vertically upwards from the surface of a planet is $2 + 10t - 4t^2$ after t seconds.

a Calculate the velocity of the stone after 3 seconds.

b Find the acceleration due to gravity.

2 A dam is being emptied. The quantity of water, V litres, remaining in the dam at any time t minutes after it starts to empty is given by $V(t) = 1000(30 - t)^3$, $t \geq 0$.

a Sketch the graph of V against t.

b Find the time at which there are:

i 2 000 000 litres of water in the dam **ii** 20 000 000 litres of water in the dam.

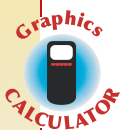
c At what rate is the dam being emptied at any time t?

d How long does it take to empty the dam?

e At what time is the water flowing out at 8000 litres per minute?

f Sketch the graphs of $y = V(t)$ and $y = V'(t)$ on the one set of axes.

3 In a certain area of Victoria the quantity of blackberries (W tonnes) ready for picking x days after 1 September is given by $W = \dfrac{x}{4000}\left(48\,000 - 2600x + 60x^2 - \dfrac{x^3}{2}\right)$, $0 \leq x \leq 60$.



a Sketch the graph of W against x for $0 \leq x \leq 60$.

b After how many days will there be 50 tonnes of blackberries ready for picking?

c Find the rate of increase of W, in tonnes per day, when $x = 20$, 40 and 60.

d Find the value of W when $x = 30$.

4 A newly installed central heating system has a thermometer which shows the water temperature as it leaves the boiler ($y\,^\circ$C). It also has a thermostat which switches off the system when $y = 65$.

The relationship between y and t, the time in minutes, is given by $y = 15 + \dfrac{1}{80}t^2(30 - t)$.

a Find the temperature at $t = 0$.

b Find the rate of increase of y with respect to t, when $t = 0$, 5, 10, 15 and 20.

c Sketch the graph of y against t for $0 \leq t \leq 20$.

5 The sweetness, S, of a pineapple t days after it begins to ripen is found to be given by $S = 4000 + (t - 16)^3$ units.

a At what rate is S increasing when $t = 0$?

b Find $\dfrac{dS}{dt}$ when $t = 4$, 8, 12, and 16.

c The pineapple is said to be unsatisfactory when our model indicates that the rate of increase of sweetness is zero. When does this happen?

d Sketch the graph of S against t up to the moment when the pineapple is unsatisfactory.

6 A slow train which stops at every station passes a certain signal box at noon. The motion of the train between the two adjacent stations is such that it is s km past the signal box at t minutes past noon, where $s = \dfrac{1}{3}t + \dfrac{1}{9}t^2 - \dfrac{1}{27}t^3$.

a Use a graphics calculator to help sketch the graphs of s against t and $\dfrac{ds}{dt}$ against t on the one set of axes. Sketch for $t \in [-2, 5]$.

Review

b Find the time of departure from the first station and the time of arrival at the second.

c Find the distance of each station from the signal box.

d Find the average velocity between the stations.

e Find the velocity with which the train passes the signal box.

7 Water is draining from a tank. The volume, V L, of water at time t (hours) is given by $V(t) = 1000 + (2 - t)^3$, $t \geq 0$ and $V(t) \geq 0$.

a What are the possible values of t?

b Find the rate of draining when:

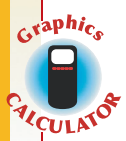
i $t = 5$ **ii** $t = 10$

8 A mountain path can be approximately described by the following rule, where y is the elevation, in metres, above the sea level and x is the horizontal distance travelled, in kilometres.

$$y = \frac{1}{5}(4x^3 - 8x^2 + 192x + 144) \quad \text{for} \quad 0 \leq x \leq 7$$

a How high above sea level is the start of the track, i.e. $x = 0$?

b When $x = 6$, what is the value of y?



c Use a graphics calculator to draw a graph of the path. Sketch this graph.

d Does this model for the path make sense for $x > 7$?

e Find the gradient of the graph for the following distances (be careful of units):

i $x = 0$ **ii** $x = 3$ **iii** $x = 7$

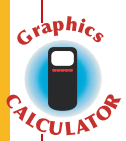
Questions 9–32 are maxima and minima problems.

9 **a** On the one set of axes sketch the graphs of $y = x^3$ and $y = 2 + x - x^2$.

b For $x \leq 0$, $2 + x - x^2 \geq x^3$. Find the value of x, $x < 0$, for which the vertical distance between the two curves is a minimum and find the minimum distance. (Hint: Consider the function with rule $y = 2 + x - x^2 - x^3$ for $x \leq 0$.)

10 The number of mosquitos, $M(x)$ in millions, in a certain area depends on the September rainfall, x, measured in mm, and is approximated by:

$$M(x) = \frac{1}{30}(50 - 32x + 14x^2 - x^3),\ 0 \leq x \leq 10$$



Find the rainfall that will produce the maximum and the minimum number of mosquitos. (First plot the graph of $y = M(x)$ using a graphics calculator.)

11 Given that $x + y = 5$ and $P = xy$ find:

a y in terms of x **b** P in terms of x

c the maximum value of P and the corresponding values of x and y.

12 Given that $2x + y = 10$ and $A = x^2y$, where $0 \leq x \leq 5$, find:

a y in terms of x **b** A in terms of x

c the maximum value of A and the corresponding values of x and y.

13 Given that $xy = 10$ and $T = 3x^2y - x^3$, find the maximum value of T for $0 < x < \sqrt{30}$.

14 The sum of two numbers x and y is 8.

a Write down an expression for y in terms of x.

b Write down an expression for s, the sum of the squares of these two numbers, in terms of x.

c Find the least value of the sum of their squares.

15 Find two positive numbers whose sum is 4, and the sum of the cube of the first and the square of the second is as small as possible.

16 A rectangular patch of ground is to be enclosed with 100 metres of fencing wire. Find the dimensions of the rectangle so that the area enclosed will be a maximum.

17 The sum of two numbers is 24. If one number is x, find the value of x such that the product of the two numbers is a maximum.

18 A factory which produces n items per hour is found to have overhead costs of $\$\left(400 - 16n + \frac{1}{4}n^2\right)$ per hour. How many items should be produced every hour to keep the overhead costs to a minimum?

19 For $x + y = 100$ prove that the product $P = xy$ is a maximum when $x = y$, and find the maximum value of P.

20 A farmer has 4 km of fencing wire and wishes to fence in a rectangular piece of land through which a straight river flows. The river is to form one side of the enclosure. How can this be done to enclose as much land as possible?

21 Two positive quantities p and q vary in such a way that $p^3q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find values of p and q that make z a minimum.

22 A beam has a rectangular cross-section of depth x cm and width y cm. The strength, S, of the beam is given by $S = 5x^2y$. The perimeter of a cross-section of the beam is 120 cm.

a Find y in terms of x. **b** Express S in terms of x.

c What are the possible values for x? **d** Sketch the graph of S against x.

e Find the values of x and y which give the strongest beam.

23 The number of salmon swimming upstream in a river to spawn is approximated by $s(x) = -x^3 + 3x^2 + 360x + 5000$, with x representing the temperature of the water in degrees (°C). (This function is valid only if $6 \leq x \leq 20$.) Find the water temperature that results in the maximum number of salmon swimming upstream.

Review

24 A piece of wire 360 cm long is used to make the twelve edges of a rectangular box for which the length is twice the breadth.

a Denoting the breadth of the box by x cm, show that the volume of the box, V cm^3, is given by $V = 180x^2 - 6x^3$.

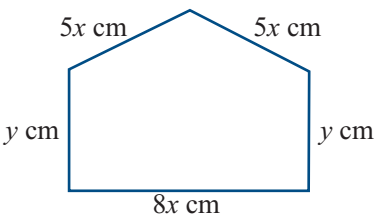
b Find the domain, S, of the function $V : S \to R$, $V(x) = 180x^2 - 6x^3$ which describes the situation.

c Sketch the graph of the function with rule $y = V(x)$.

d Find the dimensions of the box that has the greatest volume.

e Find the values of x for which $V = 20\,000$. Give values correct to 2 decimal places.

25 A piece of wire of length 90 cm is bent into the shape shown in the diagram.



a Show that the area, A cm^2, enclosed by the wire is given by $A = 360x - 60x^2$.

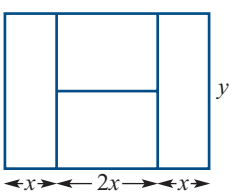
b Find the values of x and y for which A is a maximum.

26 A piece of wire 100 cm in length is to be cut into two pieces, one piece of which is to be shaped into a circle and the other into a square.

a How should the wire be cut if the sum of the enclosed areas is to be a minimum?

b How should the wire be used to obtain a maximum area?

27 A roll of tape 36 metres long is to be used to mark out the edges and internal lines of a rectangular court of length $4x$ metres and width y metres, as shown in the diagram. Find the length and width of the court for which the area is a maximum.



28 A rectangular chicken run is to be built on flat ground. A 16-metre length of chicken wire will be used to form three of the sides; the fourth side of length x metres, will be part of a straight wooden fence.

a Let y be the width of the rectangle. Find an expression for A, the area of the chicken run in terms of x and y.

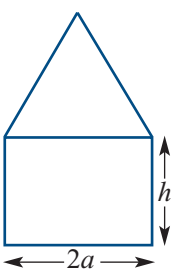
b Find an expression for A in terms of x.

c Find the possible values of x.

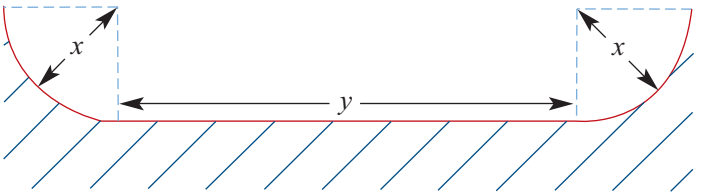
d Sketch the graph of A against x for these values of x.

e What is the largest area of ground the chicken run can cover?

29 The diagram illustrates a window that consists of an equilateral triangle and a rectangle. The amount of light that comes through the window is directly proportional to the area of the window. If the perimeter of such a window must be 8000 mm, find the values of h and a (correct to the nearest mm) which allow the maximum amount of light to pass.



30 The diagram shows a cross-section of an open drainage channel. The flat bottom of the channel is y metres across and the sides are quarter circles of radius x metres. The total length of the bottom plus the two curved sides is 10 metres.



a Express y in terms of x.

b State the possible values that x can take.

c Find an expression for A, the area of the cross-section, in terms of x.

d Sketch the graph of $y = A(x)$, for possible values of x.

e Find the value of x which maximises A.

f Comment on the cross-sectional shape of the drain.

31 A cylinder closed at both ends has a total surface area of 1000 cm^2. The radius of the cylinder is x cm and the height h cm. Let V cm^3 be the volume of the cylinder.

a Find h in terms of x.

b Find V in terms of x.

c Find $\dfrac{dV}{dx}$.

d Find $\{x: \dfrac{dV}{dx} = 0\}$.

e Sketch the graph of V against x for a suitable domain.

f Find the maximum volume of the cylinder.

g Find the value(s) of x and h for which $V = 1000$, correct to 2 decimal places.

32 A cylindrical aluminium can able to contain half a litre of drink is to be manufactured. The volume of the can must therefore be 500 cm^3.

a Find the radius and height of the can which will use the least aluminium and, therefore, be the cheapest to manufacture.

b Give reasons why many cans are not this shape.

CHAPTER

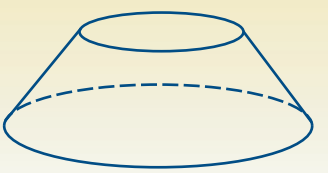
18

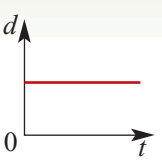
Revision

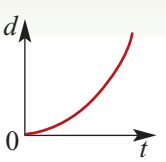
Revision of Chapters 15–17

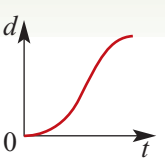
18.1 Multiple-choice questions

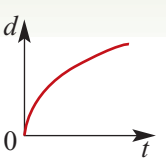
1 The bowl shown in the diagram is being filled with water from a tap flowing at a constant rate. The graph which shows how the depth, d, of water changes with time, t, is

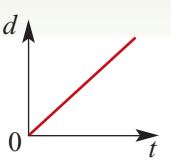


A


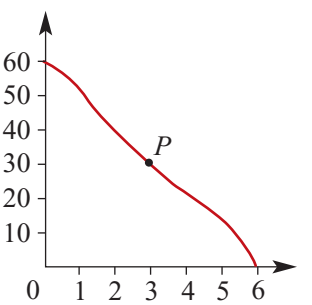
B


C


D


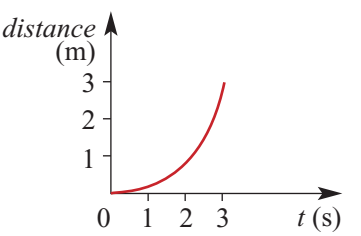
E


2 P is the point (3, 30). The gradient of the graph at P is closest to



A -10 **B** -0.1 **C** 0
D 1 **E** 10

3 For the distance–time graph shown, the average speed from $t = 0$ to $t = 3$ is



A 0.1 m/s **B** 1 m/s **C** $\frac{1}{3}$ m/s
D –3 m/s **E** 3 m/s

4 Given $f(x) = 3x^2 + 1$, the average rate of change of $f(x)$ with respect to x for the interval [0, 2] is

A 6 **B** 6.5 **C** 12 **D** 13 **E** 14

5 On a particular day, the temperature at 7.00 am was 10 °C and at 12.00 noon the temperature was 23.5 °C. The average rate of increase of temperature, °C/hour, with respect to time is

A 13.5 **B** 2.7 **C** 4.7 **D** 4 **E** 5

6 If $y = 5x^2 + 1$, then $\frac{dy}{dx}$ is equal to

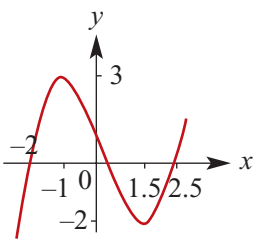
A $10x$ **B** $10x + 1$ **C** $\frac{5x^3}{3} + x$

D $10x^2 + x$ **E** $\frac{5x^3}{3} + x + c$

7 If $f(x) = x^2$ then the value of $f(5 + h) - f(5)$ is

A $2x$ **B** 25 **C** h^2 **D** $10h + h^2$ **E** $(5 + h)^2$

8 The graph shown has zero gradient for



A $x = -2$ and $x = 0$ and $x = 2.5$
B $x = -1$ and $x = 1.5$
C $x \in [-1, 1.5]$
D $x = 3$ and $x = -2$
E $x = 0$ only

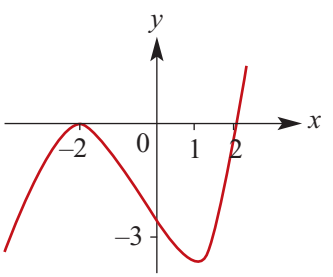
9 Water is flowing into a tank. The volume, V m^3, of water in the tank after t minutes is given by $V = 3t^2 + 4t + 2$. The instantaneous rate of change of the volume (m^3/min) of water at time $t = 2$ is

A $6t + 4$ **B** 22 **C** 16 **D** 10 **E** $t^3 + 2t^2$

10 Given that $\frac{f(3 + h) - f(3)}{h} = 2h^2 + 2h$ then $f'(3)$ is equal to:

A 0 **B** 14 **C** 2 **D** 5 **E** 6

11 For the graph shown the gradient is positive for

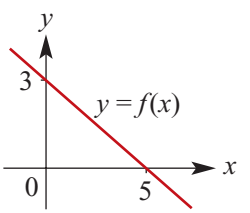


A $x \in (-\infty, 2)$
B $x \in (-2, 2)$
C $x \in (-\infty, -2) \cup (1, \infty)$
D $x \in (-\infty, -2) \cup (2, \infty)$
E $x \in (-\infty, -2)$

12 The graph of the function whose rule is $f(x) = x^3 - x^2 - 5$ has stationary points at x equals

A $\frac{2}{3}$ only **B** 0 and $\frac{2}{3}$ **C** 0 and $-\frac{2}{3}$ **D** $-\frac{1}{3}$ and 1 **E** $\frac{1}{3}$ and –1

13 For the graph shown, $f'(2)$ is equal to



A $-\frac{3}{5}$ **B** $\frac{5}{3}$ **C** –1.5
D 5 **E** –1

Revision

14 The gradient of the curve with equation $y = 2x^3 - 3x^2$ at the point $(1, -1)$ is

A -3 **B** -1 **C** 0 **D** 1 **E** 12

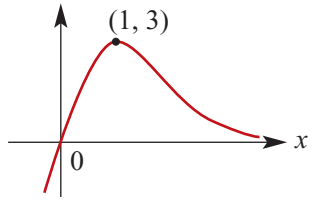
15 If $y = 7 + 2x - x^2$, then the maximum value of y is

A $-4\frac{1}{2}$ **B** $4\frac{1}{2}$ **C** 8 **D** $8\frac{1}{2}$ **E** 9

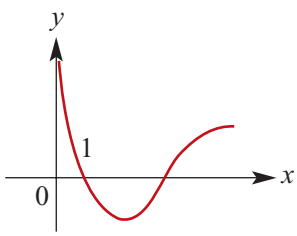
16 A ball was thrown vertically upwards and the height, s metres, it reached after t seconds was given by $s = 28t - 16t^2$. The greatest height reached by the ball was

A $12\frac{1}{4}$ m **B** $14\frac{1}{8}$ m **C** $16\frac{3}{4}$ m **D** $10\frac{1}{2}$ m **E** $25\frac{3}{8}$ m

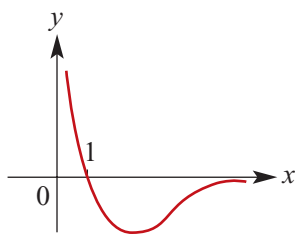
17 The graph of f is shown. The graph of f' is best represented by



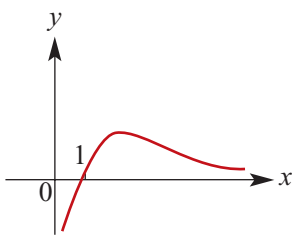
A



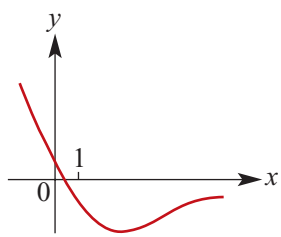
B



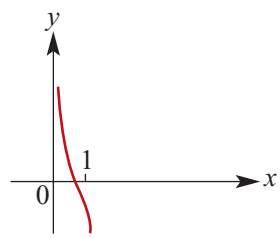
C



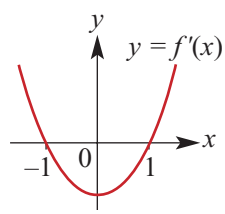
D



E



18 The graph of the gradient function for $f(x) = 2x^3 - 6x$ is shown. This means that $f(x)$ has a local maximum at



A $(1, -4)$ **B** $(0, 0)$ **C** $(0, -6)$
D $(-3, 0)$ **E** $(-1, 4)$

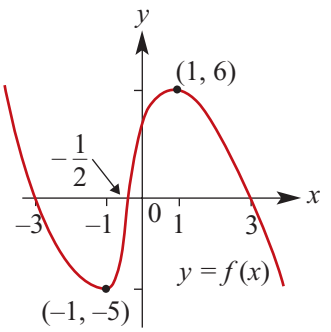
19 If $y = \frac{x^2}{2}(x^2 + 2x - 4)$ then $\frac{dy}{dx}$ equals

A $4x^3 + 6x^2 - 8x$ **B** $2x^2 + 2x$ **C** $2x^3 + 3x^2 - 4x$
D $2x^3 + 4x^2 - 8x$ **E** $x^3 + 2x^2 - 4x$

20 The derivative of $5 + 3x^2$ is

A $5x + x^3$ **B** $6x$ **C** $5x + 6x$ **D** $6x^3$ **E** $5 + x^3$

21 The graph shown has stationary points at (1, 6) and (–1, –5). The graph is that of a function with rule $y = f(x)$. $f'(x) < 0$ for



A $\left(-3, -\frac{1}{2}\right) \cup (3, \infty)$

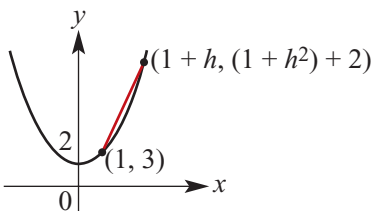
B $(-1, 1)$

C $\left(-3, -\frac{1}{2}\right)$

D $(-\infty, -5) \cup (1, \infty)$

E $(-\infty, -1) \cup (1, \infty)$

22 The gradient of the chord AB joining points $A\ (1, 3)$ and $B(1 + h, (1 + h)^2 + 2)$ is



A $1 + h$ **B** $h^2 - 2h$

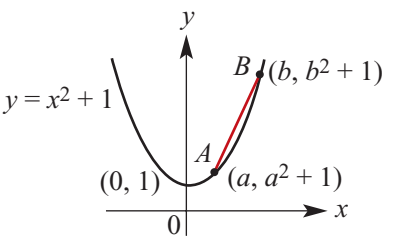
C $h^2 + 2h$ **D** $2 + h$

E 2

23 The gradient at the point (1, –2) of the curve with equation $y = x^2(2x - 3)$ is

A 0 **B** $\frac{2}{3}$ **C** $1\frac{1}{3}$ **D** $\frac{\pi}{2}$ **E** 4

24 The gradient of the chord AB is



A $b + a$ **B** $2a$

C $b - a$ **D** $a - b$

E $\lim_{h \to 0} \dfrac{(x + h)^2 + 1 - (x^2 + 1)}{h}$

25 The derivative function of $f(x) = 3x^3 + 6x^2 - x + 1$ with respect to x has rule

A $f'(x) = \frac{3}{4}x^4 + 2x^2 - \frac{x^2}{2} + x$ **B** $f'(x) = 1 + 12x + 9x^2$

C $f'(x) = 9x^2 + 12x - 1$ **D** $f'(x) = 6x + 12x - 1$

E $f'(x) = 3x^2 + 2x - 1$

26 Suppose $A = 4xy$ and $y + 3x = 10$. Which one of the following equations can be used to directly calculate the value of x so that the value of A is a maximum?

A $10x - x^2 = 0$ **B** $10 - 3x^2 = 0$ **C** $10 - 3x = 0$

D $5 - 3x = 0$ **E** $5 - x = 0$

27 The gradient of the graph of $y = x^2 + 3$ at the point with coordinates (3, 12) is

A 24 **B** 6 **C** 3 **D** 4 **E** 12

Revision

28 The function $y = x^3 + 5x^2 - 8x$ has

A a local minima when $x = -4$ and a local maxima when $x = \frac{2}{3}$
B a local maxima when $x = -4$ and a local minima at $x = \frac{2}{3}$
C a local maxima when $x = 4$ and a local minima when $x = -\frac{2}{3}$
D a local minima when $x = 4$ and a local maxima when $x = -\frac{2}{3}$
E a stationary point of inflexion when $x = -4$ and a local maximum when $x = \frac{2}{3}$

29 The gradient of the curve $y = f(x)$ at the point where $x = 1$ is

A $\lim\limits_{h \to 0} \dfrac{(1+h) - 1}{h}$ **B** $\lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$ **C** $\dfrac{f(1+h) - f(1)}{h}$

D $\dfrac{f(1)}{h}$ **E** $\dfrac{1}{h} \lim\limits_{h \to 0} (f(1+h) - f(1))$

30 The minimum value of the function $y = x^2 + 4x - 3$ is

A -2 **B** 2 **C** -7 **D** -15 **E** -3

31 The gradient of the normal to the curve $y = x^2$ at the point where $x = 2$ is equal to

A 2 **B** 4 **C** $-\dfrac{1}{2}$ **D** $-\dfrac{1}{4}$ **E** $\dfrac{1}{2}$

32 If $y = \dfrac{2x+5}{x}$ then $\dfrac{dy}{dx}$ equals

A $2 - \dfrac{5}{x^2}$ **B** 2 **C** $\dfrac{2}{x}$ **D** $2 + \dfrac{5}{x^2}$ **E** $\dfrac{-5}{x^2}$

33 The function $y = x^2 - 3x - 4$ is decreasing when

A $x \in \left(-\infty, \dfrac{3}{2}\right)$ **B** $x \in \left(-\dfrac{3}{2}, \infty\right)$ **C** $x \in (-1, 4)$

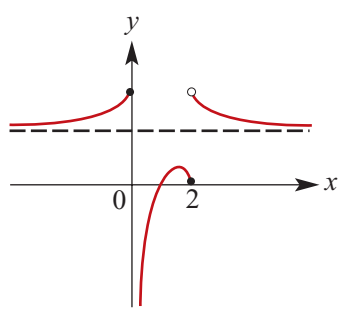
D $x \in (-\infty, 1) \cup (4, \infty)$ **E** $x \in \left(\dfrac{3}{2}, \infty\right)$

34 $\lim\limits_{x \to 0} \dfrac{x^2 - x}{x}$ is

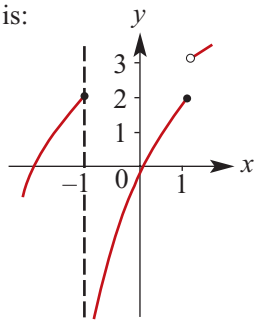
A -1 **B** 0 **C** 1 **D** ∞ **E** undefined

35 The graph shown in the diagram is

A discontinuous at $x = 2$ only
B discontinuous at $x = 0$ only
C discontinuous at $x = 0$ and $x = 2$
D continuous for all real values of x
E continuous for all $x < 0$ but discontinuous elsewhere



36 The function with the graph as shown in the diagram is:



A continuous for all real values of x

B discontinuous for all real values of x

C discontinuous at $x = -1$ and $x = 1$

D discontinuous for all $x \neq -1$ but continuous elsewhere

E discontinuous for all $x \neq 1$ but continuous elsewhere

18.2 Extended-response questions

Graphics CALCULATOR

1 An object follows a path (in a vertical plane) described by the equation $y = x - 0.01x^2$, where x is the horizontal distance travelled and y is the height above ground level of the particle when it has travelled a distance x. The object travels from (0, 0), a point at ground level.

a What is the horizontal distance travelled by the particle before it returns to ground level?

b Find $\frac{dy}{dx}$.

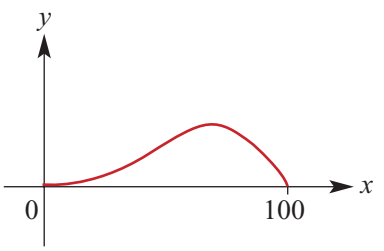
c Find the value of x for which $\frac{dy}{dx} = 0$ and the corresponding y-value.

d Sketch the graph of y against x.

e State the coordinates of the point on the path for which the gradient is:

i $\frac{1}{2}$ **ii** $-\frac{1}{2}$

2 A designer of rollercoasters decides to use a polynomial model for a new construction. He tries the polynomial $y = -0.0001(x^3 - 100x^2)$, where $x \in [0, 100]$. The graph is as shown.



a What are the coordinates of the highest point reached?

b Find the gradient of the curve at:

i $x = 20$ **ii** $x = 80$ **iii** $x = 100$

c Describe the ride the rollercoaster would provide.

d What alteration would you suggest?

3 A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the length of all its edges is 20 cm.

a Find an expression in terms of x for the height of the block.

b Show that the volume, V cm^3, is given by $V = 15x^2 - 12x^3$.

c What values of x should be considered, i.e. find the domain of the function with the rule $V = 15x^2 - 12x^3$?

d Find $\frac{dV}{dx}$.

Revision

e Find $\{x: \frac{dV}{dx} = 0\}$ and hence find the maximum volume possible.

f Sketch the graph of V against x for the values of x determined in part c.

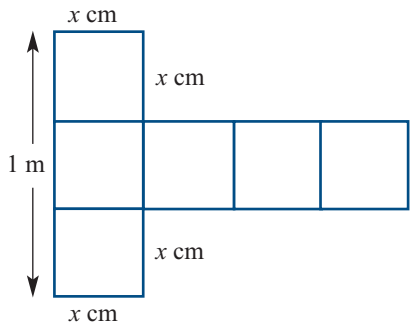
4 A stone is projected vertically upwards with a speed of 30 m/s. Its height h m, above the ground after t seconds ($t < 6$) is given by $h = 30t - 5t^2$.

a Find $\frac{dh}{dt}$.

b Find the maximum height reached.

c Sketch the graph of h against t.

5 A box is made from the net shown.



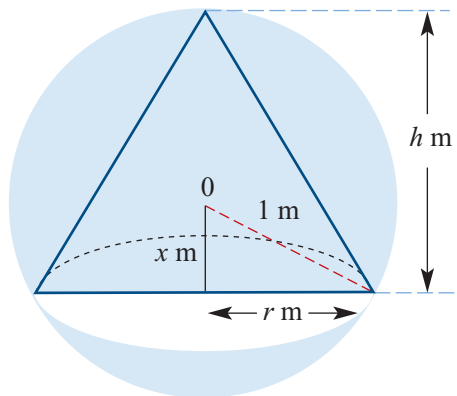
a Write down an expression for the surface area of the net in terms of x.

b Write down an expression for, V cm^3, the volume of the made-up box in terms of x.

c Sketch the graph of V against x.

d Find the dimensions and the volume of a box with maximum volume.

6 A right circular cone lies inside a sphere of radius 1 m as shown. The centre of the sphere, 0, lies x m from the base of the cone.



a The volume of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$.
Find:

i r in terms of x

ii h in terms of x

b Show that $V = \frac{\pi}{3}(1 + x - x^2 - x^3)$.

c State a suitable domain for the function with rule $V = \frac{\pi}{3}(1 + x - x^2 - x^3)$.

d **i** Find $\frac{dV}{dx}$.

ii Find $\{x: \frac{dV}{dx} = 0\}$.

iii State the maximum possible volume of the cone.

e Sketch the graph of V against x.

7 The number of insects in a colony at time t days after 1 January 1993 is approximated by the function with rule $P(t) = 1000 \times 2^{\frac{t}{20}}$, where $t = 0$ corresponds to 1 January 1993. This rule for the population is valid for the entire year.

a Find the approximate number of insects in the colony on 1 January.

b Find the approximate number of insects in the colony on 10 January (i.e. when $t = 9$).

c For what values of t is $P(t)$ equal to:

i 4000 **ii** 6000? (Give answer correct to 2 decimal places.)

d Find $P(20)$ and $P(15)$ and hence calculate the average rate of change of P with respect to time for the interval of time $[15, 20]$, giving your answer correct to 2 decimal places.

e **i** Find the average rate of change of P with respect to t for the interval $[15, 15 + h]$, in terms of h.

ii Explain how the instantaneous rate of change of P with respect to t, for $t = 15$, could be found by numerical methods.

8 A rectangular block, the length of whose base is twice its width, has total surface area of 300 m^2. The width of the base is x cm and the height of the block is h cm.

a Find h in terms of x, with h the subject of the formula.

b Find V, the volume of the block, in terms of x.

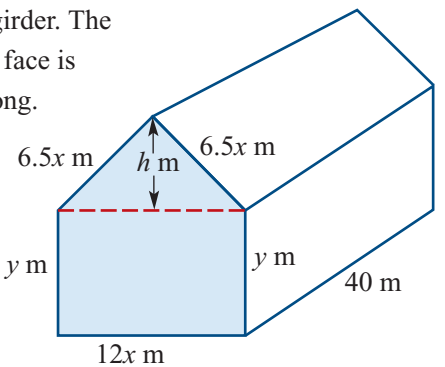
c Find $\frac{dV}{dx}$.

d State the positive values of x for which $V > 0$.

e Find the maximum value of V and the value of x for which this occurs.

f For the values of x established in part d, sketch the graph of V against x.

9 The metal frame of the front face of a building is as shown. Each length represents a steel girder. The total length of girder used for the front face is 70 metres. The building is 40 metres long.



a If $x = 2$, find the value of y.

b Show that $y = \frac{70 - 25x}{2}$.

c **i** Find the height h (marked on the diagram) in terms of x.

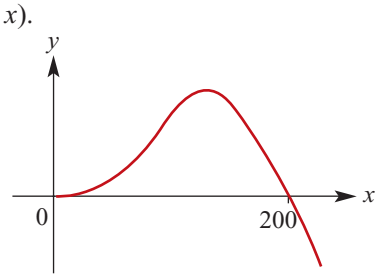
ii Show that the area of the front face of the building (shaded in the diagram) in terms of x and y is $A = 15x^2 + 12xy$.

d Find the volume, V m^3, of the building in terms of x.

e **i** For what value of x is the volume of the building a maximum? What is the corresponding value of y?

ii State the corresponding maximum volume of the building.

10 The equation of the curve is of the form $y = kx^2(a - x)$.



a Find the values of a and k given that when $x = 170$, $y = 8.67$, and when $x = 200$, $y = 0$.

b **i** Find the value of x for which there is a local maximum.

ii State the value of y which corresponds to this value of x.

Revision

c When $x = 105$ find:

i the y-value **ii** the gradient of the curve at this point.

d **i** Find the equation of the tangent at the point where $x = 105$.

ii State the y-axis intercept of the tangent.

e Find the average rate of change of y with respect to x for the interval $[0, 105]$.

f Carefully sketch the graph of $y = kx^2(a - x)$, for $x \in [0, 200]$ and on the same set of axes sketch the graph of the tangent at $x = 105$.

11 The population density (number of residents per unit area) of many cities depends on the distance from the city centre. For a particular city, the population density P in thousands of people per square kilometre at a distance of r kilometres from the centre is given approximately by $P = 10 + 40r - 20r^2$.

a What is the population density in the centre of the city?

b What are the possible values for r?

c Sketch a graph of P against r.

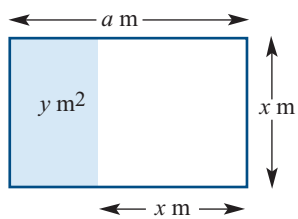
d **i** Find $\frac{dP}{dr}$.

ii Evaluate $\frac{dP}{dr}$ when $r = 0.5$, 1 and 2.

iii Sketch a graph of $\frac{dP}{dr}$ against r.

e Where is the population density greatest?

12 The diagram shows a rectangle with sides a m and x m. The area of the shaded region is y m^2.



a Find an expression for y in terms of x. In the following a is a constant.

b Find the set of possible values for x in terms of a.

c Find the maximum value of y and the corresponding value of x (in terms of a).

d Explain briefly why this value of y is a maximum.

e **i** Sketch the graph of y against x when $a = 9$.

ii State the set of possible values for y.

13 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by $V(t) = 0.6\left(20t^2 - \frac{2t^3}{3}\right)$, $0 \leq t \leq 20$.

a Find the volume of water in the flask when:

i $t = 0$ **ii** $t = 20$

b Find the rate of flow of water into the flask, i.e. find $\frac{dV}{dt}$.

c Sketch the graph of $V(t)$ against t for $t \in [0, 20]$.

d Sketch the graph of $V'(t)$ against t for $t \in [0, 20]$.

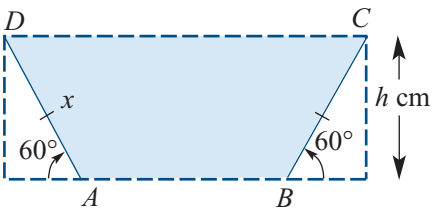
14 The function $y = ax^3 + bx^2$, where a and b are constants, has a stationary point at $(1, -1)$.

a Using the fact that $(1, -1)$ lies on the curve, form an equation involving a and b.

b Using the fact that $(1, -1)$ is also a stationary point, form a second equation and solve the two equations simultaneously to find a and b.

c Sketch the graph of y against x.

15 A trough is to be made by bending a long sheet of metal 80 cm wide to give a trapezoid cross-section with sides of equal length x cm inclined at 60° to the horizontal as shown.



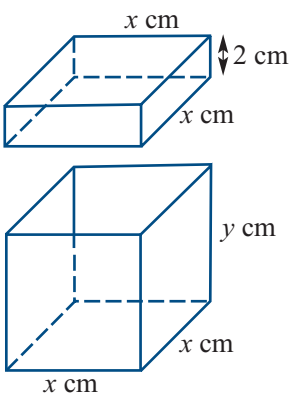
a Find:

i length AB in terms of x **ii** h in terms of x

b Find the cross-sectional area (shaded) in terms of x.

c Find the value of x for which the cross-sectional area is a maximum.

16 The diagrams opposite show an open cardboard box of dimensions x cm, x cm, y cm and the lid for the box of dimensions x cm, x cm, 2 cm. The total amount of cardboard used for the box and the lid is 1400 cm^2. Let V cm^3 be the volume of the box.



a Find y in terms of x.

b Find V in terms of x.

c Find $\frac{dV}{dx}$.

d Find $\{x: \frac{dV}{dx} = 0\}$.

e Sketch the graph of V against x.

f Find the maximum volume of the box.

g Find the values of x and y such that $V = 1000$.

CHAPTER

19

Differentiation Techniques

Objectives

- To differentiate functions having **negative integer powers**.
- To understand and use the **chain rule**.
- To differentiate **rational powers**.
- To find **second derivatives** of functions.
- To use **limits to define continuity at a point**.
- To use differentiation techniques to **sketch graphs of functions**.
- To apply differentiation techniques to **solving problems**.

The work in this chapter is not part of the content for VCE Mathematical Methods Units 1 & 2. It is included as a useful foundation for VCE Mathematical Methods Units 3 & 4.

19.1 Differentiating x^n where n is a negative integer

In earlier chapters the differentiation of polynomial functions has been introduced and applications of differentiation of such functions have been considered. In this section we add to the family of functions for which we can find the derived functions. In particular, we will consider functions which involve linear combinations of powers of x where a power may be a negative integer.

e.g. $f: R \setminus \{0\} \to R,\ f(x) = x^{-1}$

$f: R \setminus \{0\} \to R,\ f(x) = 2x + x^{-1}$

$f: R \setminus \{0\} \to R,\ f(x) = x + 3 + x^{-2}$

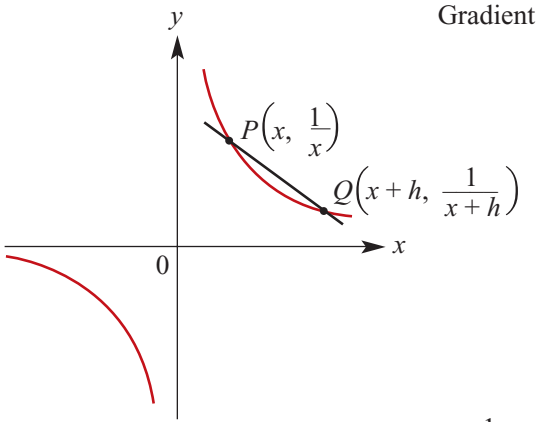
Note: We have reintroduced the function notation which emphasises the need for consideration of domain.

CAS APPENDIX

Example 1

Let $f: R \setminus \{0\} \to R$, $f(x) = \frac{1}{x}$. Find $f'(x)$ by first principles.

Solution



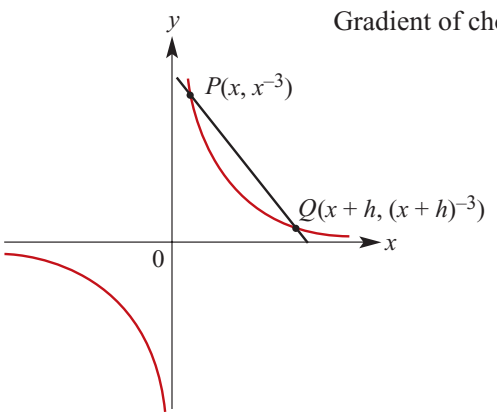
$$\text{Gradient of chord } PQ = \frac{f(x+h)-f(x)}{x+h-x}$$
$$= \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$
$$= \frac{x-(x+h)}{(x+h)x} \times \frac{1}{h}$$
$$= \frac{-h}{(x+h)x} \times \frac{1}{h}$$
$$= \frac{-1}{(x+h)x}$$

The gradient of the curve at $P = \lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2} = x^{-2}$, i.e. $f'(x) = -x^{-2}$

Example 2

Let $f: R \setminus \{0\} \to R$, $f(x) = x^{-3}$. Find $f'(x)$ by first principles.

Solution



$$\text{Gradient of chord } PQ = \frac{(x+h)^{-3}-x^{-3}}{h}$$
$$= \frac{x^3-(x+h)^3}{(x+h)^3x^3} \times \frac{1}{h}$$
$$= \frac{x^3-(x^3+3x^2h+3xh^2+h^3)}{(x+h)^3x^3} \times \frac{1}{h}$$
$$= \frac{-3x^2h-3xh^2-h^3}{(x+h)^3x^3} \times \frac{1}{h}$$
$$= \frac{-3x^2-3xh-h^2}{(x+h)^3x^3}$$

The gradient of the curve at $P = \lim_{h \to 0} \frac{-3x^2-3xh-h^2}{(x+h)^3x^3}$

$$= \frac{-3x^2}{x^6}$$
$$= \frac{-3}{x^4}$$
$$= -3x^{-4}$$

i.e. $f'(x) = -3x^{-4}$

We are now in a position to state the generalisation of the result we found in Chapter 16.

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, n is a non-zero integer.
For $f(x) = c$, $f'(x) = 0$, where c is a constant.
We note that for $n \leq -1$ we take the domain of f to be $R \setminus \{0\}$, and for $n \geq 1$ we take the domain of f to be R.

Example 3

Find the derivative of $x^4 - 2x^{-3} + x^{-1} + 2, x \neq 0$.

Solution

$$\begin{aligned} \text{If } f(x) &= x^4 - 2x^{-3} + x^{-1} + 2, x \neq 0 \\ f'(x) &= 4x^3 - 2(-3x^{-4}) + (-x^{-2}) + 2(0) \\ &= 4x^3 + 6x^{-4} - x^{-2}, \ x \neq 0 \end{aligned}$$

Example 4

Find the derivative f' of $f: R \setminus \{0\} \to R$, $f(x) = 3x^2 - 6x^{-2} + 1$.

Solution

$$\begin{aligned} f': R \setminus \{0\} \to R, f'(x) &= 3(2x) - 6(-2x^{-3}) + 1(0) \\ &= 6x + 12x^{-3} \end{aligned}$$

Example 5

Find the gradient of the curve determined by the function $f: R \setminus \{0\}, f(x) = x^2 + \frac{1}{x}$ at the point (1, 2).

Solution

$$\begin{aligned} f': R \setminus \{0\} \to R, f'(x) &= 2x + (-x^{-2}) \\ &= 2x - x^{-2} \\ \text{and } f'(1) &= 2 - 1 \\ &= 1 \end{aligned}$$

The gradient of the curve is 1 at the point (1, 2).

Example 6

Show that the gradient of the function $f: R \setminus \{0\} \to R$, $f(x) = x^{-3}$ is always negative.

Solution

$f': R \setminus \{0\} \to R,\ f'(x) = -3x^{-4}$

$= \dfrac{-3}{x^4}$

x^{-4} is positive for all x and thus $f'(x) < 0$ for all $x \neq 0$.

Exercise 19A

Examples 3, 4

1 Differentiate each of the following with respect to x:

a $3x^{-2} + 5x^{-1} + 6$ **b** $\dfrac{3}{x^2} + 5x^2$ **c** $\dfrac{5}{x^3} + \dfrac{4}{x^2} + 1$

d $3x^2 + \dfrac{5}{3}x^{-4} + 2$ **e** $6x^{-2} + 3x$ **f** $\dfrac{3x^2 + 2}{x}$

2 Find the derivative of each of the following:

a $\dfrac{3z^2 + 2z + 4}{z^2}, z \neq 0$ **b** $\dfrac{3 + z}{z^3}, z \neq 0$ **c** $\dfrac{2z^2 + 3z}{4z}, z \neq 0$

d $9z^2 + 4z + 6z^{-3}, z \neq 0$ **e** $9 - z^{-2}, z \neq 0$ **f** $\dfrac{5z - 3z^2}{5z}, z \neq 0$

Examples 1, 2

3 **a** Carefully sketch the graph of $f(x) = \dfrac{1}{x^2}, x \neq 0$.

b Let P be the point $(1, f(1))$ and Q the point $(1 + h, f(1 + h))$. Find the gradient of the chord PQ.

c Hence find the gradient of the curve $f(x) = \dfrac{1}{x^2}$ at $x = 1$.

d Find the equation of the normal to the curve with equation $y = x^{-2}$ at the point $(1, 1)$.

Example 5

4 Find the gradient of each of the following curves at the given point:

a $y = x^{-2} + x^3, x \neq 0$, at $\left(2, 8\frac{1}{4}\right)$ **b** $y = \dfrac{x - 2}{x}, n \neq 0$, at $\left(4, \frac{1}{2}\right)$

c $y = x^{-2} - \dfrac{1}{x}, x \neq 0$, at $(1, 0)$ **d** $y = x(x^{-1} + x^2 - x^{-3}), x \neq 0$, at $(1, 1)$.

5 Find the x-coordinate of the point on the curve with equation $f(x) = x^{-2}$ and gradient:

a 16 **b** -16

19.2 The chain rule

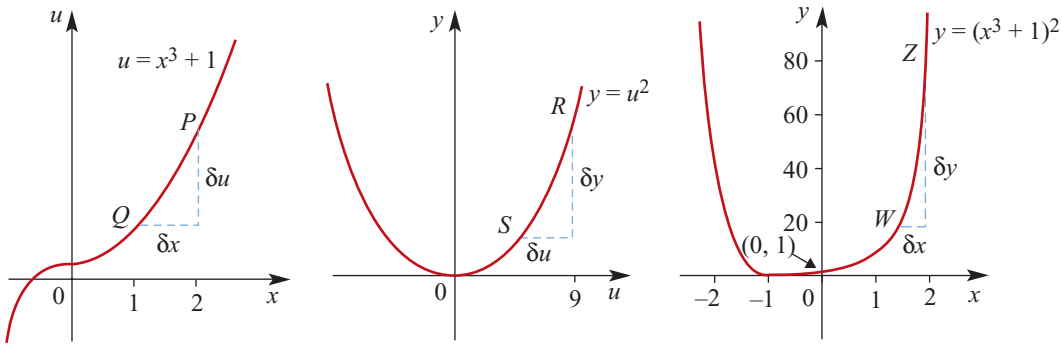
An expression such as $f(x) = (x^3 + 1)^2$ may be differentiated by expanding and then differentiating each term. This method is a great deal more tiresome for an expression such as $f(x) = (x^3 + 1)^{30}$. We transform $f(x) = (x^3 + 1)^2$ into two simpler functions defined by:

$$h(x) = x^3 + 1 (= u) \quad \text{and} \quad g(u) = u^2 (= y)$$

which are 'chained' together: $x \xrightarrow{h} u \xrightarrow{g} y$

We may exploit this connection to differentiate. We use Leibniz notation to explore this idea.

Consider $f(x) = (x^3 + 1)^2$ at the point where $x = 2$.



When $x = 2$, $u = 9$
P is the point (2, 9)

When $u = 9$, $y = 81$
R is the point (9, 81)

When $x = 2$, $y = 81$
Z is the point (2, 81)

A section of a spreadsheet (below) illustrates the connection between the gradients of the chords PQ, RS and WZ. It can be seen that as δx gets smaller so does δu and $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$. By considering δu becoming smaller and hence δx becoming smaller, it is seen that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This is called the **chain rule** for differentiation.

x	u	y	δx	δu	δy	$\frac{\delta u}{\delta x}$	$\frac{\delta y}{\delta u}$	$\frac{\delta u}{\delta x} \times \frac{\delta y}{\delta u}$	$\frac{\delta y}{\delta x}$
1.60000	5.09600	25.96922	0.40000	3.90400	55.03078	9.76000	14.09600	137.57696	137.57696
1.80000	6.83200	46.67622	0.20000	2.16800	34.32378	10.84000	15.83200	171.61888	171.61888
1.90000	7.85900	61.76388	0.10000	1.14100	19.23612	11.41000	16.85900	192.36119	192.36119
1.99000	8.88060	78.86504	0.01000	0.11940	2.13496	11.94010	17.88060	213.49614	213.49614
1.99900	8.98801	80.78425	0.00100	0.01199	0.21575	11.99400	17.98801	215.74816	215.74816
1.99990	8.99880	80.97840	0.00010	0.00120	0.02160	11.99940	17.99880	215.97480	215.97480
1.99999	8.99988	80.99784	0.00001	0.00012	0.00216	11.99994	17.99988	215.99748	215.99748

From the spreadsheet it can be seen that the gradient of $u = x^3 + 1$ at $x = 2$ is 12, and the gradient of $y = u^2$ at $u = 9$ is 18. The gradient of $y = (x^3 + 1)^2$ at $x = 2$ is 216. The chain rule is used to confirm this.

$$\frac{du}{dx} = 3x^2 \quad \text{and at } x = 2 \quad \frac{du}{dx} = 12$$

$$\frac{dy}{du} = 2u \quad \text{and at } u = 9 \quad \frac{dy}{du} = 18$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 216$$

Example 7

Find the derivative of $y = (3x+4)^{20}$.

Solution

Let $u = 3x + 4$ then $y = u^{20}$

So $\dfrac{du}{dx} = 3$ and $\dfrac{dy}{du} = 20u^{19}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= 20u^{19} \cdot 3$$
$$= 60(3x+4)^{19}$$

Example 8

Find the gradient of the curve with equation $y = \dfrac{16}{3x^2+1}$ at the point (1, 4).

Solution

Let $u = 3x^2 + 1$ then $y = 16u^{-1}$

So $\dfrac{du}{dx} = 6x$ and $\dfrac{dy}{du} = -16u^{-2}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= -16u^{-2} \cdot 6x$$
$$= \frac{-96x}{(3x^2+1)^2}$$

$\therefore$ at $x = 1$ the gradient is $\dfrac{-96}{16} = -6$.

Example 9

Differentiate $y = (4x^3 - 5x)^{-2}$.

Solution

Let $u = 4x^3 - 5x$

then $y = u^{-2}$

and $\dfrac{du}{dx} = 12x^2 - 5$ and $\dfrac{dy}{du} = -2u^{-3}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= -(2u^{-3}) \cdot (12x^2 - 5)$$
$$= \frac{-2(12x^2-5)}{(4x^3-5x)^3}$$

Example 10

Use the chain rule to prove $\frac{dy}{dx} = nx^{n-1}$ for $y = x^n$ and n a negative integer.
(Assume the result for n a positive integer and $x > 0$.)

Solution

Let n be a negative integer and $y = x^n$. Then $y = \frac{1}{x^{-n}}$ as $-n$ is a positive integer.

Let $u = x^{-n}$ then $y = \frac{1}{u} = u^{-1}$

Thus $\frac{du}{dx} = -nx^{-n-1}$ ($-n$ is a positive integer) and $\frac{dy}{du} = -u^{-2}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= -u^{-2} \cdot (-nx^{-n-1})$$
$$= nx^{-n-1}(x^{-n})^{-2}$$
$$= nx^{-n-1}(x^{2n})$$
$$= nx^{n-1}$$

With function notation the chain rule is stated as:

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

where $f \circ g(x) = f(g(x))$.

Example 11

CAS APPENDIX

Given that $f(x) = (x^2 + 1)^3$, find $f'(x)$.

Solution

Now $f = k \circ g$ where $k(x) = x^3$ and $g(x) = x^2 + 1$
It follows that $k'(x) = 3x^2$ and $g'(x) = 2x$
By the chain rule $f'(x) = k'(g(x))g'(x)$
We have in this case $f'(x) = 3(g(x))^2\, 2x$
which yields $f'(x) = 6x(x^2 + 1)^2$

Skillsheet TEACHER

Exercise 19B

Examples 7,8,9

1 Differentiate each of the following with respect to x:

a $(x - 1)^{30}$ **b** $(x^5 - x^{10})^{20}$ **c** $(x - x^3 - x^5)^4$

d $(x^2 + 2x + 1)^4$ **e** $(x^2 + 2x)^{-2}, x \neq -2, 0$ **f** $\left(x^2 - \frac{2}{x}\right)^{-3}, x \neq 0$

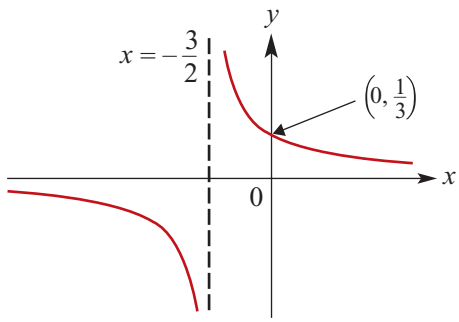
2 **a** Find the derivative of $f: R \to R$, $f(x) = (2x^3 + 1)^4$.

b Find the gradient of $f: R \to R$, $f(x) = (2x^3 + 1)^4$ at the point (1, 81).

3 **a** Find the gradient of the curve with equation $y = \dfrac{1}{x+3}$ at the point $\left(1, \frac{1}{4}\right)$.

b Find the gradient of the curve with equation $y = \dfrac{1}{(x+3)^3}$, at the point $\left(1, \frac{1}{64}\right)$.

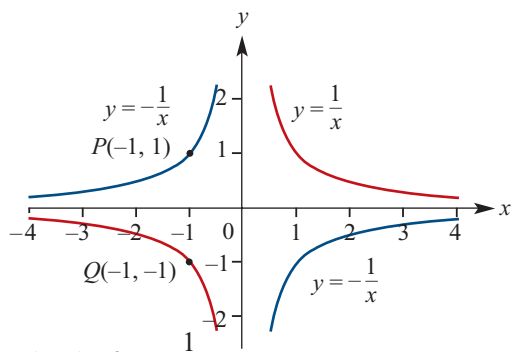
4 The diagram is a sketch graph of a function with $f(x) = \dfrac{1}{2x+3}$.



a Find the gradient of the curve at the points $\left(0, \frac{1}{3}\right)$.

b Find the coordinates of the points on the curve for which the gradient is $-\dfrac{2}{9}$.

5 The diagram is a sketch graph of the functions $y = \dfrac{1}{x}$ and $y = -\dfrac{1}{x}$.



a Find the gradient of $y = \dfrac{1}{x}$ at the point $\left(2, \frac{1}{2}\right)$.

b Without further calculation state the gradient of $y = -\dfrac{1}{x}$ at $\left(2, \frac{-1}{2}\right)$.

c Find the equation of the tangent at the point (1, 1) of $y = \dfrac{1}{x}$.

d Find the equation of the tangent at the point (1, –1) of $y = -\dfrac{1}{x}$.

e Find the equations of the tangents at points P and Q and find their point of intersection.

f Draw sketch graphs of $y = \dfrac{1}{x}$ and $y = -\dfrac{1}{x}$ on the same set of axes and draw in the four tangents.

19.3 Differentiating rational powers $(x^{p/q})$

Using the chain rule in the form $\dfrac{dy}{du} = \dfrac{dy}{dx} \cdot \dfrac{dx}{du}$

with $y = u$ $\quad 1 = \dfrac{dy}{dx} \cdot \dfrac{dx}{dy}$

and thus $\quad \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}, \quad$ for $\quad \dfrac{dx}{dy} \neq 0$

Let $y = x^{\frac{1}{n}}$, where $n \in Z \setminus \{0\}$ and $x > 0$.

Then $y^n = x$ and $\dfrac{dx}{dy} = ny^{n-1}$.

From the above results $\frac{dy}{dx} = \frac{1}{ny^{n-1}}$

$$= \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}}$$

$$= \frac{1}{n}x^{\frac{1}{n}-1}$$

For $y = x^{\frac{1}{n}}$, $\frac{dy}{dx} = \frac{1}{n}x^{\frac{1}{n}-1}$; $n \in Z \setminus \{0\}$ and $x > 0$.

This result may now be extended to any rational power.

Let $y = x^{\frac{p}{q}}$, where $p, q \in Z \setminus \{0\}$.

Write $y = \left(x^{\frac{1}{q}}\right)^p$. Let $u = x^{\frac{1}{q}}$. Then $y = u^p$.

The chain rule yields $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= pu^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1}$$

$$= p\left(x^{\frac{1}{q}}\right)^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1}$$

$$= \frac{p}{q}x^{\frac{p}{q}-\frac{1}{q}}x^{\frac{1}{q}-1}$$

$$= \frac{p}{q}x^{\frac{p}{q}-1}$$

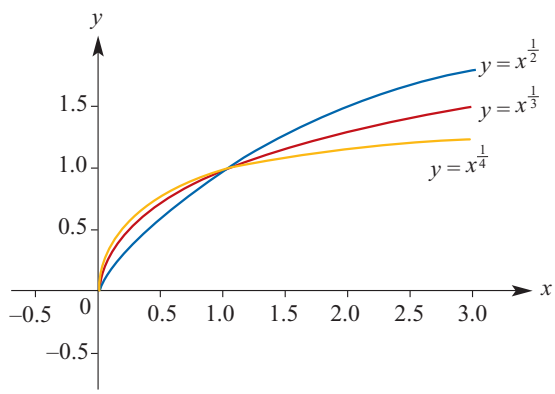
Thus the result for any non-zero rational power has been established and, in fact, it is true for any non-zero real power.

For $f(x) = x^a$, $f'(x) = ax^{a-1}$, for $x > 0$ and $a \in R$.

These results have been stated for $x > 0$, as $(-3)^{\frac{1}{2}}$ is not defined although $(-2)^{\frac{1}{3}}$ is.

The graphs of $y = x^{\frac{1}{2}}$, $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{4}}$ are shown.

The domain of each has been taken to be R^+.

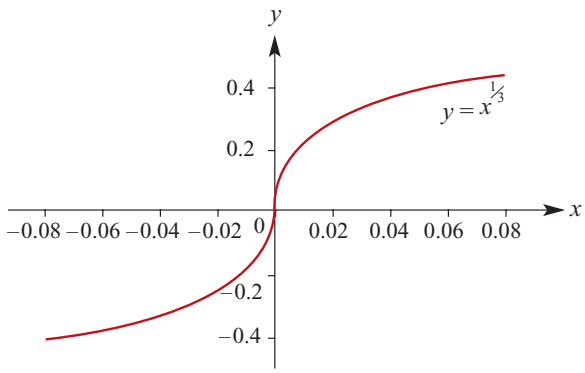


The figure to the right is the graph of the function $f: R \to R,\ f(x) = x^{\frac{1}{3}}$.

Note that the values shown here are $-0.08 \leq x \leq 0.08$.

From this it can be seen that the tangent at the origin of $y = x^{\frac{1}{3}}$ is on the y-axis.

Use a graphics calculator to investigate graphs of this type further.



Example 12

Find the derivative of each of the following with respect to x:

a $4x^{\frac{2}{3}}$ **b** $x^{\frac{1}{5}} - 2x^{-3}$

Solution

a Let $y = 4x^{\frac{2}{3}}$, then $\dfrac{dy}{dx} = 4 \times \dfrac{2}{3} \times x^{\frac{2}{3}-1} = \dfrac{8}{3}x^{\frac{-1}{3}}$

b Let $y = x^{\frac{1}{5}} - 2x^{-3}$, then $\dfrac{dy}{dx} = \dfrac{1}{5}x^{\frac{1}{5}-1} - 2 \times -3x^{-3-1}$

$= \dfrac{1}{5}x^{\frac{-4}{5}} + 6x^{-4}$

Exercise 19C

Example 12

1 Find the derivative of each of the following with respect to x:

a $x^{\frac{1}{3}}$ **b** $x^{\frac{3}{2}}, x > 0$ **c** $x^{\frac{5}{2}} - x^{\frac{3}{2}}, x > 0$

d $2x^{\frac{1}{2}} - 3x^{\frac{5}{3}}, x > 0$ **e** $x^{-\frac{5}{6}}, x > 0$ **f** $x^{-\frac{1}{2}} - 4, x > 0$

2 Find the derivative of each of the following with respect to x:

a $\sqrt{1+x^2}$ **b** $\sqrt[3]{x+x^2}$ **c** $(1+x^2)^{-\frac{1}{2}}$ **d** $(1+x)^{\frac{1}{3}}$

3 **a** Find the gradient of $y = x^{\frac{1}{3}}$ at each of the following points:

i $\left(\frac{1}{8}, \frac{1}{2}\right)$ **ii** $\left(-\frac{1}{8}, -\frac{1}{2}\right)$ **iii** $(1, 1)$ **iv** $(-1, -1)$

b Comment on your results.

4 Consider the graphs of $y = x^{\frac{1}{2}}$ and $y = x^{\frac{1}{3}}$ for $x > 0$.

a Find $\left\{x : x^{\frac{1}{2}} < x^{\frac{1}{3}}\right\}$.

b Find the values for x for which the gradient of $y = x^{\frac{1}{2}}$ is greater than the gradient of $y = x^{\frac{1}{3}}$.

5 Differentiate each of the following with respect to x:

a $(2-5\sqrt{x})^2$ **b** $(3\sqrt{x}+2)^2$ **c** $\dfrac{2+\sqrt{x}}{x^2}$

d $\dfrac{x^2+2}{\sqrt{x}}$ **e** $3\sqrt{x}\,(x^2+2)$

19.4 The second derivative

For $f: R \to R$, $f(x) = 2x^2 + 4x + 1$, the derived function has rule $f'(x) = 4x + 4$,
i.e. the derivative of $2x^2 + 4x + 1$ is $4x + 4$.
The second derivative of $2x^2 + 4x + 1$ is 4, i.e. the derivative of $4x + 4$.
The function notation for the second derivative is $f''(x) = 4$.
In the Leibniz notation
if $y = 2x^2 + 4x + 1$, $\dfrac{dy}{dx} = 4x + 4$ and the second derivative is written $\dfrac{d^2y}{dx^2}$, and $\dfrac{d^2y}{dx^2} = 4$.

Example 13

For each of the following find $f''(x)$.

a $f(x) = 3x^3 + 2x^{-1} + 1, x \neq 0$ **b** $f(x) = x^{\frac{1}{2}} + 3x^{-4} + 1, x > 0$

c $f(x) = x^4 + x^{-\frac{3}{2}} + 1, x > 0$

Solution

a $f'(x) = 9x^2 - 2x^{-2}, x \neq 0$
$f''(x) = 18x + 4x^{-3}, x \neq 0$

b $f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} - 12x^{-5}, x > 0$
$f''(x) = -\dfrac{1}{4}x^{-\frac{3}{2}} + 60x^{-6}, x > 0$

c $f'(x) = 4x^3 - \dfrac{3}{2}x^{-\frac{5}{2}}, x > 0$
$f''(x) = 12x^2 + \dfrac{15}{4}x^{-\frac{7}{2}}, x > 0$

If $S(t)$ denotes the displacement of a body at time t,
then $S'(t)$ gives the velocity at time t,
and $S''(t)$ gives acceleration at time t.

Exercise 19D

Example 13 **1** Find $f''(x)$ for each of the following:

a $f(x) = x^3 + 2x + 1$ **b** $f(x) = 3x + 2$ **c** $f(x) = (3x + 1)^4$

d $f(x) = x^{\frac{1}{2}} + 3x^3, x > 0$ **e** $f(x) = (x^6 + 1)^3$ **f** $f(x) = 5x^2 + 6x^{-1} + 3x^{\frac{3}{2}}$

Example 13 **2** For each of the following find $\dfrac{d^2y}{dx^2}$:

a $y = 3x^3 + 4x + 1$ **b** $y = 6$ **c** $y = 6x^2 + 3x + 1$

d $y = (6x + 1)^4$ **e** $y = (5x + 2)^4$ **f** $y = x^3 + 2x^2 + 3x^{-1}$

3 The height of a stone is $(20t - 4.9t^2)$ metres at a time t seconds after it is thrown. What is the acceleration?

4 After t seconds, the x-coordinate of a particle moving along the x-axis is given by $x(t) = 4t - 3t^3$, where the units on the x-axis are in metres.

a Find:

i the x-coordinate after 2 seconds of motion

ii the velocity of the particle at the start

iii the velocity of the particle after half a second

iv the velocity of the particle after 2 seconds.

b When is the acceleration zero?

c What is the average velocity during the first 2 seconds?

19.5 Sketch graphs

In Chapter 17 it was demonstrated how calculus may help us sketch the graphs of polynomial functions. In this section we show how the same techniques may be applied to non-polynomial functions.

For example: Let $f: R \setminus \{0\} \to R$, $f(x) = \dfrac{1}{x^2} + x$.

This function is not defined at $x = 0$.

1 Consider the behaviour of the function as the magnitude of x becomes large.

This section of a spreadsheet demonstrates the behaviour of increasing x. This has been done by using increasing powers of 2.

We see that as $x \to \infty$, $f(x) \to x$.

This shows us that the line $y = x$ is an oblique asymptote for the graph.

It is clear also that as $x \to -\infty$, $f(x) \to x$.

x	$f(x)$
1	2.000000000000000
2	2.250000000000000
4	4.062500000000000
8	8.015625000000000
16	16.003906250000000
32	32.000976562500000
64	64.000244140625000
128	128.000061035156250
256	256.000015258789063
512	512.000003814697266
1024	1024.00000095367432
2048	2048.00000023841858

The behaviour of the function as the magnitude of x approaches 0 is demonstrated below. A spreadsheet has been used and the values of x considered are 2^0, 2^{-1}, 2^{-2}, 2^{-3}, etc.

a

x	$f(x)$
1.0000000000	2.000000000000
0.5000000000	4.500000000000
0.2500000000	16.250000000000
0.1250000000	64.125000000000
0.0625000000	256.062500000000
0.0312500000	1024.031250000000
0.0156250000	4096.015625000000
0.0078125000	16384.007812500000
0.0039062500	65536.003906250000

Spreadsheet a shows the behaviour as x approaches 0 from the right.
$f(x)$ becomes increasingly large.
We write $\lim_{x \to 0^+} f(x) = \infty$.

b

x	$f(x)$
–1.0000000000	0.000000000000
–0.5000000000	3.500000000000
–0.2500000000	15.750000000000
–0.1250000000	63.875000000000
–0.0625000000	255.937500000000
–0.0312500000	1023.968750000000
–0.0156250000	4095.984375000000
–0.0078125000	16383.992187500000
–0.0039062500	65535.996093750000

Spreadsheet b shows the behaviour as x approaches 0 from the left.
Again $f(x)$ becomes increasingly large.
We write $\lim_{x \to 0^-} f(x) = \infty$.

2 Consider the axis intercepts.

The function cannot have a y-axis intercept, as it is not defined at $x = 0$. It would cross the x-axis when $\frac{1}{x^2} + x = 0$, i.e. when $x^3 = -1$, which implies $x = -1$.

3 Consider points of zero gradient (turning points).

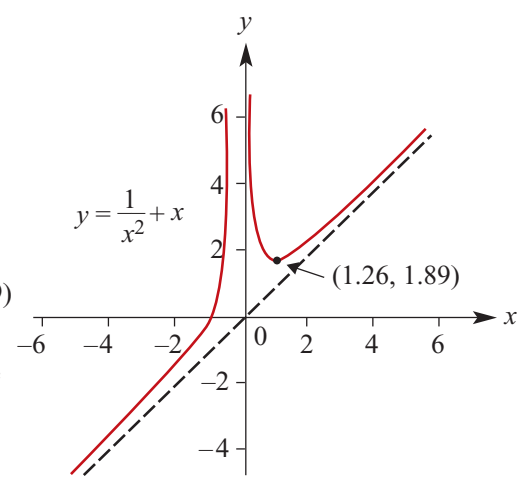
For $f(x) = \frac{1}{x^2} + x$ we have $f'(x) = \frac{-2}{x^3} + 1$ and $f'(x) = 0$ implies $x = 2^{\frac{1}{3}}$, i.e. there is a point on the curve at $x = 2^{\frac{1}{3}}$ at which the gradient is zero.

As the graph is not continuous for R, it is advisable to use a gradient chart for $x > 0$ only.

x		$2^{\frac{1}{3}}$	
$f'(x)$	–	0	+
shape of f	\	—	/

Thus there is a minimum at the point (1.26, 1.89) (coordinates correct to 2 decimal places).

The spreadsheet above shows how rapidly the graph of $y = \frac{1}{x^2} + x$ moves towards $y = x$.
At $x = 4$, $f(x) = 4.0625$ and at $x = 8$, $f(x) = 8.015625$.



Example 14

For $f: R \setminus \{-1\} \to R$, $f(x) = \dfrac{x^2+3}{x+1}$, sketch the graph.

Solution

By division, $f(x) = x - 1 + \dfrac{4}{x+1}$.

Behaviour as magnitude of x increases

As $x \to \infty$, $f(x) \to x - 1$, from above.

As $x \to -\infty$, $f(x) \to x - 1$, from below.

There is an oblique asymptote with equation $y = x - 1$.

Behaviour as $x \to -1$

$\lim\limits_{x \to -1^+} f(x) = \infty$ and $\lim\limits_{x \to -1^-} f(x) = -\infty$

There is a vertical asymptote $x = -1$.

Axis intercepts

When $x = 0$, $f(x) = 3$. There is no x-axis intercept as $x^2 + 3 \neq 0$ for all $x \in R$.

Turning points

For $f(x) = x - 1 + \dfrac{4}{x+1}$

$$f'(x) = 1 - \frac{4}{(x+1)^2}$$

Where $f'(x) = 0$

$$1 - \frac{4}{(x+1)^2} = 0$$

which implies $(x+1)^2 = 4$

i.e. $x + 1 = 2$ or $x + 1 = -2$

$\therefore x = 1$ or $x = -3$

$f(1) = 2$ and $f(-3) = -6$

We consider two gradient charts.

First for $x > -1$.

x		1	
$f'(x)$	–	0	+
shape of f	↘	—	↗

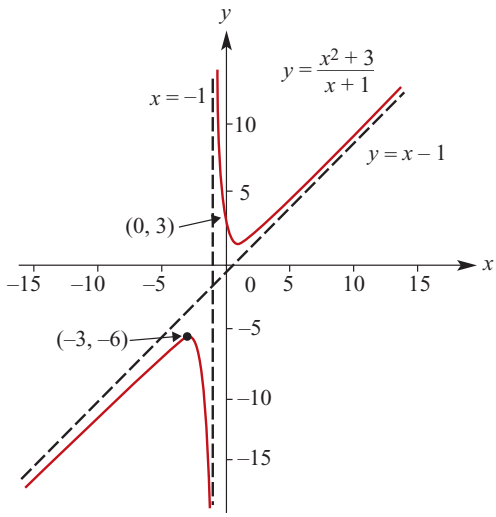
$\therefore$ a minimum at (1, 2)

Next, for $x < -1$.

x		–3	
$f'(x)$	+	0	–
shape of f	↗	—	↘

$\therefore$ a maximum at (–3, –6)

We are now in a position to sketch the graph.



Exercise 19E

1 The equation of a curve is $y = 4x + \frac{1}{x}$.

a Find the coordinates of the turning points.

b Find the equation of the tangent to the curve at the point where $x = 2$.

2 Find the x-coordinates of the points on the curve $y = \frac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

3 Find the gradient of the curve $y = \frac{2x - 4}{x^2}$ at the point where the curve crosses the x-axis.

4 For the curve $y = x - 5 + \frac{4}{x}$, find:

a the coordinates of the points of intersection with the axes

b the equations of all asymptotes **c** the coordinates of all turning points.

Use this information to sketch the curve.

5 If x is positive, find the least value of $x + \frac{4}{x^2}$.

6 For positive values of x, sketch the graph of $y = x + \frac{4}{x}$ and find the least value of y.

Example 14

MC TEST

D&D TEST

7 Sketch the graphs of each of the following, indicating the coordinates of the axes intercepts and the turning point, and the equations of asymptotes.

a $y = x + \frac{1}{x}, x \neq 0$ **b** $y = \frac{1}{x^2} - x, x \neq 0$

c $y = x + 1 + \frac{1}{x+3}, x \neq 3$ **d** $y = x^3 + \frac{243}{x}, x \neq 0$

e $y = x - 5 + \frac{1}{x}, x \neq 0$ **f** $y = \frac{x^2 - 4}{x+2}, x \neq -2$

Chapter summary

- The general result for differentiating functions, including powers of x with negative integers:
  For $f(x) = x^n$, $f'(x) = nx^{n-1}$, n a non-zero integer. For $f(x) = 1$, $f'(x) = 0$.
  We note that for $n \leq -1$, we take the domain of f to be $R \setminus \{0\}$, and for $n \geq 1$ we take the domain of f to be R.
- The chain rule is often used to differentiate some more complicated functions by transforming the original function into two simpler functions:
  e.g. $f(x)$ is transformed to $h(x)$ and $g(u)$, which are 'chained' together as
  $$x \xrightarrow{h} u \xrightarrow{g} y$$
- Using Leibniz notation the chain rule is stated as $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.
- With function notation the chain rule is stated as
  $(f \circ g)'(x) = f'(g(x))g'(x) \quad \text{where} \quad (f \circ g)(x) = f(g(x))$
- The general result for any non-zero real power:
  For $f(x) = x^a$, $f'(x) = ax^{a-1}$, for $x > 0$ and $a \in R$.
- The function for the second derivative is f''.
  In Leibniz notation it is written as $\frac{d^2y}{dx^2}$.
- When using calculus as an aid to sketching graphs of polynomial and non-polynomial functions, the following should be considered:
  1 The behaviour of the function as the magnitude of x becomes large
  2 The axis intercepts
  3 The points of zero gradient (turning points)

Multiple-choice questions

1 If $f(x) = \frac{4x^4 - 12x^2}{3x}$ then $f'(x)$ equals

A $\frac{16x^3 - 24x}{3}$ **B** $4x^2 - 4$ **C** $\frac{16x^3 - 24x}{3x}$

D $4x^2 - 8x$ **E** $\frac{8x^3 - 16x}{3x}$

2 If $f(x) = 2x^{\frac{p}{q}}$, where p and q are integers, $f'(x)$ equals

A $2x^{\frac{(p-q)}{q}}$ **B** $2px^{\frac{p}{q}-1}$ **C** 2 **D** $\frac{2p}{q}x^{\frac{(p-q)}{q}}$ **E** $\frac{2p}{q}x$

3 For $f: R \setminus \{2\} \to R$, where $f(x) = 4 + \frac{4}{2-x}$, $f'(x) > 0$ for

A $R \setminus \{2\}$ **B** R **C** $x < 2$ **D** $x > 2$ **E** $x > 4$

Review

4 A particle moves in a straight line so that its position x cm from a fixed point O at time t s ($t \geq 0$) is given by $x = -t^3 + 7t^2 - 14t + 6$. The particle's acceleration at $t = 3$ is

A -4 cm/s^2 **B** 3 cm/s^2 **C** 4 cm/s^2 **D** 8 cm/s^2 **E** 0 cm/s^2

5 Let $y = f(g(x))$, where $g(x) = x^3$, then $\frac{dy}{dx}$ equals

A $3x^2 f'(x^3)$ **B** $3x^2 f(x^3)$ **C** $2xf(x)f'(x^3)$

D $2f(x)f'(x^3)$ **E** $3x^2$

6 The graph defined by the rule $f(x) = x + \frac{1}{x}$ has a local minimum at $(a, f(a))$. The value of a is

A -1 **B** 2 **C** $-\frac{5}{2}$ **D** $\frac{5}{2}$ **E** 1

7 Which of the following is not true for the curve of $y = f(x)$, where $f(x) = x^{\frac{1}{5}}$?

A The gradient is defined for all real numbers.

B The curve passes through the origin.

C The curve passes through the points with coordinates (1, 1) and $(-1,-1)$.

D For $x > 0$ the gradient is positive.

E For $x > 0$ the gradient is decreasing.

8 Which of the following is not true for $y = f(x)$, where $f(x) = x^{\frac{3}{4}}$?

A The maximal domain of the function is $R^+ \cup \{0\}$.

B $f(x) > x$ for all $x > 1$.

C The curve of $y = f(x)$ passes through the points with coordinates (1, 1).

D For $x > 0$ the gradient of the curve is positive.

E For $x > 0$ the gradient of the curve is decreasing.

9 The derivative of $(5x^2 + 2x)^n$ is

A $n(10x + 2)(5x^2 + 2x)^{n-1}$ **B** $(5x^2 + 2x)^{n-1}$ **C** $(10x + 2)^n$

D $n(5x^2 + 2x)^{n-1}$ **E** $10x^{2n-1} + 2x^{n-1}$

10 The graph of the function with rule $y = \frac{k}{2(x^2 + 1)}$ has gradient 1 when $x = 1$. The value of k is

A 1 **B** -1 **C** 4 **D** -4 **E** $-\frac{1}{4}$

Short-answer questions (technology-free)

1 Find the derivative of each of the following with respect to x:

a x^{-4} **b** $2x^{-3}$ **c** $-\frac{1}{3x^2}$ **d** $-\frac{1}{x^4}$

e $\frac{3}{x^5}$ **f** $\frac{x^2 + x^3}{x^4}$ **g** $\frac{3x^2 + 2x}{x^2}$ **h** $5x^2 - \frac{2}{x}$

2 Find the derivative of each of the following with respect to x:

a $x^{\frac{1}{2}}$ **b** $\sqrt[3]{x}$ **c** $-\frac{2}{x^{\frac{1}{3}}}$

d $x^{\frac{4}{3}}$ **e** $x^{-\frac{1}{3}}$ **f** $x^{-\frac{1}{3}} + 2x^{\frac{3}{5}}$

3 Differentiate each of the following with respect to x:

a $(2x+3)^2$ **b** $2(3x+4)^4$ **c** $(3-2x)^{-\frac{1}{2}}$ **d** $\dfrac{1}{3+2x}$

e $\dfrac{1}{(2x-1)^{\frac{2}{3}}}$ **f** $\dfrac{3}{\sqrt{2+x^2}}$ **g** $\left(2x^2-\dfrac{3}{x^2}\right)^{\frac{1}{3}}$

4 Find the gradient of each of the following curves at the given point:

a $y=\sqrt{x}$; $(9, 3)$ **b** $y=\dfrac{1}{2x+1}$; $(0, 1)$ **c** $y=\dfrac{2}{x^2}$; $\left(4, \dfrac{1}{8}\right)$

d $y=3+\dfrac{2}{x}$; $(1, 5)$ **e** $y=\sqrt{x+1}$; $(8, 3)$ **f** $y=(x^2-7x-8)^3$; $(8, 0)$

5 Find the coordinates of the point(s) on the curve with equation $y=\dfrac{1}{x}$ for which the gradient is -4.

6 Find the coordinates of the point(s) on the curve with equation $y=\sqrt{x}$ for which the gradient is 2.

Extended-response questions

1 A solid circular cylinder has radius r cm and height h cm. It has a fixed volume of 400 cm^3.

a Find h in terms of r.

b Show that the total surface area, A cm^2, of the cylinder is given by $A=2\pi r^2+\dfrac{800}{r}$.

c Find $\dfrac{dA}{dr}$.

d Solve the equation $\dfrac{dA}{dr}=0$ for r.

e Find correct to 3 significant figures the minimum surface area of the cylinder.

f Sketch the graph of A against r.

2 A rectangle has sides of length x cm and y cm and the area of the rectangle is 16 cm^2.

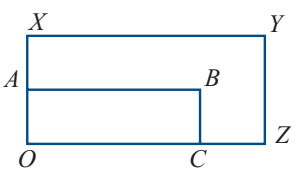
a Find y in terms of x.

b Show that the perimeter, P cm, is given by $P=2x+\dfrac{32}{x}$.

c Find the value of x for which the value of P is a minimum and find this value of P.

d Sketch the graph of P against x for $x>0$.

3 The area of rectangle $OABC$ is 120 cm^2. Let the length of OC be x cm, $CZ=5$ cm and $AX=7$ cm.



a Find the length of OA in terms of x.

b Find the length of OX in terms of x.

c Find the length of OZ in terms of x.

d Find the area, A cm^2, of rectangle $OXYZ$ in terms of x.

e Find the value of x for which the area, A cm^2, is a minimum.

Review

4 The curve with equation $y = \sqrt{x+2}$ meets the x-axis at A and the y-axis at B.

a Find the coordinates of A and B.

b By using the chain rule find $\frac{dy}{dx}$.

c **i** Find the gradient of the curve where $x = -1$.

ii Find the equation of the tangent at the point where $x = -1$.

iii If the tangent meets the x-axis at C and the y-axis at D, find the distance CD.

d Find the values of x for which $\frac{dy}{dx} < 1$.

5 An open rectangular box of height h cm has a horizontal rectangular base with side lengths x cm and $2x$ cm. The volume of the box is 36 cm^3:

a Express h in terms of x.

b Show that the total surface area of the box is given by $A = 2x^2 + \frac{108}{x}$.

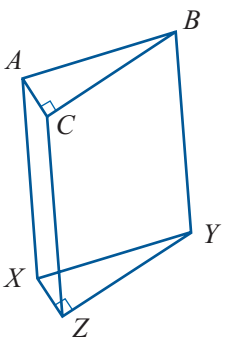
c Calculate the values of x and h which make the total surface area a minimum.

d Sketch the graph of A against x for $x > 0$.

6 The prism shown in the diagram has a triangular cross-section. The 'ends' of the prism shown are congruent right-angled triangles with the right angles at C and Z.

$AX = CZ = BY = y$ cm, $AC = XZ = 3x$ cm and $CB = ZY = 4x$ cm

The volume of the prism is 1500 cm^3.



a Express y in terms of x.

b Show that the total surface area, S cm^2, is given by $S = 12x^2 + \frac{3000}{x}$.

c Find $\frac{dS}{dx}$.

d Find the minimum value of S.

CHAPTER

20

Integration

Objectives

- To **antidifferentiate polynomial functions**.
- To **identify families of curves** with the same gradient function.
- To **find the definite integral** of an algebraic function.
- To **determine the area** between a curve and the *x*-axis.
- To use **numerical methods** for finding areas under a graph.

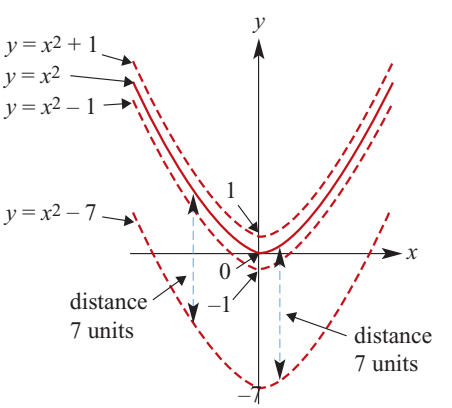
20.1 Antidifferentiation of polynomial functions

The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$, it is clear that the unknown expression could be x^2. The process of finding a function from its derivative is called **antidifferentiation**.

Consider the polynomial functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$.

We have $f'(x) = 2x$ and $g'(x) = 2x$, i.e. the two different functions have the same derived function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be **antiderivatives** of $2x$. If two functions have the same derivative on an interval, then they differ by a constant. If two functions have the same derived function then the graph of one function is obtained by a translation parallel to the y-axis of the other. The following illustrates several antiderivatives of $2x$.



Each of the graphs is a translation of $y = x^2$ parallel to the y-axis.

The general antiderivative of $2x$ is $x^2 + c$, where c is an arbitrary real number. We use the notation of Leibniz to state this with symbols:

$$\int 2x\,dx = x^2 + c$$

This is read as 'the general antiderivative of $2x$ with respect to x is equal to x^2 plus c' or 'the *indefinite* integral of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all antiderivatives and to emphasise this we write:

$$\int 2x\,dx = \{f(x)\colon f'(x) = 2x\} = \{x^2 + c\colon c \in R\}$$

The set notation is not commonly used, but it should be clearly understood that there is not a unique antiderivative for a given function. We do not use the set notation but it is advisable to keep it in mind when considering further results.

In general:
If $F'(x) = f(x)$, $\int f(x)dx = F(x) + c$, where c is an arbitrary real number.

We know that:

$f(x) = x^3$ implies $f'(x) = 3x^2$
$f(x) = x^8$ implies $f'(x) = 8x^7$
$f(x) = x^{20}$ implies $f'(x) = 20x^{19}$
$f(x) = x$ implies $f'(x) = 1$

Reversing this process we have:

$\int 3x^2 dx = x^3 + c$, where c is an arbitrary constant
$\int 8x^7 dx = x^8 + c$, where c is an arbitrary constant
$\int 20x^{19} dx = x^{20} + c$, where c is an arbitrary constant
$\int 1 dx = x + c$, where c is an arbitrary constant

and from this:

$$\int x^2 dx = \frac{1}{3}x^3 + c$$
$$\int x^7 dx = \frac{1}{8}x^8 + c$$
$$\int x^{19} dx = \frac{1}{20}x^{20} + c$$
$$\int 1 dx = x + c$$

From this we see that:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \in N \cup \{0\}$$

We also record the following results which follow immediately from the corresponding results for differentiation in Chapter 16.

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

and $$\int kf(x)dx = k\int f(x)dx, \text{ where } k \text{ is a real number}$$

Example 1

Find the general antiderivative (indefinite integral) of each of the following:

a $3x^5$ **b** $3x^2 + 4x^3 + 3$

Solution

a
$$\int 3x^5 dx = 3\int x^5 dx$$
$$= 3 \times \frac{x^6}{6} + c$$
$$= \frac{x^6}{2} + c$$

b
$$\int 3x^2 + 4x^3 + 3dx$$
$$= 3\int x^2 dx + 4\int x^3 dx + 3\int 1dx$$
$$= 3\left(\frac{x^3}{3}\right) + \frac{4x^4}{4} + \frac{3x}{1} + c$$
$$= x^3 + x^4 + 3x + c$$

In many situations we are given extra information that helps us to find a unique antiderivative.

Example 2

It is known that $f'(x) = x^3 + 4x^2$ and $f(0) = 0$. Find $f(x)$.

Solution

$$\int x^3 + 4x^2 dx = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

$$\therefore \quad f(x) = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

and as $f(0) = 0$, $c = 0$

$$\therefore \quad f(x) = \frac{x^4}{4} + \frac{4x^3}{3}$$

Example 3

If the gradient at a point (x, y) on a curve is given by $5x$ and the curve passes through $(0, 6)$ find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$.

Then $f'(x) = 5x$

$$\int 5x dx = \frac{5x^2}{2} + c$$

$$\therefore f(x) = \frac{5x^2}{2} + c$$

This describes the family of curves for which $f'(x) = 5x$. Here we are given the additional information that the curve passes through (0, 6), i.e. $f(0) = 6$.

Hence $6 = c$ and $f(x) = \frac{5x^2}{2} + 6$

Example 4

Find y in terms of x if:

a $\frac{dy}{dx} = x^2 + 2x$ and $y = 1$ when $x = 1$ **b** $\frac{dy}{dx} = 3 - x$ and $y = 2$ when $x = 4$

Solution

a $\int x^2 + 2x\,dx$

$= \frac{x^3}{3} + x^2 + c$

$\therefore \quad y = \frac{x^3}{3} + x^2 + c$

and, as $y = 1$ when $x = 1$,

$1 = \frac{1}{3} + 1 + c \quad$ and $\quad c = -\frac{1}{3}$

$\therefore \quad y = \frac{x^3}{3} + x^2 - \frac{1}{3}$

b $\int 3 - x\,dx = 3x - \frac{x^2}{2} + c$

and $\quad y = 2$ when $x = 4$

$\therefore \quad 2 = 3 \times 4 - \frac{4^2}{2} + c$

i.e. $\quad c = -2$

$\therefore \quad y = 3x - \frac{x^2}{2} - 2$

Exercise 20A

Skillsheet TEACHER

Example 1

1 Find:

a $\int \frac{1}{2}x^3 dx$ **b** $\int 3x^2 - 2dx$ **c** $\int 5x^3 - 2x dx$

d $\int \frac{4}{5}x^3 - 2x^2 dx$ **e** $\int (x-1)^2\,dx$ **f** $\int x\left(x + \frac{1}{x}\right)dx, x \neq 0$

g $\int 2z^2(z-1)dz$ **h** $\int (2t-3)^2 dt$ **i** $\int (t-1)^3 dt$

Example 4

2 Find y in terms of x in each of the following:

a $\frac{dy}{dx} = 2x - 1$ and $y = 0$ when $x = 1$ **b** $\frac{dy}{dx} = 3 - x$ and $y = 1$ when $x = 0$

c $\frac{dy}{dx} = x^2 + 2x$ and $y = 2$ when $x = 0$ **d** $\frac{dy}{dx} = 3 - x^2$ and $y = 2$ when $x = 3$

e $\frac{dy}{dx} = 2x^4 + x$ and $y = 0$ when $x = 0$

3 $\frac{dV}{dt} = t^2 - t$ when $t > 1$ and $V = 9$ when $t = 3$.

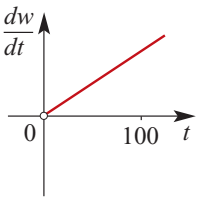
a Find V in terms of t. **b** Calculate the value of V when $t = 10$.

Example 3

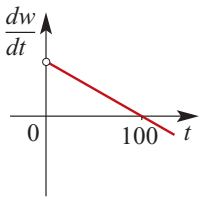
4 The gradient at any point $(x, f(x))$, on the curve with equation $y = f(x)$, is given by $3x^2 - 1$. Find $f(x)$, if the curve passes through the point (1, 2), i.e. $f(1) = 2$.

5 **a** Which one of the following graphs represents $\frac{dw}{dt} = 2000 - 20t, t > 0$?

A



B



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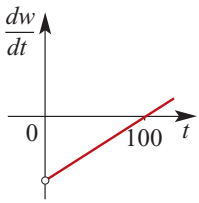
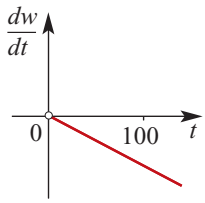
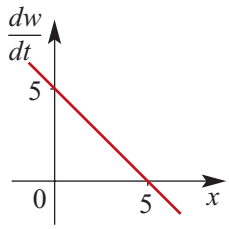


D



b Find w in terms of t, if when $f = 0$, $w = 100\,000$.

6 The graph shows $\frac{dy}{dx}$ against x for a certain curve with equation $y = f(x)$.
Find $f(x)$, given that the point (0, 4) lies on the curve.



7 Find the equation of the curve $y = f(x)$ which passes through the point $(2, -6)$ and for which $f'(x) = x^2(x - 3)$.

8 The curve $y = f(x)$ for which $f'(x) = 4x + k$, where k is a constant, has a turning point at $(-2, -1)$.

a Find the value of k.

b Find the coordinates of the point at which the curve meets the y-axis.

9 Given that $\frac{dy}{dx} = ax^2 + 1$ and that when $x = 1$, $\frac{dy}{dx} = 3$ and $y = 3$, find the value of y when $x = 2$.

10 The curve for which $\frac{dy}{dx} = 2x + k$, where k is a constant, is such that the tangent at (3, 6) passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k **b** the equation of the curve

11 The curve $y = f(x)$ for which $f'(x) = 16x + k$, where k is a constant, has a stationary point at (2, 1).

a Find the value of k. **b** Find the value of $f(x)$ when $x = 7$.

12 Suppose that a point moves along some unknown curve $y = f(x)$ in such a way that at each point (x, y) on the curve the tangent line has slope x^2. Find an equation for the curve, given that it passes through (2, 1).

20.2 Antidifferentiation of algebraic expressions with rational exponents

(This section is not listed in the VCE study design for Mathematical Methods Units 1 & 2 but can be used as a preparation for Mathematical Methods Units 3 & 4.)

In Chapter 19 it was shown that

if $y = x^{\frac{p}{q}}$ where $p, q \in Z \setminus \{0\}$

then $\frac{dy}{dx} = \frac{p}{q}x^{\frac{p}{q}-1}$

For example, we know that

$f(x) = x^{-2}$ implies $f'(x) = -2x^{-3}$

$f(x) = x^{\frac{1}{2}}$ implies $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$f(x) = x^{\frac{3}{2}}$ implies $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

Reversing this process we have

$\int -2x^{-3}dx = x^{-2} + c$, where c is an arbitrary constant

$\int \frac{1}{2}x^{-\frac{1}{2}}dx = x^{\frac{1}{2}} + c$, where c is an arbitrary constant

$\int \frac{3}{2}x^{\frac{1}{2}}dx = x^{\frac{3}{2}} + c$, where c is an arbitrary constant

and from this

$\int x^{-3}dx = -\frac{1}{2}x^{-2} + c$

$\int x^{-\frac{1}{2}}dx = 2x^{\frac{1}{2}} + c$

$\int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + c$

From this we see that

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c, \ r \in Q \setminus \{-1\}$$

Note: This definition can only be applied for suitable values of x for a given value of r. For example if $r = \frac{1}{2}$, $x \in R^+$ is a suitable restriction.

For $r = -2$, $x \in R \setminus \{0\}$ and if $r = 3$, $x \in R$.

Example 5

Find y in terms of x if:

a $\frac{dy}{dx} = \frac{1}{x^2}$ and $y = 1$ where $x = 1$

b $\frac{dy}{dx} = 3\sqrt{x}$ and $y = 2$ when $x = 4$

c $\frac{dy}{dx} = x^{\frac{3}{4}} + x^{-\frac{3}{4}}$ and $y = 0$ when $x = 0$

Solution

a $\int \frac{1}{x^2}dx = \int x^{-2}dx$

$= \frac{x^{-1}}{-1} + c$

$\therefore \quad y = \frac{-1}{x} + c$

and,

as $y = 1$ when $x = 1$,

we have $c = 2$

$\therefore \quad y = -\frac{1}{x} + 2$

b $\int 3\sqrt{x}dx = 3\int x^{\frac{1}{2}}dx$

$= 3 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$\therefore y = 2x^{\frac{3}{2}} + c$ and,

as $y = 2$ when $x = 4$,

$2 = 2 \times (4)^{\frac{3}{2}} + c$

$\therefore 2 = 2 \times 8 + c$

and $c = -14$

Thus $y = 2x^{\frac{3}{2}} - 14$

c $\int x^{\frac{3}{4}} + x^{-\frac{3}{4}}dx$

$\therefore y = \frac{4}{7}x^{\frac{7}{4}} + 4x^{\frac{1}{4}} + c$

and, as $y = 0$ when $x = 0, c = 0$

$\therefore y = \frac{4}{7}x^{\frac{7}{4}} + 4x^{\frac{1}{4}}$

Exercise 20B

Skillsheet TEACHER

1 Find:

a $\int 3x^{-2}dx$ **b** $\int 2x^{-4} + 6xdx$ **c** $\int \sqrt{x}(2 + x)dx$

d $\int 3x^{\frac{1}{3}} - 5x^{\frac{5}{4}}dx$ **e** $\int \frac{3z^4 + 2z}{z^3}dz$ **f** $\int 3x^{\frac{3}{4}} - 7x^{\frac{1}{2}}dx$

Example 5

2 Find y in terms of x for each of the following:

a $\frac{dy}{dx} = x^{\frac{1}{2}} + x$ and $y = 6$ when $x = 4$ **b** $\frac{dy}{dx} = \frac{1}{x^2}$ and $y = 1$ when $x = 1$

c $\frac{dy}{dx} = 3x + \frac{1}{x^2}$ and $y = 5$ when $x = 1$

3 A curve with equation $y = f(x)$ passes through the point (2, 0) and $f'(x) = 3x^2 - \frac{1}{x^2}$. Find $f(x)$.

4 Find s in terms of t, if $\frac{ds}{dt} = 3t - \frac{8}{t^2}$ and $s = 1\frac{1}{2}$ when $t = 1$.

5 Given that $\frac{dy}{dx} = \frac{a}{x^2} + 1$ and when $x = 1$ $\frac{dy}{dx} = 3$ and $y = 3$, find the value of y when $x = 2$.

6 The curve for which $\frac{dy}{dx} = ax$, where a is a constant, is such that the tangent at (1, 2) passes through the origin.

Find the gradient of this tangent and hence determine:

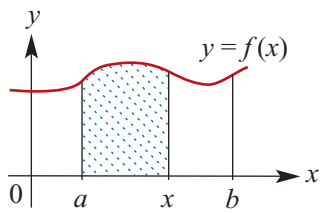
a the value of a **b** the equation of the curve.

7 Find the equation of the curve which passes through the point $(-1, 2)$ and has the property that for each point (x, y) on the curve, the gradient equals the square of the distance between the point and the y-axis.

20.3 Area – the definite integral

This section can serve as a preparation for Mathematical Methods Units 3 & 4. It is not required for Mathematical Methods Units 1 & 2.

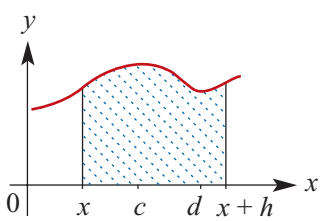
Let $f: [a, b] \to R$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$.



We define the function $F(x)$ geometrically by saying that it is the measure of the area under the curve between a and x. We thus have $F(a) = 0$. It will be shown that $F'(x) = f(x)$.

Consider the quotient $\dfrac{F(x+h) - F(x)}{h}$, $h > 0$.

By our definition of $F(x)$, $F(x + h) - F(x)$ is the area between x and $x + h$.



Let c be the point in $[x, x + h]$ such that $f(c) \geq f(z)$ for all $z \in [x, x + h]$ and let d be the point in the same interval such that $f(d) \leq f(z)$ for all $z \in [x, x + h]$.

Thus $f(d) \leq f(z) \leq f(c)$ for all $z \in [x, x + h]$.

Therefore $hf(d) \leq F(x + h) - F(x) \leq hf(c)$.

That is, the shaded region has an area less than the area of the rectangle with base h and height $f(c)$ and an area greater than the area of the rectangle with base h and height $f(d)$.

Dividing by h: $\quad f(d) \leq \dfrac{F(x+h) - F(x)}{h} \leq f(c)$

As $h \to 0$ both $f(c)$ and $f(d)$ approach $f(x)$. Thus we have shown $F'(x) = f(x)$.

We know that if $G(x)$ is an antiderivative of $f(x)$ then $F(x) = G(x) + k$, where k, is a constant.

Let $x = a$. We then have

$$0 = F(a) = G(a) + k, \quad \text{i.e. } k = -G(a)$$

Thus $F(x) = G(x) - G(a)$ and letting $x = b$ yields

$F(b) = G(b) - G(a)$ (equation 1)

> The area under the curve $y = f(x)$ between a and b is $G(b) - G(a)$, where $G(x)$ is an antiderivative of $f(x)$.

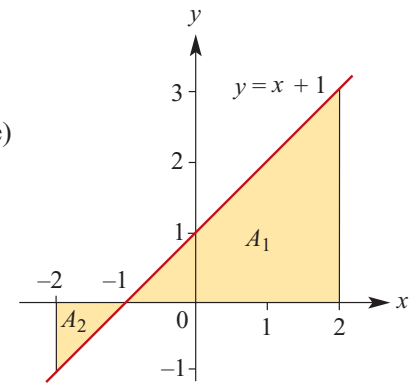
A similar argument could be used if $f(x) < 0$ for all $x \in [a, b]$, but in this case we must take F to be the negative of the area under the curve.

Signed area

For $f(x) = x + 1$:

$$A_1 = \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2} \qquad \text{(area of a triangle)}$$

$$A_2 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$



The total area $= A_1 + A_2 = 5$

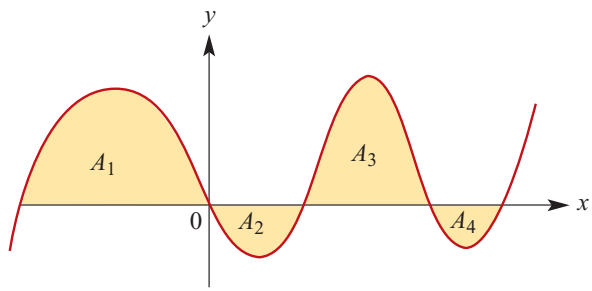
The **signed area** $= A_1 - A_2 = 4$

Regions below the x-axis have *negative signed areas*.

Regions above the x-axis have *positive signed areas*.

The total area of the shaded region is $A_1 + A_2 + A_3 + A_4$.

The signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$.



We will denote the signed area between the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis by $\int_a^b f(x)dx$.

If $f(x) \geq 0$ for $x \in [a, b]$ then $\int_a^b f(x)dx$ is the area enclosed between the x-axis and the curve $y = f(x)$ and the lines $x = a$ and $x = b$. We thus have (by equation 1)

$$\int_a^b f(x)dx = G(b) - G(a), \text{ where } G \text{ is an antiderivative of } f.$$

This result holds generally and is known as the Fundamental Theorem of Integral Calculus.

Let f be a continuous function on the interval $[a, b]$. Then

$$\int_a^b f(x)dx = G(b) - G(a), \text{ where } G \text{ is any antiderivative of } f.$$

To facilitate setting out we write $G(b) - G(a) = \Big[G(x)\Big]_a^b$

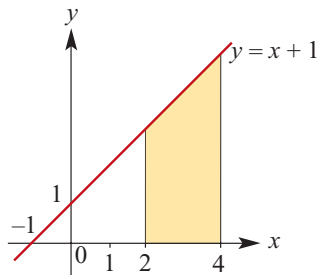
$\int_a^b f(x)dx$ is called the **definite integral** from a to b.

The number a is called the **lower limit of integration** and b is called the **upper limit of integration**. The function f is called the **integrand**.

We note that if $f(x) \geq 0$ for all $x \in [a, b]$ then the area between b and a is given by $\int_a^b f(x)dx$, and if $f(x) < 0$ for $x \in [a, b]$ then the area between b and a is given by $-\int_a^b f(x)dx$.

Example 6

Find the area of the shaded region.



Solution

$$\text{Area} = \int_2^4 x + 1 dx$$

An antiderivative of $x + 1$ is $\frac{x^2}{2} + x$

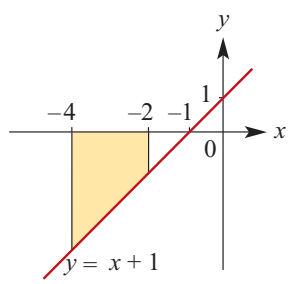
We write $\int_2^4 x + 1\, dx = \left[\frac{x^2}{2} + x\right]_2^4$

$$= \left(\frac{4^2}{2} + 4\right) - \left(\frac{2^2}{2} + 2\right)$$
$$= 12 - 4$$
$$= 8$$

The shaded region has area 8 square units.

Example 7

Find the area of the shaded region.
(**Note:** The negative is introduced as the area is the negative of the integral from -4 to -2.)



Solution

$$\text{Area} = -\int_{-4}^{-2} x + 1 dx$$
$$= -\left[\frac{x^2}{2} + x\right]_{-4}^{-2}$$
$$= -(0 - 4)$$

The area of the shaded region is 4 square units.

Now consider $\int_{-4}^{0} f(x)dx$, where $f(x) = x + 1$

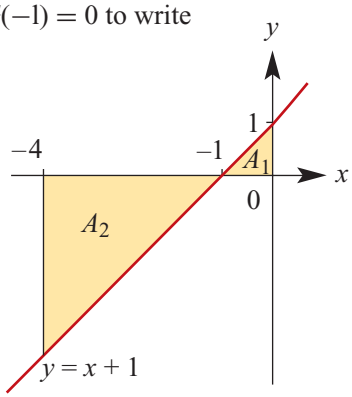
$$= G(0) - G(-4)$$

To explain what this represents we use the fact that $G(-1) - G(-1) = 0$ to write

$$\int_{-4}^{0} f(x)dx = G(0) - G(-1) + G(-1) - G(-4)$$
$$= \int_{-1}^{0} f(x)dx + \int_{-4}^{-1} f(x)dx$$
$$A_1 = \int_{-1}^{0} f(x)dx,\ A_2 = -\int_{-4}^{-1} f(x)dx$$

Thus $\int_{-4}^{0} f(x)dx = A_1 + (-A_2)$

$$= \frac{1}{2} + \left(-4\frac{1}{2}\right) = -4$$

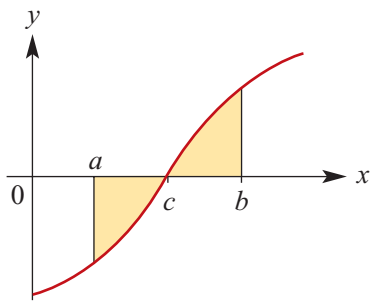


This is the signed area.

The actual area $A_1 + A_2 = \frac{1}{2} + 4\frac{1}{2} = 5$

Note: The areas of examples 6 and 7 may also be determined geometrically.

In summary

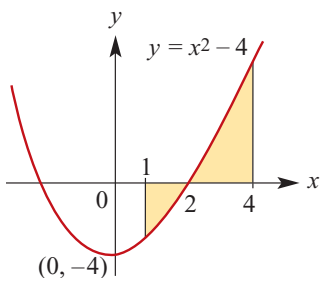


- If $f(x) \geq 0$ for all $x \in [a, b]$ the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)dx$.
- If $f(x) \leq 0$ for all $x \in [c, d]$ the area of the region contained between the curve, the x-axis and the lines $x = c$ and $x = d$ is given by $-\int_c^d f(x)dx$.
- If $c \in (a, b), f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$ then the area of the shaded region is given by $\int_c^b f(x)dx + -\int_a^c f(x)dx$.

Note: the important aspect is the sign of $f(x)$ in the given interval.

Example 8

Find the area of the shaded region.



Solution

$$\text{Area} = \int_2^4 x^2 - 4dx + -\int_1^2 x^2 - 4dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_2^4 - \left[\frac{x^3}{3} - 4x\right]_1^3$$

$$= \frac{64}{3} - 16 - \left(\frac{8}{3} - 8\right) - \left[\left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right)\right] = \frac{56}{3} - 8 - \left[\frac{7}{3} - 4\right]$$

$$= \frac{37}{3}$$

The area is $\frac{37}{3}$ square units.

Properties of the definite integral

1 $\int_a^a f(x)dx = 0$

2 $\int_a^b f(x)dx = \int_c^b f(x)dx + \int_a^c f(x)dx,\ a < c < b$

3 $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

4 $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5 $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Example 9

Evaluate each of the following definite integrals:

a $\int_2^3 x^2 dx$ **b** $\int_3^2 x^2 dx$ **c** $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}} dx$

Solution

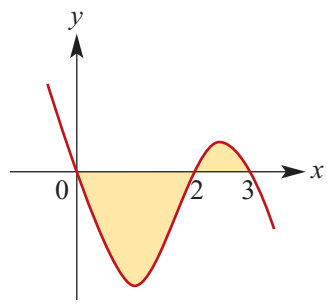
a
$$\int_2^3 x^2 dx$$
$$= \left[\frac{x^3}{3}\right]_2^3$$
$$= \frac{27}{3} - \frac{8}{3}$$
$$= 9 - 2\frac{2}{3}$$
$$= 6\frac{1}{3}$$

b
$$\int_3^2 x^2 dx$$
$$= \left[\frac{x^3}{3}\right]_3^2$$
$$= \frac{8}{3} - \frac{27}{3}$$
$$= -6\frac{1}{3}$$

c
$$\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}} dx$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right]_0^1$$
$$= \frac{2}{3} + \frac{2}{5}$$
$$= \frac{16}{15}$$

Example 10

Find the area enclosed by the graph of $y = x(2 - x)(x - 3)$ and the x-axis.



Solution

$$y = x(-x^2 + 5x - 6)$$
$$= -x^3 + 5x^2 - 6x$$

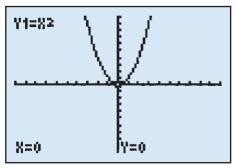
Note: There is no need to find the coordinates of stationary points.

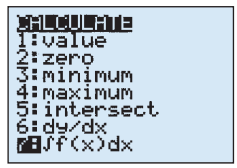
$$\text{Area} = \int_2^3 (-x^3 + 5x^2 - 6x)dx + -\int_0^2 (-x^3 + 5x^2 - 6x)dx$$
$$= \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_2^3 - \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_0^2$$
$$= \left(\frac{-81}{4} + 45 - 27\right) - \left(-4 + \frac{40}{3} - 12\right) - \left(-4 + \frac{40}{3} - 12\right)$$
$$= \frac{-81}{4} + 18 + 32 - \frac{80}{3}$$
$$= 50 - \frac{243 + 320}{12}$$
$$= 3\frac{1}{12}$$

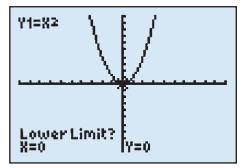
Area is $3\frac{1}{12}$ square units.

Using a graphics calculator

Enter **Y1** $= X^2$ in the **Y=** menu and select **6:Zstandard** from the **ZOOM** menu.
Select integration **(7:** $\int f(x)dx$**)** from the **CALC** menu.
The screen will appear as shown.

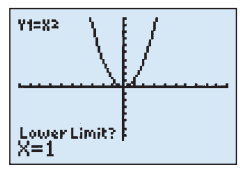




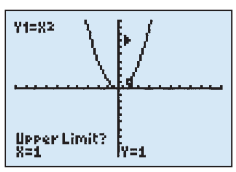


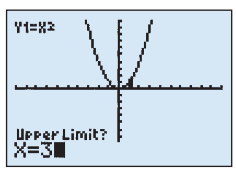
Enter **1** as the required value for lower limit for integration.
and press ENTER.

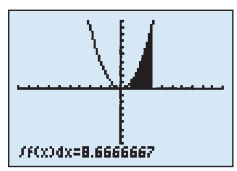
This value can also be chosen by using the calculator's arrow keys.



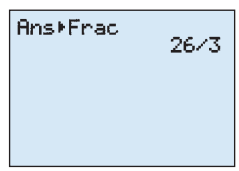
The sequence of screens necessary to complete the integration is shown below.







The integral of x^2 between 1 and 3 has been evaluated.
The exact answer can be found in the Home screen.



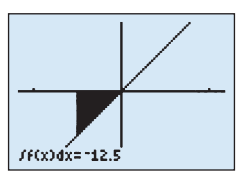
The integration can also be performed by choosing **9:fnInt(** from the **MATH** menu.

Example 11

Find $\int x dx$ with lower limit -5 and upper limit 0 by using **Y1=** X.

Solution

$\int_{-5}^{0} f(x)dx \approx -12.5$



Exercise 20C

Example 9

1 Evaluate each of the following:

a $\int_1^2 x^2 dx$ **b** $\int_{-1}^3 x^3 dx$ **c** $\int_0^1 x^3 - x dx$

d $\int_{-1}^2 (x+1)^2 dx$ **e** $\int_1^2 x^3 dx$ **f** $\int_1^4 x + 2x^2 dx$

g $\int_0^2 x^3 + 2x^2 + x + 2 dx$ **h** $\int_1^4 2x + 5 dx$

2 Evaluate each of the following:

a $\int_1^2 \frac{(2-x)(2+x)}{x^2} dx$ **b** $\int_1^4 2x - 3\sqrt{x} dx$ **c** $\int_1^3 \frac{4x^2+9}{x^2} dx$

d $\int_1^4 6x - 3\sqrt{x} dx$ **e** $\int_1^4 \frac{x^2-1}{x^2} dx$ **f** $\int_1^4 \frac{2x - 3\sqrt{x}}{x} dx$

3 Given that $\int_1^5 h(x) dx = 4$, evaluate:

a $\int_1^5 2h(x) dx$ **b** $\int_1^5 h(x) + 3 dx$ **c** $\int_5^1 h(x) dx$

4 Given that $\int_2^5 f(x) dx = 12$, evaluate:

a $\int_5^2 f(x) dx$ **b** $\int_2^5 3f(x) dx$ **c** $\int_2^4 f(x) + 4 dx + \int_4^5 f(x) dx$

Example 6

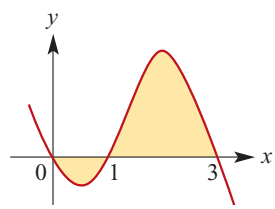
5 On a graph of $y = x^2$, shade the region corresponding to $\int_1^3 x^2 dx$ and calculate its value.

6 On a graph of the line $y = 2t + 3$ shade the trapezium between the line and the t-axis bounded by $t = 1$ and $t = 5$. Use geometry to find the area of this trapezium and verify your result by integration.

7 The figure shows part of the graph of the curve with equation $y = x(x-1)(3-x)$.

Example 8

Calculate the area of the shaded region.



8 Calculate the values of $\int_1^3 f(x) dx$, $\int_3^4 f(x) dx$, $\int_1^4 f(x) dx$ for:

a $f(x) = 6x$ **b** $f(x) = 6 - 2x$

What is the relationship between your three answers in each case?

9 Sketch the graph of $y = 5x - x^2 - 4$ and find the area enclosed by the x-axis and the portion of the curve above the x-axis.

10 Sketch the graph of $y = x(10 - x)$ and hence find the area enclosed between the x-axis and the portion of the curve above the x-axis.

11 Sketch the graph of $y = x(x-2)(x+1)$ and find the area of the region contained between the graph and the x-axis. (Do not attempt to find the coordinates of the turning points.)

12 Find the area bounded by the x-axis and the graph of each of the following functions:

a $x^2 - 2x$ **b** $(4-x)(3-x)$ **c** $(x+2)(7-x)$

d $x^2 - 5x + 6$ **e** $3 - x^2$ **f** $x^3 - 6x^2$

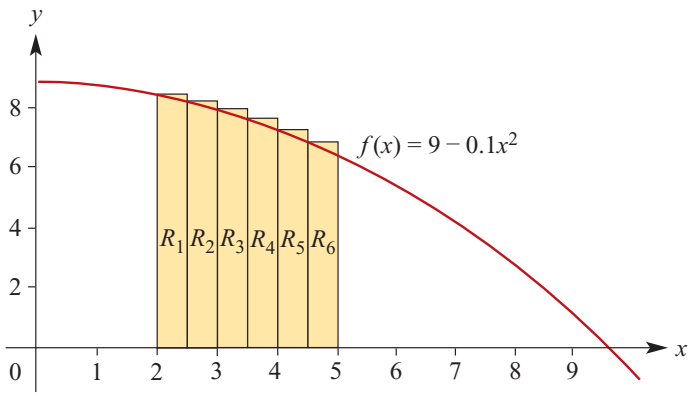
20.4 Numerical methods for finding areas

This section is not required for Mathematical Methods Units 1 & 2 but can be used as a preparation for Mathematical Methods Units 3 & 4.

In this section we consider three methods for determining the area under a graph.

The left-endpoint estimate

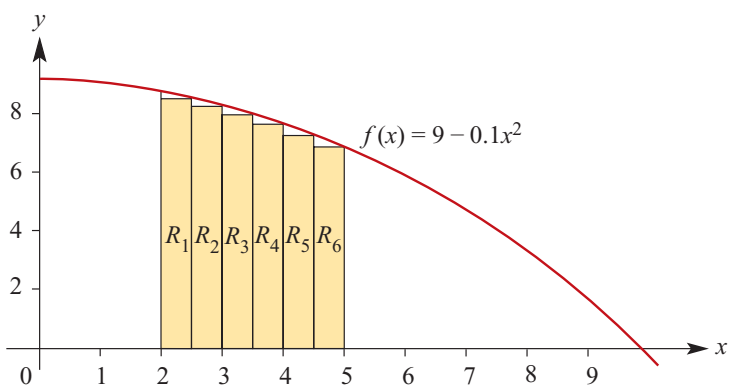
We first find an approximation for the area of the shaded region by dividing the region into rectangles as illustrated. The width of each rectangle is 0.5.



The area of rectangle $R_1 = 0.5\,f(2.0) = 0.5 \times 8.60 = 4.30$ square units.
The area of rectangle $R_2 = 0.5\,f(2.5) = 0.5 \times 8.38 = 4.19$ square units.
The area of rectangle $R_3 = 0.5\,f(3.0) = 0.5 \times 8.10 = 4.05$ square units.
The area of rectangle $R_4 = 0.5\,f(3.5) = 0.5 \times 7.78 = 3.89$ square units.
The area of rectangle $R_5 = 0.5\,f(4.0) = 0.5 \times 7.40 = 3.70$ square units.
The area of rectangle $R_6 = 0.5\,f(4.5) = 0.5 \times 6.98 = 3.49$ square units.
The sum of the areas of the rectangles is 23.62 square units.

The left-endpoint estimate will be larger than the actual area for a graph that is decreasing over the interval and smaller than the actual area for a graph that is increasing.

The right-endpoint estimate



Again the rectangles have width 0.5 and we find:

The area of rectangle $R_1 = 0.5\,f(2.5) = 0.5 \times 8.38 = 4.19$ square units.

The area of rectangle $R_2 = 0.5\,f(3.0) = 0.5 \times 8.10 = 4.05$ square units.

The area of rectangle $R_3 = 0.5\,f(3.5) = 0.5 \times 7.78 = 3.89$ square units.

The area of rectangle $R_6 = 0.5\,f(5.0) = 0.5 \times 6.5 = 3.25$ square units.

The sum of the areas of the rectangles is 22.67 square units.

This is called the right-endpoint estimate for the area.

It is an approximation for $\int_2^5 (9 - 0.1x^2)dx$

For f decreasing over $[a, b]$:
left-endpoint estimate $\geq$ true area $\geq$ right-endpoint estimate

For f increasing over $[a, b]$:
left-endpoint estimate $\leq$ true area $\leq$ right-endpoint estimate

From this it can be seen that a further estimate for the area may be achieved by the average of the two estimates:

$$\text{Average area} = \frac{\text{left-endpoint estimate} + \text{right-endpoint estimate}}{2}$$

For the example discussed above, average area = 23.15 square units.

It is clear that if narrower strips are chosen we obtain an estimate that is closer to the true value. This is time-consuming to do by hand, but a computer program or spreadsheet makes the process quite manageable. In the spreadsheet shown below the right-endpoint estimate is calculated using rectangles of width 0.1 units.

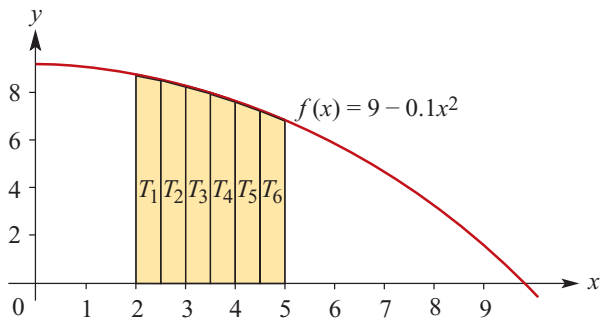
x	$y = f(x)$	$f(x) \times 0.1$	x	$y = f(x)$	$f(x) \times 0.1$
2.1	8.559	0.8559	3.6	7.704	0.7704
2.2	8.516	0.8516	3.7	7.631	0.7631
2.3	8.471	0.8471	3.8	7.556	0.7556
2.4	8.424	0.8424	3.9	7.479	0.7479
2.5	8.375	0.8375	4	7.4	0.74
2.6	8.324	0.8324	4.1	7.319	0.7319
2.7	8.271	0.8271	4.2	7.236	0.7236
2.8	8.216	0.8216	4.3	7.151	0.7151
2.9	8.159	0.8159	4.4	7.064	0.7064
3	8.1	0.81	4.5	6.975	0.6975
3.1	8.039	0.8039	4.6	6.884	0.6884
3.2	7.976	0.7976	4.7	6.791	0.6791
3.3	7.911	0.7911	4.8	6.696	0.6696
3.4	7.844	0.7844	4.9	6.599	0.6599
3.5	7.775	0.7775	5	6.5	0.65

Sum of areas of rectangles = 22.9945, therefore $\int_2^5 (9 - 0.1x^2)dx \approx 22.9945$.

The trapezoidal estimate

For this estimate we work with trapeziums instead of rectangles.

The area of a trapezium $= \frac{1}{2}(a+b)h$, where a and b are lengths of the parallel sides and h their distance apart.



The area of trapezium $T_1 = \frac{1}{2}(f(2.0) + f(2.5))0.5 = 4.24375$ square units.

The area of $T_2 = \frac{1}{2}(f(2.5) + f(3.0))0.5 = 4.11875$ square units.

The area of $T_3 = \frac{1}{2}(f(3.0) + f(3.5))0.5 = 3.96875$ square units.

The area of $T_4 = \frac{1}{2}(f(3.5) + f(4.0))0.5 = 3.79375$ square units.

The area of $T_5 = \frac{1}{2}(f(4.0) + f(4.5))0.5 = 3.59375$ square units.

The area of $T_6 = \frac{1}{2}(f(4.5) + f(5.00))0.5 = 3.36875$ square units.

The sum of the areas $= 23.0875$ square units.

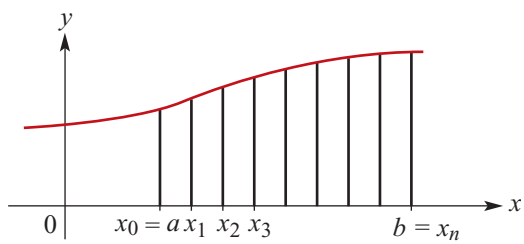
This method is called the **trapezoidal rule**.

We see from adding that:

$$\text{the required area} = \frac{1}{2}(f(2) + f(5) + 2[f(2.5) + f(3) + f(3.5) + f(4) + f(4.5)])$$

Once again we have found an approximation for $\int_2^5 (9 - 0.1x^2)dx$.

If the interval $[a, b]$ on the x-axis is divided into n equal sub-intervals $[a, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, . . . $[x_{n-1}, x_n]$ as illustrated, the three methods may be summarised as follows.



The left-endpoint estimate

$$L_n = \frac{b-a}{n}[f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$$

The right-endpoint estimate

$$R_n = \frac{b-a}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)]$$

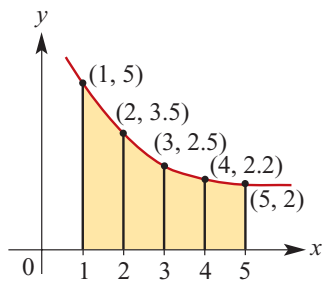
The trapezoidal estimate

$$T_n = \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots 2f(x_{n-1}) + f(x_n)]$$

These methods are not limited to situations in which the graph is either increasing or decreasing for all the interval but may be used to determine the area under any continuous curve. They may be applied to any continuous function on an interval $[a, b]$ to determine an approximate value of $\int_a^b f(x)dx$.

Exercise 20D

1 To find the area of the region shaded in the figure, calculate:

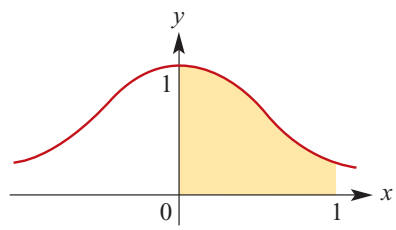


a the left-endpoint estimate
b the right-endpoint estimate
c the trapezoidal estimate.
(Use sub-intervals as shown.)

2 Calculate an approximation to an area under the graph of $y = x(3 - x)$ between $x = 0$ and $x = 4$ using strips of width:

i 0.5 **ii** 0.2

3 The graph is that of $y = \frac{1}{1 + x^2}$. It is known that the area of the shaded region is $\frac{\pi}{4}$. Apply the trapezoidal rule with strips of width 0.25 and hence find an approximate value for π.



4 A table of values is given for the rule $y = f(x)$.

x	0	1	2	3	4	5	6	7	8	9	10
y	3	3.5	3.7	3.8	3.9	3.9	4.0	4.0	3.7	3.3	2.9

Find the area enclosed by the graph of $y = f(x)$, the lines $x = 0$ and $x = 10$, and the x-axis by using:

a the left-endpoint estimate **b** the trapezoidal estimate.

5 Use the trapezium rule to find approximate values for:

a $\int_0^2 2^x dx$ (Use intervals of width 0.5.)

b $\int_0^{0.9} \frac{1}{\sqrt{1 - x^2}} dx$ (Use intervals of width 0.1.)

6 A man takes soundings at intervals of 3 metres across a river 30 metres wide. Using the trapezium rule find an estimate for the area of the cross-section of the river's channel.

Distance from bank in metres	0	3	6	9	12	15	18	21	24	27	30
Depth of sounding in metres	1	2	3	4	5	5	6	4	4	2	2

20.5 The fundamental theorem of calculus revisited

We restate the theorem.

Let f be a function continuous on $[a, b]$.

If F is any one antiderivative of f on $[a, b]$ then $\int_a^b f(x)dx = F(b) - F(a)$

In Section 20.3 we defined the definite integral in terms of area, with areas above the x-axis counted positively and those below counted negatively. Our discussion was based mainly on intuitive ideas and we wish to treat this more formally. But more importantly we wish to consider how a definite integral can be defined as the limit of a sum, as this is a technique that is used widely. Our discussion remains informal as a proof of the result is beyond the scope of this book.

We first introduce a notation to help us express sums. We will do this through examples.

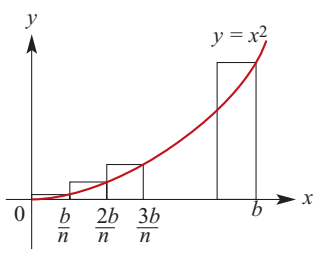
$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$$

$$\sum_{i=1}^{3} y_i^2 = y_1^2 + y_2^2 + y_3^2$$

$$\sum_{i=1}^{4} x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + x_4 f(x_4)$$

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n$$

(Σ is the upper case of the Greek letter 'sigma' which is used in mathematics to denote *sum*.)



Consider the graph of $y = x^2$.

Divide the interval $[0, b]$ into n sub-intervals. This gives a sequence of closed intervals.

$$\left[0, \frac{b}{n}\right], \left[\frac{b}{n}, \frac{2b}{n}\right], \left[\frac{2b}{n}, \frac{3b}{n}\right], \ldots, \left[\frac{(n-1)b}{n}, b\right]$$

As in the previous section, the right-endpoint estimate

$$= \frac{b-0}{n}\left[f\left(\frac{b}{n}\right) + f\left(\frac{2b}{n}\right) + f\left(\frac{3b}{n}\right) + \cdots + f(b)\right]$$

$$= \frac{b}{n}\left[\frac{b^2}{n^2} + \frac{4b^2}{n^2} + \frac{9b^2}{n^2} + \cdots + b^2\right]$$

$$= \frac{b^3}{n^3}[1 + 4 + 9 + \cdots + n^2]$$

$$= \frac{b^3}{n^3}\sum_{i=1}^{n} i^2$$

There is a rule for working out the sum of the first n square numbers:

$$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\therefore \text{Right-endpoint estimate} = \frac{b^3}{n^3} \times \frac{n}{6}(n+1)(2n+1)$$

$$= \frac{b^3}{6n^2}(2n^2+3n+1)$$

$$= \frac{b^3}{6}\left(2+\frac{1}{2n}+\frac{1}{6n^2}\right)$$

As n becomes very large, the terms $\frac{1}{2n}$ and $\frac{1}{6n^2}$ become very small.

We write: $\lim_{n\to\infty} \frac{b^3}{6}\left(2+\frac{1}{2n}+\frac{1}{6n^2}\right) = \frac{b^3}{3}$

We read this as 'the limit of the sum as n approaches infinity is $\frac{b^3}{3}$'.

It can also be shown that working with the left-endpoint estimate for n rectangles and considering the limit as n approaches infinity gives $\frac{b^3}{3}$.

This result was first achieved by Archimedes.

This may also be applied in general to a continuous function f in an interval $[a, b]$.

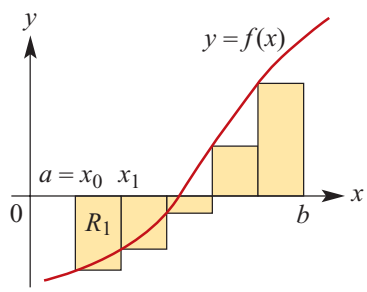
For convenience we will consider an increasing function.

When considering the left-endpoint estimate for this graph we note the areas of rectangles are signed areas.

For example:

$$\text{Area of } R_1 = (x_1 - x_0)f(x_0) \text{ and } f(x_0) < 0$$

Thus, when the left-endpoint or right-endpoint estimation is found, it is an estimation of the signed area.



The left-endpoint estimate for a function f in an interval $[a, b]$ with n sub-intervals

$$= \frac{b-a}{n}\sum_{i=0}^{n} f(x_i)$$

If the limit as $n \to \infty$ exists, we can state the following definition:

$$\lim_{n\to\infty} \frac{b-a}{n}\sum_{i=0}^{n} f(x_i) = \int_a^b f(x)dx$$

The procedure indicated could also have used the right-endpoint estimate, as the left- and right-endpoint estimates will converge to the one limit as n approaches infinity.

Definite integrals may be defined as the limit of suitable sums and the fundamental theorem of calculus holds true under this definition.

MC TEST D&D TEST

Chapter summary

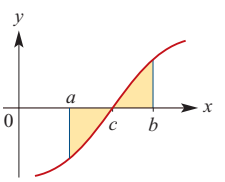
- To find the **general antiderivative** F:
  If $F'(x) = f(x)$
  then $\int f(x)dx = F(x) + c$, where c is an arbitrary real number.

- This gives the following general results:

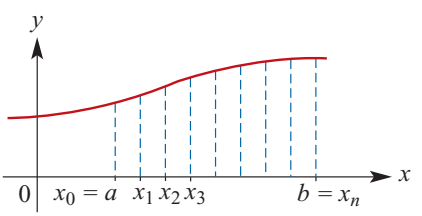
$$\int x^r dx = \frac{x^{r+1}}{r+1} + c, \quad r \in Q \setminus \{-1\}$$
$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$
$$\int kf(x)dx = k\int f(x)dx, \text{ where } k \text{ is a real number}$$

- The **fundamental theorem of integral calculus** states that $\int_a^b f(x)dx = G(b) - G(a)$, where G is any antiderivative of f, and $\int_a^b f(x)dx$ is called the definite integral from a to b. The number a is called the *lower limit of integration* and b is called the *upper limit of integration*.
  The function f is called the *integrand*.

- If $f(x) \geq 0$ for all $x \in [a, b]$ the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)dx$.
  If $f(x) \leq 0$ for all $x \in [c, d]$ the area of the region contained between the curve, the x-axis and the lines $x = c$ and $x = d$ is given by $-\int_c^d f(x)dx$.
  If $c \in [a, b]$, $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by $\int_c^b f(x)dx + -\int_a^c f(x)dx$



- Three numerical methods are used for finding areas.
  The interval $[a, b]$ on the x-axis is divided into n equal sub-intervals $[a, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$.



  - **The left-endpoint estimate**

$$L_n = \frac{b-a}{n}[f(x_0) + f(x_1) + \cdots f(x_{n-1})]$$

  - **The right-endpoint estimate**

$$R_n = \frac{b-a}{n}[f(x_1) + f(x_2) + \cdots f(x_n)]$$

  - **The trapezoidal estimate**

$$T_n = \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots 2f(x_{n-1}) + f(x_n)]$$

Review

Multiple-choice questions

1 An antiderivative of $x^3 + 3x$ is

A $x(x^2 + 3)$ **B** $3(x^2 + 1)$ **C** $\frac{x^4}{4} + \frac{3x^2}{2} + c$

D $x^3 + 3 + c$ **E** $3x^3 + x^2 + c$

2 $\int \sqrt{x} + x dx$ equals

A $\frac{\sqrt{x^2} + x^2}{2} + c$ **B** $x^{\frac{3}{2}} + x^2 + c$ **C** $\frac{1}{2}x + 1$

D $\frac{2x^{\frac{3}{2}}}{3} + \frac{1}{2}x^2 + c$ **E** 2

3 $3\int x^{-4}\, dx$ equals

A $\frac{-1}{x^3} + c$ **B** $-x^3 + c$ **C** $-12x^{-3} + c$

D $-12x^{-5} + c$ **E** $\frac{-3x^{-4}}{5} + c$

4 An expression for y, if $\frac{dy}{dx} = 2x + 5$ and $y = 1$ when $x = 0$, is

A $y = x^2 + 5x$ **B** $y = 2$ **C** $y = x^2 + 5x - 1$

D $y = x^2 + 5x + 1$ **E** $y = x^2 + 5x - 5$

5 Given that $f'(x) = 5x^4 - 9x^2$ and $f(1) = 2$, then $f(x)$ is equal to

A $x^5 - 3x^2 + 2$ **B** $x^5 - 3x^3 + 4$ **C** $x^5 - 9x^2 + 5$

D $x^5 - 3x^3$ **E** $x^5 - 3x^2 + 2$

6 An expression for y, if $\frac{dy}{dx} = \frac{4}{x^3}$ and $y = 0$ when $x = 1$, is

A $y = \frac{-2}{x^2}$ **B** $y = \frac{-2}{x^2} + 2$ **C** $y = \frac{-2}{3x^2} - \frac{2}{3}$

D $y = \frac{-4}{x^2} - 2$ **E** $y = 2x$

7 If $F'(x) = f(x)$, then $\int_3^5 f(x)dx$ is

A $f(5) - f(3)$ **B** $f(5) + c$ **C** $f(5) - f(3) + c$

D $F(5) - F(3)$ **E** $F(5) + F(3)$

8 $\int_0^2 (3x^2 - 2x)dx =$

A 12 **B** 4 **C** 8 **D** 32 **E** 2

9 An equivalent expression for $\int_0^2 3f(x) + 2dx$ is

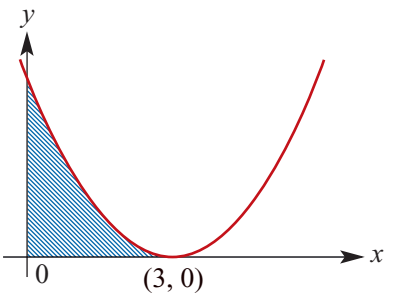
A $3\int_0^2 f(x)dx + 2x$ **B** $3\int_0^2 f(x)dx + 2x$ **C** $3\int_0^2 f(x)dx + 4$

D $2f'(x) + 3$ **E** $\int_0^2 f(x) + 6$

10 The graph with the equation $y = k(x-3)^2$ is shown. If the area of the shaded region is 36 square units, the value of k is

A 4 **B** -4 **C** 9

D $\frac{4}{7}$ **E** 32



Short-answer questions (technology-free)

1 Find:

a $\int \frac{1}{2} dx$ **b** $\int \frac{1}{2} x^2 dx$ **c** $\int (x^2 + 3x) dx$ **d** $\int (2x+3)^2 dx$

e $\int a t dt$ **f** $\int \frac{1}{3} t^3 dt$ **g** $\int (t+1)(t-2) dt$ **h** $\int (2-t)(t+1) dt$

2 The curve with equation $y = f(x)$ passes through the point $(3, -1)$ and $f'(x) = 2x + 5$. Find $f(x)$.

3 The curve with equation $y = f(x)$ passes through the origin. Let $f'(x) = 3x^2 - 8x + 3$.

a Find $f(x)$. **b** Find the intercepts of the curve with the x-axis.

4 Find:

a $\int 2x^{-3} dx$ **b** $\int \sqrt{x}(x-2) dx$ **c** $\int \frac{3x^2 + 2x}{x} dx$

d $\int \frac{3x+1}{x^3} dx$ **e** $\int 5x - 2\sqrt{x} dx$ **f** $\int 5x^{\frac{3}{4}} - 2x^{\frac{1}{3}} dx$

g $\int 2 - \sqrt{x} dx$ **h** $\int \frac{3x^2 + 2x}{x^4} dx$

5 Find s in terms of t, if $\frac{ds}{dt} = t + 3 - \frac{1}{t^2}$ and $s = 6$ when $t = 1$.

6 Evaluate each of the following definite integrals:

a $\int_1^2 2x dx$ **b** $\int_2^5 2 dx$ **c** $\int_3^5 (3x^2 + 2x) dx$

d $\int_1^5 (x^2 + 2x) dx$ **e** $\int_{-3}^{-2} 5 dx$

7 Evaluate each of the following definite integrals:

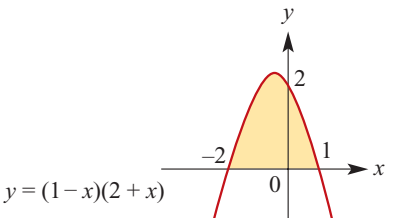
a $\int_1^4 \sqrt{x} dx$ **b** $\int_1^4 x^3 - 2x dx$ **c** $\int_1^2 \frac{1}{x^2} dx$

d $\int_1^4 \frac{2}{x^3} dx$ **e** $\int_0^1 \sqrt{x}(x+1) dx$

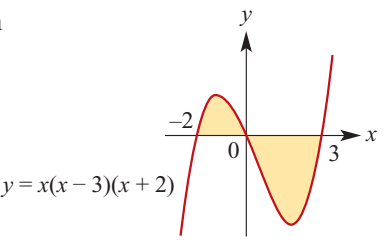
8 On a graph of $y = x^3$, shade the region corresponding to $\int_1^2 x^3 dx$ and calculate its value.

Review

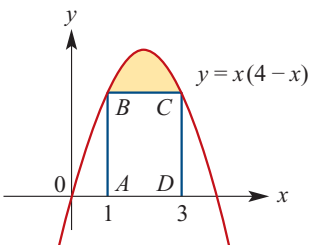
9 The figure shows the graph of $y = (1 - x)(2 + x)$.
Find the area of the shaded region.



10 The figure shows the graph of the curve with equation $y = x(x - 3)(x + 2)$.
Calculate the area of the shaded region.



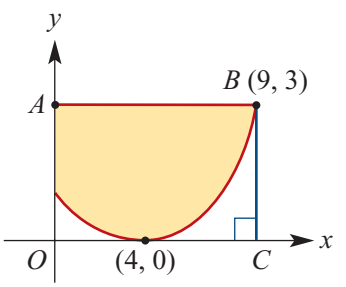
11 **a** Find the coordinates of points B and C.
b Find the area of rectangle $ABCD$.
c Find the area of the shaded region.



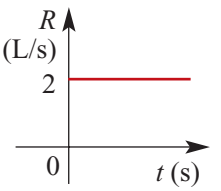
Extended-response questions

1 The slope of a children's slide is given by $\frac{dy}{dx} = \frac{9}{32}(x^2 - 4x)$ for $x \in [0, 4]$, where the origin is taken to be at ground level beneath the highest point of the slide, which is 3 m above the ground. All units are in metres.
a Find the equation of the curve which describes the slide.
b Sketch the curve of the slide, labelling stationary points.
c Does the slope of the slide ever exceed 45°?

2 A swimming pool has a cross-sectional area as shown.
a Find the area of the rectangle $OABC$.
b Find the equation of the curve given that it is of the form $y = k(x - 4)^2$.
c Find the total area of the region enclosed between the curve and the x-axis for $x \in [0, 9]$.
d Find the area of the cross-section of the pool (i.e. the shaded region).



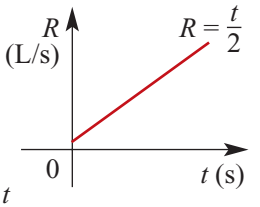
3 **a** Water flows into a container at the constant rate (R) of 2 litres/second. A graph of the rate of flow is as shown.



i How much water has flowed into the container after 1 minute?

ii Illustrate this quantity as an area under the graph of $R = 2$.

b Water flows into a container at a rate R L/s, where $R = \frac{t}{2}$ and t is the time measured in seconds. The graph is as shown.

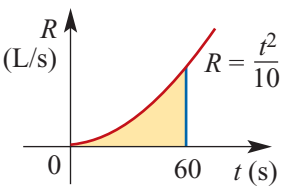


i How much water has flowed into the container after 1 minute?

ii Illustrate this quantity as an area under the graph of $R = \frac{t}{2}$.

iii How much water has flowed into the container after a minutes?

c Water flows into a container at a rate R L/s, where $R = \frac{t^2}{10}$ and t is the time measured in seconds.



i Find the area of the shaded region.

ii What does the area represent?

iii After how many seconds will 10 000 litres have flowed into the container?

4 **a** **i** A car travels on a straight road at 60 km/h for 2 hours. Sketch the speed–time graph illustrating this.

ii Shade the region which indicates the total distance travelled by the car after 2 hours.

b **i** Sketch the speed–time graph of a car travelling for 5 minutes, if the car starts from rest and accelerates at a rate of 0.3 km/min^2.

ii How far has the car travelled at the end of 5 minutes?

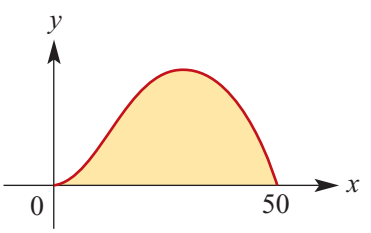
c A particle starts from a point O and travels at a velocity V m/s given by $V = 20t - 3t^2$, where t is the time the particle has been travelling measured in seconds.

i Find the acceleration of the particle at time t.

ii Sketch the graph of V against t for $0 \leq t \leq \frac{20}{3}$.

iii How far has the particle travelled after 6 seconds? Illustrate this quantity by shading a suitable region under the graph.

5 A large mound of earth has a constant cross-sectional area. The cross-section is described by the rule $y = \frac{x^2}{1000}(50 - x)$, where y denotes the height of the mound in metres at a distance of x metres from the edge. (x is measured from 0.)



a Find the height of the mound when:

i $x = 10$ **ii** $x = 40$

Review

b Find the gradient of the boundary curve $y = \frac{x^2}{1000}(50 - x)$ when:

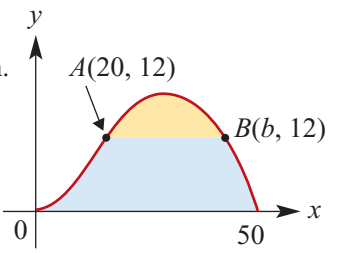
i $x = 10$ **ii** $x = 40$

c **i** Find the value of x for which the height of the mound is a maximum.

ii Find the maximum height of the mound.

d Find the cross-sectional area of the mound.

e It is decided to take the 'top' off the mound as shown.



i Point A has coordinates (20, 12). Find the coordinates of point B.

ii Find values p, q and R such that the area of the cross-section of the top is determined by $\int_q^p \frac{x^2}{1000}(50 - x)dx - R$.

6 **a** The curve with equation $y = f(x)$ passes through the point with coordinates (1, 6) and $f'(x) = 6x + 3$.

i Find the gradient of the curve at the point with coordinates (1, 6).

ii Find the equation of the tangent to the curve $y = f(x)$ at this point.

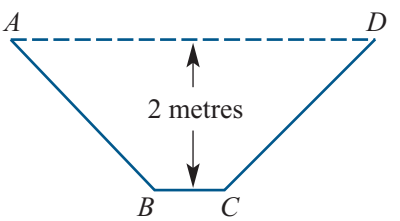
iii Find the equation of the curve.

b The curve with equation $y = f(x)$ is such that the tangent to the curve at the point with coordinates (2, 10) passes through the origin and $f'(x) = 6x + k$.

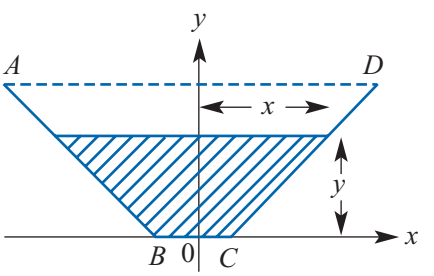
i Find the gradient of this tangent in terms of k.

ii Find the value of k. **iii** Find the equation of the curve.

7 **a** An irrigation channel 2 metres deep is constructed. Its cross-section, trapezium $ABCD$, is shown in the diagram. $AD = 5$ metres and $BC = 1$ metre. Calculate the area enclosed by this cross-section.



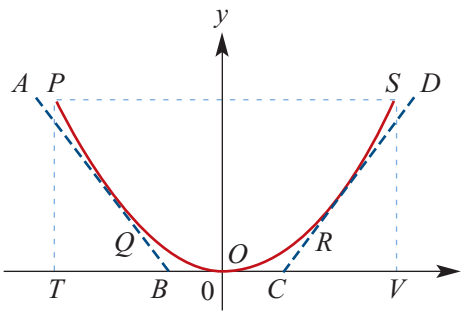
b The cross-section is placed symmetrically on coordinate axes as shown.



i Find the equation of the line CD.

ii Calculate the area of the cross-section of the water (shaded) when the water is y metres deep.
Give your answer in terms of x.

c In an attempt to improve water flow a metal chute is added. Its cross-section has the shape of a parabola *PQORS* and it just touches the original channel at *Q*, *O* and *R*.

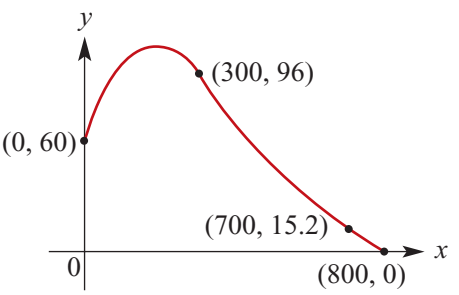


i Given that the *x*-coordinate of *R* is 1, find the equation of the parabola and find the coordinates of *P* and *S*.

ii Calculate the area of the cross-section *PQORS*.

(Hint: You could find the area of rectangle *PTVS* and subtract the area 'under' the parabola.)

8 A small hill has a cross-section as shown. The coordinates of four points are given. Measurements are in metres.



a Find the equation of the cubic.

b Find the maximum height of the hill correct to the nearest metre.

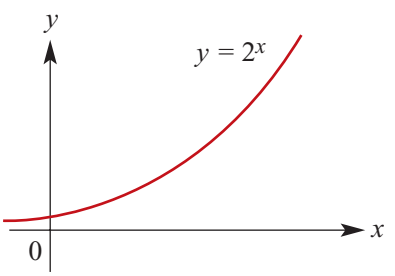
c **i** Using a graphics calculator, plot the graph of the gradient function.

ii State the coordinates of the point on the curve where the magnitude of the gradient is a maximum.

d Use a graphics calculator to find the cross-sectional area correct to the closest square metre.

9 **a** Use a graphics calculator to plot the graph of $g(x) = \int_0^x f(t)dt$, where $f(t) = 2^t$ for $x \in [0, 5]$.

b Solve the equation $\int_0^x f(t)dt = 10$.



CHAPTER

21

Revision

Revision of Chapters 19–20

21.1 Multiple-choice questions

1 The derivative of $y = \frac{1}{x^5}$ is equal to

A $\frac{1}{5x^4}$ **B** $-\frac{4}{x^4}$ **C** $\frac{5}{x^4}$ **D** $-\frac{5}{x^6}$ **E** $-\frac{5}{x^4}$

2 If $f(x) = \sqrt{9x^2 + 4}$ then $f'(x)$ is equal to

A $18x(9x^2 + 4)^{\frac{1}{2}}$ **B** $3x + 2$ **C** $\sqrt{18x}$

D $-9x(9x^2 + 4)^{-\frac{1}{2}}$ **E** $9x(9x^2 + 4)^{-\frac{1}{2}}$

3 If $f(x) = (3x^2 - 7)^4$ then $f'(x)$ is equal to

A $6x(3x^2 - 7)^3$ **B** $(6x - 7)^4$ **C** $24x(3x^2 - 7)^3$

D $24x(6x - 7)^3$ **E** $(6x)^4$

4 If $f(x) = 4x^3 - 3x + 7 - \frac{2}{x}$, then $f'(1)$ equals

A -2 **B** 18 **C** 7 **D** 11 **E** 14

5 The derivative of $\frac{2}{3 + x}$ is

A $\frac{2}{(3 + x)^2}$ **B** $\frac{2}{3}$ **C** $\frac{1}{2}$ **D** $\frac{2}{3 - x}$ **E** $\frac{-2}{(x + 3)^2}$

6 The derivative of $\frac{x - 1}{\sqrt{x}}$ is

A $2\sqrt{x}$ **B** $\frac{x + 1}{x\sqrt{x}}$ **C** $\frac{3x - 1}{2\sqrt{x}}$ **D** $\frac{x + 1}{2x\sqrt{x}}$ **E** $\frac{3x - 1}{2x\sqrt{x}}$

7 $\int 3x^2 + 6dx =$

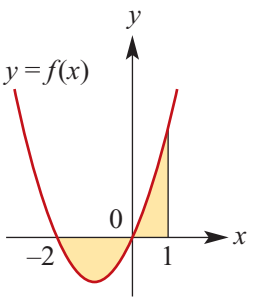
A $x^3 + 6x + c$ **B** $6x$ **C** $6x + c$

D $\frac{x^3}{3} + \frac{6}{x} + c$ **E** $x^3 + 6x + c$

8 $\int_1^3 x - 2dx =$

A 0 **B** 1 **C** $-\frac{1}{2}$ **D** $\frac{1}{2}$ **E** 2

9 The total area of the shaded regions is determined by



A $\int_{-2}^{1} x^2 + 2xdx$

B $\int_0^1 x^2 + 2xdx + \int_{-2}^{0} x^2 + 2xdx$

C $\int_0^1 x^2 + 2xdx + \int_0^{-2} x^2 + 2xdx$

D $\int_0^{-2} x^2 + 2xdx$

E $\int_1^0 (x^2 + 2x) + \int_0^{-2} x^2 + 2xdx$

10 If $f'(x) = 6x^2 + 3$ and $f(1) = 7$ then $f(x)$ equals

A $2x^3 + 3x$ **B** $12x$ **C** 84

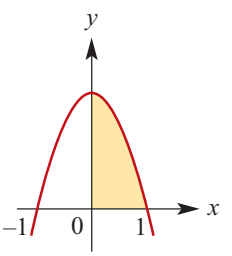
D $2x^3 + 3x + 2$ **E** $2x^3 + 3x + 7$

11 $\int_1^4 5f(x) + 2dx$ can be written as

A $4\int_1^4 f(x)dx + 2f(x)$ **B** $\int_1^4 f(x) + 10dx$ **C** $3f'(x) - 2x + c$

D $\int_1^4 5f(x)dx - 5\int_1^4 2dx$ **E** $5\int_1^4 f(x)dx + 6$

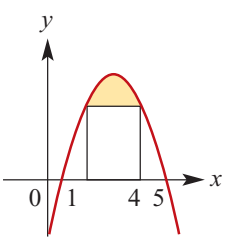
12 The graph with the equation $y = k(1 - x^2)$ is shown.
If the area of the shaded region is 40 square units then k is



A $-\frac{80}{3}$ **B** $\frac{80}{3}$ **C** 60

D 10 **E** $\frac{2}{3}$

13 The curve with equation $y = 5x - x^2$ is shown.
The area of the shaded region is



A 5.4 square units **B** 4.5 square units

C 16.5 square units **D** 6.0 square units

E 12 square units

Revision

14 $4x^2(2x+1)dx$ is equal to

A $\frac{x^3}{3}\left(\frac{x^2}{2}+x\right)+c$ **B** $4(x^4+x^3)+c$ **C** $2x^4+\frac{4x^3}{3}+c$

D $\frac{x^4}{4}+\frac{x^3}{3}+c$ **E** $\frac{x^2(x+1)^2}{2}$

15 The total area of the shaded region is

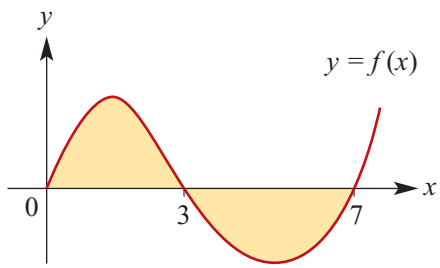
A $\int_0^7 f(x)dx$

B $\int_7^0 f(x)dx$

C $\int_0^3 f(x)dx - \int_3^7 f(x)dx$

D $\int_0^3 f(x)dx - \int_7^3 f(x)dx$

E $\int_0^3 f(x)dx + \int_3^7 f(x)dx$



16 If $g(x)$ is such that $g'(x) = f(x)$ then $\int_1^3 f(x)dx$ is equal to

A $f(3)-f(1)$ **B** $f(3)+c$ **C** $g(3)-g(1)+c$

D $f(3)-f(1)+c$ **E** $g(3)-g(1)$

Answers

Chapter 1

Exercise 1A

1 a 3 **b** 9 **c** 1 **d** -8 **e** 5 **f** 2 **g** $\frac{5}{3}$
h $\frac{-7}{2}$ **i** $\frac{7}{3}$ **j** $\frac{20}{3}$ **k** $\frac{-10}{3}$ **l** $\frac{14}{5}$

2 a $a+b$ **b** $a-b$ **c** $\frac{b}{a}$ **d** ab **e** $\frac{bc}{a}$

3 a 7 **b** 5 **c** -3 **d** 14 **e** $\frac{7}{2}$ **f** $\frac{14}{3}$
g 48 **h** $\frac{3}{2}$ **i** 2 **j** 3 **k** 7 **l** 2

4 a $\frac{4}{3}$ **b** -5 **c** 2

5 a -1 **b** 18 **c** $\frac{6}{5}$ **d** 23 **e** 0 **f** 10
g 12 **h** 8 **i** $-\frac{14}{5}$ **j** $\frac{12}{5}$ **k** $\frac{7}{2}$

6 a $\frac{-b}{a}$ **b** $\frac{e-d}{c}$ **c** $\frac{c}{a}-b$ **d** $\frac{b}{c-a}$
e $\frac{ab}{b+a}$ **f** $a+b$ **g** $\frac{b-d}{a-c}$ **h** $\frac{bd-c}{a}$

7 a -18 **b** -78.2 **c** 16.75
d 28 **e** 34 **f** $\frac{3}{26}$

Exercise 1B

1 a $x+2=6, 4$ **b** $3x=10, \frac{10}{3}$
c $3x+6=22, \frac{16}{3}$ **d** $3x-5=15, \frac{20}{3}$
e $6(x+3)=56, \frac{19}{3}$ **f** $\frac{x+5}{4}=23, 87$

2 $A=\$8$, $B=\$24$, $C=\$16$
3 14 and 28 **4** 8 kg **5** 1.3775 m^2
6 49, 50, 51 **7** 17, 19, 21, 23 **8** 4200 L
9 21 **10** 3 km **11** 9 and 12 dozen
12 7.5 km/h **13** 3.6 km **14** 30, 6

Exercise 1C

1 a $x=-1, y=-1$ **b** $x=5, y=21$
c $x=-1, y=5$

2 a $x=8, y=-2$ **b** $x=-1, y=4$
c $x=7, y=\frac{1}{2}$

3 a $x=2, y=-1$ **b** $x=2.5, y=-1$
c $m=2, n=3$ **d** $x=2, y=-1$
e $s=2, t=5$ **f** $x=10, y=13$
g $x=\frac{4}{3}, y=\frac{7}{2}$ **h** $p=1, q=-1$
i $x=-1, y=\frac{5}{2}$

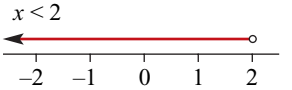
Exercise 1D

1 25, 113 **2** 22.5, 13.5
3 a \$70 **b** \$12 **c** \$3
4 a \$168 **b** \$45 **c** \$15
5 17 and 28 **6** 44 and 12
7 5 pizzas, 25 hamburgers
8 Started with 60 and 50; finished with 30 each
9 \$17 000 **10** 120 shirts and 300 ties
11 360 Outbacks and 300 Bush Walkers
12 Mydney = 2800; Selbourne = 3200
13 20 kg at \$10, 40 kg at \$11 and 40 kg at \$12.

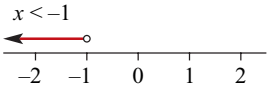
Exercise 1E

1 a $x<1$ **b** $x>13$ **c** $x\geq 3$ **d** $x\leq 12$
e $x\leq -6$ **f** $x>3$ **g** $x>-2$
h $x\geq -8$ **i** $x\leq \frac{3}{2}$

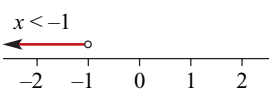
2 a $x<2$



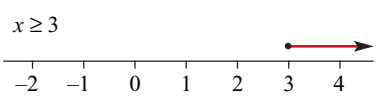
b $x<-1$



c $x<-1$

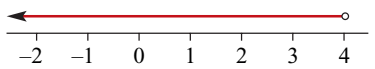


d $x\geq 3$

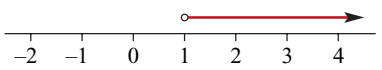


Answers

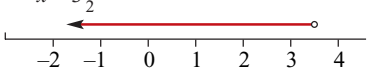
e $x < 4$



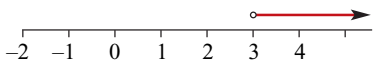
f $x > 1$



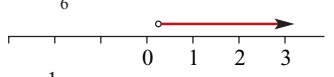
g $x < 3\frac{1}{2}$



h $x \geq 3$



i $x > \frac{1}{6}$



3 a $x > \frac{-1}{2}$ **b** $x < 2$ **c** $x > -5$

4 $3x < 20, x < \frac{20}{3}$, 6 pages **5** 87

Exercise 1F

1 a 18 **b** 9 **c** 3 **d** −18
e 3 **f** 81 **g** 5 **h** 20

2 a $S = a + b + c$ **b** $P = xy$ **c** $C = 5p$
d $T = dp + cq$ **e** $T = 60a + b$

3 a 15 **b** 31.4 **c** 1000
d 12 **e** 314 **f** 720

4 a $V = \frac{c}{p}$ **b** $a = \frac{F}{m}$
c $P = \frac{I}{rt}$ **d** $r = \frac{w - H}{C}$
e $t = \frac{S - lP}{Pr}$ **f** $r = \frac{R(V - 2)}{V}$

5 a $T = 48$ **b** $b = 8$ **c** $h = 3.82$ **d** $b = 10$

6 a $(4a + 3w)$ m **b** $(h + 2b)$ m
c $3wh$ m^2 **d** $(4ah + 8ab + 6wb)$ m^2

7 a i $T = 2\pi(p + q) + 4h$ **ii** $88\pi + 112$
b $p = \frac{A}{h} - q$

8 a $D = \frac{2}{3}$ **b** $b = 2$
c $n = \frac{60}{29}$ **d** $r = 4.8$

9 a $D = \frac{1}{2}bc(1 - k^2)$ **b** $k = \sqrt{1 - \frac{2D}{bc}}$
c $k = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$

10 a $P = 4b$ **b** $A = 2bc - c^2$ **c** $b = \frac{A + c^2}{2c}$

11 a $b = \frac{a^2 - a}{2}$ **b** $x = \frac{-ay}{b}$
c $r = \sqrt{3q - p^2x^2}$ **d** $v = \sqrt{u^2\left(1 - \frac{x^2}{y^2}\right)}$

Multiple-choice questions

1 D **2** D **3** C **4** A **5** C
6 C **7** B **8** B **9** A **10** B

Short-answer questions (technology-free)

1 a 1 **b** $\frac{-3}{2}$ **c** $\frac{-2}{3}$ **d** −27
e 12 **f** $\frac{44}{13}$ **g** $\frac{1}{8}$ **h** 31

2 a $t = a - b$ **b** $\frac{cd - b}{a}$ **c** $\frac{d}{a} + c$
d $\frac{cb - a}{c - 1}$ **e** $\frac{2b}{c - a}$ **f** $\frac{1 - cd}{ad}$

3 a $x < \frac{2}{3}$ **b** $x \leq 148\frac{1}{2}$
c $x < \frac{22}{29}$ **d** $x \geq \frac{-7}{17}$

4 $x = 2(z + 3t), -10$

5 a $d = e^2 + 2f$ **b** $f = \frac{d - e^2}{2}$ **c** $f = \frac{1}{2}$

6 $x = \frac{a^2 + b^2 + 2ab}{ac + bc} = \frac{a + b}{c}$ **7** $x = \frac{ab}{a - b - c}$

Extended-response questions

1 a $c = \frac{-10}{9}$ **b** $F = 86$ **c** $x = -40$
d $x = -62.5$ **e** $x = \frac{-160}{13}$ **f** $k = 5$

2 a $r = \frac{2uv}{u + v}$ **b** $m = \frac{v}{u}$

3 a $T = 6w + 6l$
b i $T = 8w$ **ii** $l = \frac{25}{6}, w = 12\frac{1}{2}$
c i $y = \frac{L - 6x}{8}$ **ii** $y = 22$
d $x = 10, y = 5$

4 a distance that Tom travelled $= ut$ km and distance Julie travelled $= vt$ km
b i $t = \frac{d}{u + v}$ h **ii** distance from A $= \frac{ud}{u + v}$ km
c $t = 1.25$ h, distance from town A $= 37.5$ km

5 a average speed $= \frac{uv}{u + v}$
b i $\frac{uT}{v}$ **ii** $\frac{vT + uT}{v}$

6 a $\frac{3}{a} + \frac{3}{b}$
c i $c = \frac{2ab}{a + b}$ **ii** $\frac{40}{3}$

7 a $\frac{x}{8}, \frac{y}{10}$ **b** $\frac{80(x + y)}{10x + 8y}$
c $x = \frac{320}{9}, y = \frac{310}{9}$

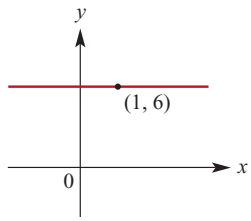
Chapter 2

Exercise 2A

1 a 4 **b** 2 **c** $\frac{1}{4}$ **d** -4 **e** 1 **f** -1
g $\frac{5}{4}$ **h** -2 **i** $\frac{-5}{4}$ **j** $\frac{4}{3}$ **k** 0

2 Any line parallel to the one shown

3



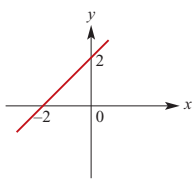
4 a $-\frac{1}{4}$ **b** $-\frac{5}{2}$ **c** -2 **d** -8 **e** 0 **f** -1
g 7 **h** 11 **i** -13 **j** 11 **k** 111 **l** 61

5 a -2 **b** $\frac{2}{5}$ **6 a** 54 **b** $\frac{5}{6}$

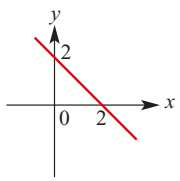
7 a $x = 4$ **b** $y = 11$

Exercise 2B

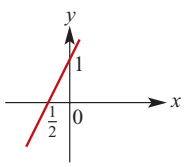
1 a



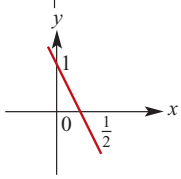
b



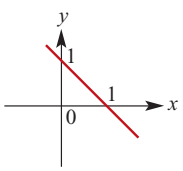
c



d



2 a



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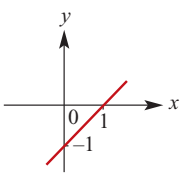

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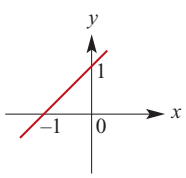

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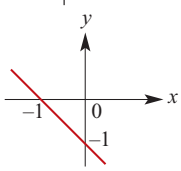

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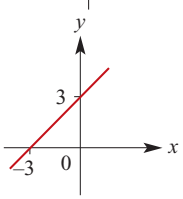

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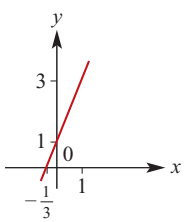

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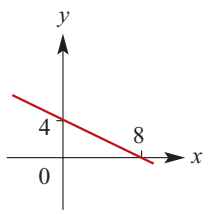

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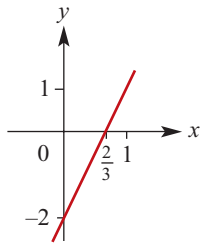

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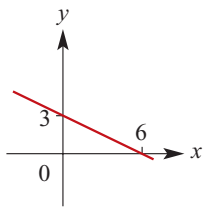

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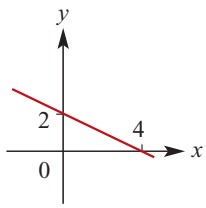

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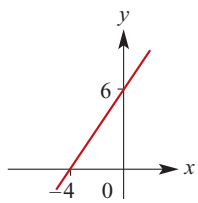
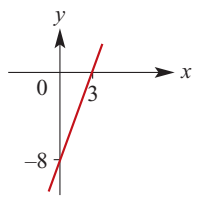
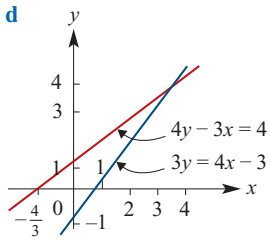


h



4 Pairs which are parallel: **a**, **b** and **c**; Non-parallel:

d



5 *c* only **6 a** -1 **b** 0 **c** -1 **d** 1

Exercise 2C

1 a $y = 3x + 5$ **b** $y = -4x + 6$ **c** $y = 3x - 4$
2 a $y = 3x - 11$ **b** $y = -2x + 9$
3 a 2 **b** $y = 2x + 6$
4 a -2 **b** $y = -2x + 6$
5 a $y = 2x + 4$ **b** $y = -2x + 8$
6 a $y = 4x + 4$ **b** $y = -\frac{2}{3}x$ **c** $y = -x - 2$
d $y = \frac{1}{2}x - 1$ **e** $y = 3\frac{1}{2}$ **f** $x = -2$

7 Some possible answers:
a $y = 4x - 3$ **b** $y = -\frac{2}{3}x - 1$ **c** $y = -x - 1$
d $y = \frac{1}{2}x + 1$ **e** $y = 4$ **f** $x = -1$
Check with your teacher for other answers.

8 a $y = \frac{3}{4}x + 9\frac{1}{2}$ **b** $y = -\frac{1}{2}x - 1$
c $y = 3$ **d** $y = -3$
9 a $y = \frac{4}{3}x + 3$ **b** $y = -\frac{1}{2}x + 3$

Answers

c $y = -0.7x + 6.7$ **d** $y = 1\frac{1}{2}x - 3$

e $y = -\frac{3}{4}x + 6$ **f** $y = -x$

10 a $y = -\frac{2}{3}x + 4$ **b** $y = -2x - 6$

c $y = -\frac{1}{2}x + 4$ **d** $y = -x + 8$

11 a $y = \frac{2}{3}x + 4$ **b** $y = \frac{2}{3}x - \frac{2}{3}$

c $y = \frac{1}{2}x + 1\frac{1}{2}$ **d** $y = -\frac{1}{2}x + 2$

e $y = x + 3.5$ **f** $y = -\frac{1}{2}x + 0.25$

12 Yes

13 AB: $y = \frac{2x}{3} + \frac{1}{3}$ BC: $y = -\frac{3x}{2} + 9$

AC: $y = \frac{1}{8}x - \frac{3}{4}$

Exercise 2D

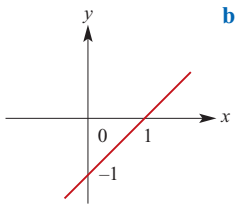
1 a (0, 4), (4, 0) **b** (0, −4) (4, 0)

c (0, −6), (−6, 0) **d** (0, 8) (−8, 0)

2 a $y = -2x + 6$ **b** $y = \frac{1}{2}x - 2$

c $y = x$ **d** $y = \frac{-1}{2}x + 3$

3 a



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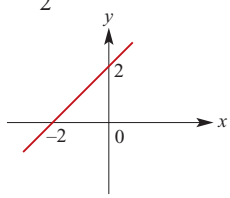

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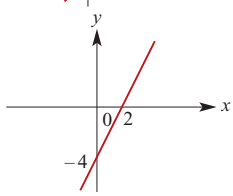
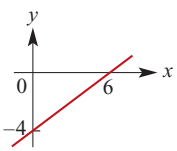


4 a



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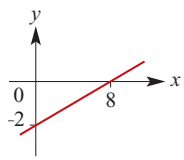

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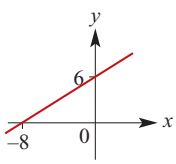

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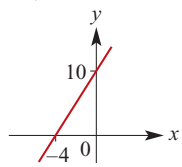

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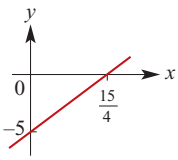

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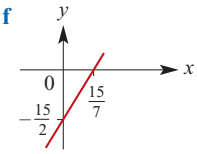


5 a $x + 3y = 11$ **b** $7x + 5y = 20$

c $2x + y = 4$ **d** $-11x + 3y = -61$

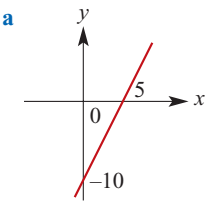
6 a $y = 2x - 9$, $m = 2$

b $y = -\frac{3}{4}x + \frac{5}{2}$, $m = -\frac{3}{4}$

c $y = -\frac{1}{3}x - 2$, $m = -\frac{1}{3}$

d $y = \frac{5}{2}x - 2$, $m = \frac{5}{2}$

7 a



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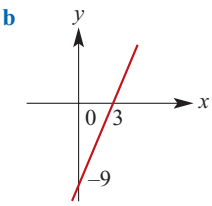

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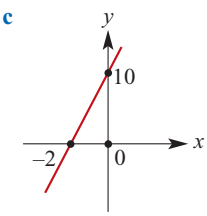
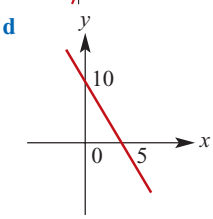


d



8 $a = -4$, $b = \frac{4}{3}$, $d = -1$, $e = \frac{14}{3}$

Exercise 2E

1 a $d = 50t$ **b** $d = 40t + 5$

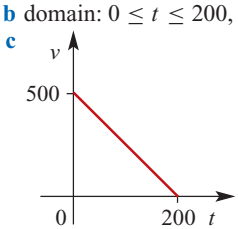
2 a $V = 5t$ **b** $V = 10 + 5t$

3 $w = 20n + 350$, possible values for $n = N \cup \{0\}$

4 a $v = 500 - 2.5t$

b domain: $0 \leq t \leq 200$, range: $0 \leq v \leq 500$

c

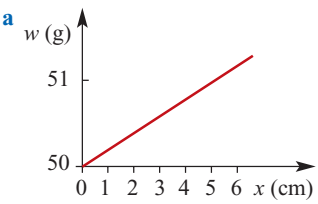


5 $C = 1.5n + 2.6$

6 a $C = 0.24x + 85$ **b** \$145

7 $d = 200 - 5t$

8 a



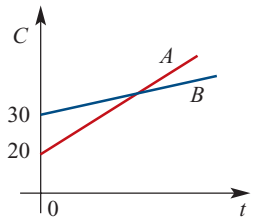
b $w = 0.2x + 50$ **c** $x = 12.5$ cm

9 a $C = 0.06n - 1$ **b** \$59

10 a $C = 5n + 175$ **b** Yes **c** \$175

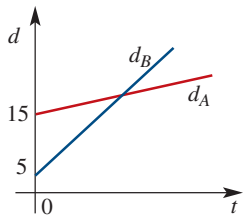
Exercise 2F

1 a



b $t = 5$

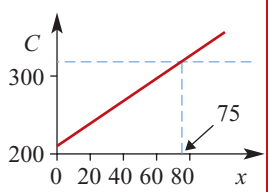
2 a



b 2.00 pm

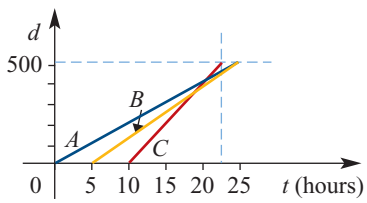
3 a $C_1 = 210 + 1.6x$
$C_2 = 330$

b



c Fixed charge method is cheaper when $x > 75$.

4 a



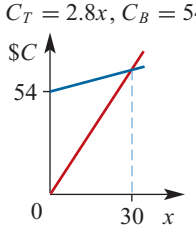
b C wins the race

d C, leaving 5 hours after B, overtakes B $13\frac{1}{2}$ hours after B had started and then overtakes A 20 hours after A had started. C wins the race with a total handicap time of $22\frac{1}{2}$ hours ($12\frac{1}{2}$ hours for journey + 10 hours handicap) with A and B deadheating for 2nd, each with a total handicap time of 25 hours.

5 Both craft will pass over the point $(5\frac{1}{3}, -4)$

6 a $C_T = 2.8x$, $C_B = 54 + x$

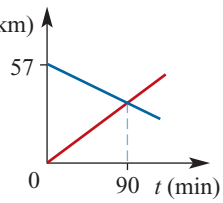
b



c >30 students

7 a $d_A = \frac{1}{3}t$, $d_M = 57 - \frac{3}{10}t$

b



c 10.30 am **d** Maureen 30 km, Anne 27 km

8 $a = 0.28$ and $b = 0.3$, $\frac{10}{3}$ m/s

Exercise 2G

1 a 135° **b** 45° **c** 26.57° **d** 135°

2 a 45° **b** 135° **c** 45° **d** 135°
e 63.43° (to 2 d.p.) **f** 116.57° (to 2 d.p.)

3 a 45° **b** 26°34′ **c** 161°34′
d 49°24′ **e** 161°34′ **f** 135°

4 a 71°34′ **b** 135° **c** 45° **d** 161°34′

5 $m_{BC} = -\frac{3}{5}$, $m_{AB} = \frac{5}{3}$

$\therefore\ m_{BC} \times m_{AB} = -\frac{3}{5} \times \frac{5}{3} = -1$

$\therefore\ \Delta ABC$ is a right-angled triangle

6 $m_{RS} = -\frac{1}{2}$, $m_{ST} = 2$ $\therefore\ RS \perp ST$

$m_{UT} = -\frac{1}{2}$, $m_{ST} = 2$ $\therefore\ UT \perp ST$ (Also need to show $SR = UT$.)

$\therefore\ RSTU$ is a rectangle.

7 $y = 2x + 2$

8 a $2x - 3y = 14$ **b** $2y + 3x = 8$

9 $l = -\frac{16}{3}$, $m = \frac{80}{3}$

Exercise 2H

1 a 7.07 **b** 4.12 **c** 5.83 **d** 13

2 29.27 **3** DN

Exercise 2I

1 a (5, 8) **b** $(\frac{1}{2}, \frac{1}{2})$
c (1.6, 0.7) **d** (–0.7, 0.85)

2 M_{AB} (3, 3). M_{BC} $(8, 3\frac{1}{2})$. M_{AC} $(6, 1\frac{1}{2})$

3 Coordinates of C are (6, 8.8)

4 a $PM = 12.04$ **b** No, it passes through $(0, 3\frac{1}{3})$

5 a (4, 4) **b** (2, –0.2) **c** (–2, 5) **d** (–4, –3)

6 $\left(\frac{1+a}{2}, \frac{4+b}{2}\right)$; $a = 9, b = -6$

Exercise 2J

1 a 34°41′ **b** 45° **c** 90° **d** 49°24′ **e** 26°33′

Multiple-choice questions

1 A **2** E **3** C **4** D **5** B
6 E **7** D **8** C **9** E **10** E

Answers

Answers

Short-answer questions (technology-free)

1 a $\frac{9}{4}$ **b** $-\frac{10}{11}$ **c** undefined **d** -1 **e** $\frac{b}{a}$ **f** $\frac{-b}{a}$

2 a $y = 4x$ **b** $y = 4x + 5$ **c** $y = 4x + 2$
d $y = 4x - 5$

3 a $a = -2$ **b** $\frac{20}{3}$

4 $4y + 3x = -7$ **5** $3y + 2x = -5$

6 $a = -3, b = 5, c = 14$

7 a $y = 11$ **b** $y = 6x - 10$ **c** $3y + 2x = -3$

8 a midpoint $= (3, 2)$, length $= 4$

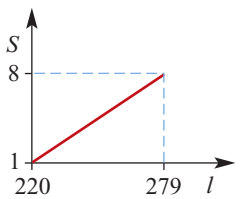
b midpoint $= \left(-\frac{1}{2}, -\frac{9}{2}\right)$, length $= \sqrt{74}$

c midpoint $= \left(5, \frac{5}{2}\right)$, length $= 5$

9 $\sqrt{3}y - x = 3\sqrt{3} - 2$ **10** $y + x = 1$ **11** $37°52'$

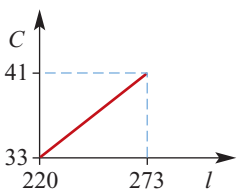
Extended-response questions

1 a



b Since the graph is a line of best fit answers may vary according to the method used; e.g. if the two end points are used then the rule is $S = \frac{7}{59}l - 25.1$ $\left(\text{or } l = \frac{59}{7}S + \frac{1481}{7}\right)$
If a least squares method is used the rule is $l = 8.46S + 211.73$.

c



d Again this is a line of best fit. If the two end points are used then
$C = \frac{8}{53}l - \frac{11}{53}$ $\left(\text{or } l = \frac{53C - 11}{8}\right)$
A least squares method gives
$l = 6.65C + 0.6166$.

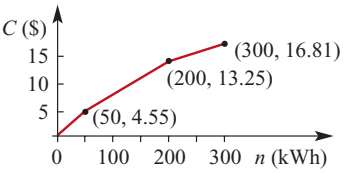
2 a $C = 110 + 38n$ **b** 12 days
c Less than 5 days

3 a Cost of the plug
b Cost per metre of the cable
c 1.8 **d** $11\frac{1}{9}$m

4 a The maximum profit (when $x = 0$)
b 43 seats
c The profit reduces by \$24 for every seat empty.

5 a **i** $C = 0.091n$ **ii** $C = 1.65 + 0.058n$
iii $C = 6.13 + 0.0356n$

b

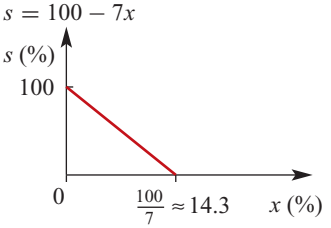


i For 30 kWh, $C = 2.73$
ii For 90 kWh, $C = 6.87$
iii For 300 kWh, $C = 16.81$
c 389.61 kWh

6 a $y = -\frac{7}{3}x + 14\frac{2}{3}$ **b** $20\frac{1}{3}$ km south

7 a $s = 100 - 7x$

b



c $\frac{5}{7}\%$ **d** $14\frac{2}{7}\%$
e Probably not a realistic model at this value of s
f $0 \leq x \leq 14\frac{2}{7}$

8 a AB, $y = x + 2$; CD, $y = 2x - 6$
b Intersection is at (8, 10), i.e. on the near bank.

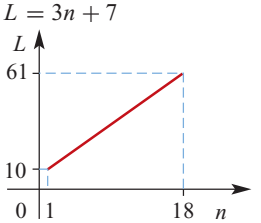
9 a $\frac{128}{19}$ **b** $y = -\frac{199}{190}x + \frac{128}{19}$
c No, since gradient of AB is $\frac{20}{19}(1.053)$, whereas the gradient of VC is –1.047

10 a No **b** $1\frac{41}{71}$ km to the east of H

11 a $y = x - 38$ **b** B (56, 18)
c $y = -2x + 166$ **d** (78, 10)

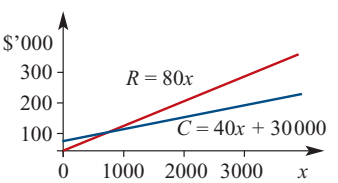
12 a $L = 3n + 7$

b



13 a $C = 40x + 30\,000$
b \$45 **c** 5000 **d** $R = 80x$

e



f 751 **g** $P = 40x - 30\,000$

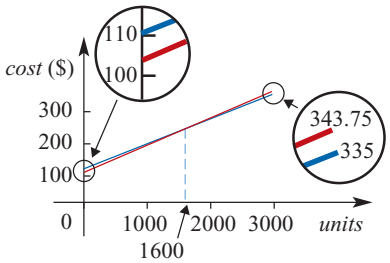
14 a Method 1: Cost = \$226.75; Method 2: Cost = \$227; ∴ Method 1 cheaper

b

	0	1000	2000	3000
Method 1	100	181.25	262.50	343.75
Method 2	110	185	260	335

Cost the same for approx. 1600 units

c



d $C_1 = 0.08125x + 100$ (1)
$C_2 = 0.075x + 110$ (2)
$x = 1600$

15 a (17, 12) **b** $3y = 2x + 2$

16 a *PD:* $y = \frac{2}{3}x + 120$; *DC:* $y = \frac{2}{5}x + 136$;
CB: $y = -\frac{5}{2}x + 600$
AB: $y = \frac{2}{5}x + 20$; *AP:* $y = -\frac{3}{5}x + 120$

b At B and C since product of gradients is -1
e.g. $m_{DC} = \frac{2}{5}$, $m_{CB} = -\frac{5}{2}$;
$\frac{2}{5} \times -\frac{5}{2} = -1$

17 a $y = 3x + 2$ **b** (0, 2) **c** $y = 3x - 8$
d (2, –2) **e** Area = 10 square units
f Area = 40 square units

Chapter 3

Exercise 3A

1 a $2x - 8$ **b** $-2x + 8$ **c** $6x - 12$
d $-12 + 6x$ **e** $x^2 - x$ **f** $2x^2 - 10x$

2 a $6x + 1$ **b** $3x - 6$ **c** $x + 1$ **d** $5x - 3$

3 a $14x - 32$ **b** $2x^2 - 11x$
c $32 - 16x$ **d** $6x - 11$

4 a $2x^2 - 11x$ **b** $3x^2 - 15x$ **c** $-20x - 6x^2$
d $6x - 9x^2 + 6x^3$ **e** $2x^2 - x$ **f** $6x - 6$

5 a $6x^2 - 2x - 28$ **b** $x^2 - 22x + 120$
c $36x^2 - 4$ **d** $8x^2 - 22x + 15$
e $x^2 - (\sqrt{3} + 2)x + 2\sqrt{3}$ **f** $2x^2 + \sqrt{5}x - 5$

6 a $x^2 - 8x + 16$ **b** $4x^2 - 12x + 9$
c $36 - 24x + 4x^2$ **d** $x^2 - x + \frac{1}{4}$
e $x^2 - 2\sqrt{5}x + 5$ **f** $x^2 - 4\sqrt{3}x + 12$

7 a $6x^3 - 5x^2 - 14x + 12$ **b** $x^3 - 1$
c $24 - 20x - 8x^2 + 6x^3$ **d** $x^2 - 9$
e $4x^2 - 16$ **f** $81x^2 - 121$
g $3x^2 + 4x + 3$ **h** $-10x^2 + 5x - 2$
i $x^2 + y^2 - z^2 - 2xy$
j $ax - ay - bx + by$

8 a i $x^2 + 2x + 1$ **ii** $(x + 1)^2$
b i $(x - 1)^2 + 2(x - 1) + 1$ **ii** x^2

Exercise 3B

1 a $2(x + 2)$ **b** $4(a - 2)$ **c** $3(2 - x)$
d $2(x - 5)$ **e** $6(3x + 2)$ **f** $8(3 - 2x)$

2 a $2x(2x - y)$ **b** $8x(a + 4y)$
c $6b(a - 2)$ **d** $2xy(3 + 7x)$
e $x(x + 2)$ **f** $5x(x - 3)$
g $-4x(x + 4)$ **h** $7x(1 + 7x)$
i $x(2 - x)$ **j** $3x(2x - 3)$
k $xy(7x - 6y)$ **l** $2xy^2(4x + 3)$

3 a $(x^2 + 1)(x + 5)$
b $(x - 1)(x + 1)(y - 1)(y + 1)$
c $(a + b)(x + y)$ **d** $(a^2 + 1)(a - 3)$
e $(x - a)(x + a)(x - b)$

4 a $(x - 6)(x + 6)$ **b** $(2x - 9)(2x + 9)$
c $2(x - 7)(x + 7)$ **d** $3a(x - 3)(x - 3)$
e $(x - 6)(x + 2)$ **f** $(7 + x)(3 - x)$
g $3(x - 1)(x + 3)$ **h** $-5(2x + 1)$

5 a $(x - 9)(x + 2)$ **b** $(y - 16)(y - 3)$
c $(3x - 1)(x - 2)$ **d** $(2x + 1)(3x + 2)$
e $(a - 2)(a - 12)$ **f** $(a + 9)^2$
g $(5x + 3)(x + 4)$ **h** $(3y + 6)(y - 6)$
i $2(x - 7)(x - 2)$ **j** $4(x - 3)(x - 6)$
k $3(x + 2)(x + 3)$ **l** $a(x + 3)(x + 4)$
m $x(5x - 6)(x - 2)$ **n** $3x(4 - x)^2$
o $x(x + 2)$

Exercise 3C

1 a 2 or 3 **b** 0 or 2 **c** 4 or 3
d 4 or 3 **e** 3 or –4 **f** 0 or 1
g $\frac{5}{2}$ or 6 **h** –4 or 4

2 a –0.65 or 4.65 **b** –0.58 or 2.58
c –2.58 or 0.58

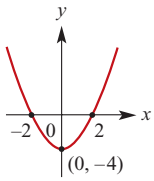
3 a 4, 2 **b** 11, –3 **c** 4, –16
d 2, –7 **e** $-\frac{3}{2}, -1$ **f** $\frac{1}{2}, \frac{3}{2}$
g –3, 8 **h** $-\frac{2}{3}, -\frac{3}{2}$ **i** $-\frac{3}{2}, 2$
j $\frac{5}{6}, 3$ **k** $-\frac{3}{2}, 3$ **l** $\frac{1}{2}, \frac{3}{5}$
m $-\frac{3}{4}, \frac{2}{3}$ **n** $\frac{1}{2}$ **o** –5, 1
p 0, 3 **q** –5, –3 **r** $\frac{1}{5}, 2$

4 4 and 9 **5** 3 **6** $2, 2\frac{3}{8}$
7 13 **8** 50 **9** 6 cm, 2 cm
10 5 **11** \$90, \$60 **12** 42

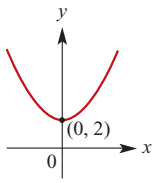
Answers

Exercise 3D

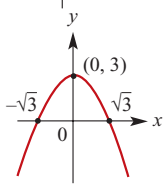
a **i** $(0, -4)$
ii $x = 0$
iii $(-2, 0), (2, 0)$



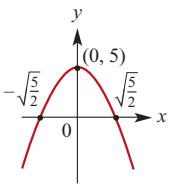
b **i** $(0, 2)$
ii $x = 0$
iii none



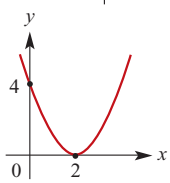
c **i** $(0, 3)$
ii $x = 0$
iii $(-\sqrt{3}, 0), (\sqrt{3}, 0)$



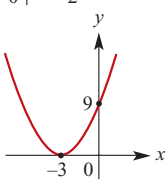
d **i** $(0, 5)$
ii $x = 0$
iii $\left(-\sqrt{\frac{5}{2}}, 0\right), \left(\sqrt{\frac{5}{2}}, 0\right)$



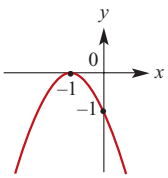
e **i** $(2, 0)$
ii $x = 2$
iii $(2, 0)$



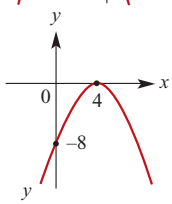
f **i** $(-3, 0)$
ii $x = -3$
iii $(-3, 0)$



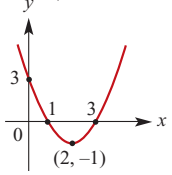
g **i** $(-1, 0)$
ii $x = -1$
iii $(-1, 0)$



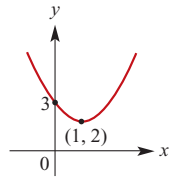
h **i** $(4, 0)$
ii $x = 4$
iii $(4, 0)$



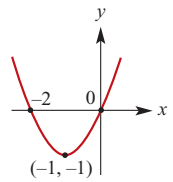
i **i** $(2, -1)$
ii $x = 2$
iii $(1, 0)(3, 0)$



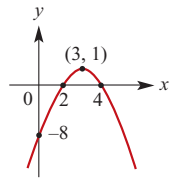
j **i** $(1, 2)$
ii $x = 1$
iii none



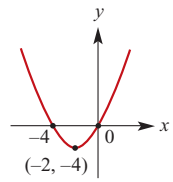
k **i** $(-1, -1)$
ii $x = -1$
iii $(-2, 0)(0, 0)$



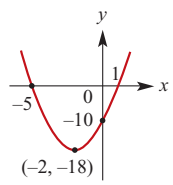
l **i** $(3, 1)$
ii $x = 3$
iii $(2, 0)(4, 0)$



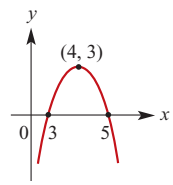
m **i** $(-2, -4)$
ii $x = -2$
iii $(-4, 0), (0, 0)$



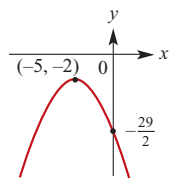
n **i** $(-2, -18)$
ii $x = -2$
iii $(-5, 0), (1, 0)$



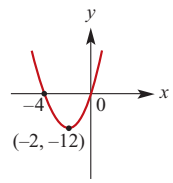
o **i** $(4, 3)$
ii $x = 4$
iii $(3, 0), (5, 0)$



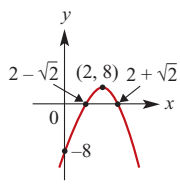
p **i** $(-5, -2)$
ii $x = -5$
iii none



q **i** $(-2, -12)$
ii $x = -2$
iii $(0, 0), (-4, 0)$



r **i** $(2, 8)$
ii $x = 2$
iii $(2 - \sqrt{2}, 0)(2 + \sqrt{2}, 0)$



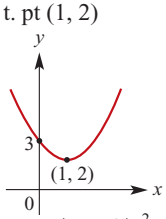
Exercise 3E

1 a $x^2 - 2x + 1$ **b** $x^2 + 4x + 4$ **c** $x^2 - 6x + 9$
d $x^2 - 6x + 9$ **e** $x^2 + 4x + 4$ **f** $x^2 - 10x + 25$
g $x^2 - x + \frac{1}{4}$ **h** $x^2 - 3x + \frac{9}{4}$

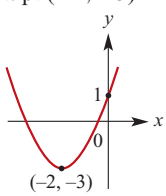
2 a $(x - 2)^2$ **b** $(x - 6)^2$ **c** $-(x - 2)^2$
d $2(x - 2)^2$ **e** $-2(x - 3)^2$ **f** $\left(x - \frac{1}{2}\right)^2$
g $\left(x - \frac{3}{2}\right)^2$ **h** $\left(x + \frac{5}{2}\right)^2$

3 a $1 \pm \sqrt{2}$ **b** $2 \pm \sqrt{6}$ **c** $3 \pm \sqrt{7}$
d $\frac{5 \pm \sqrt{17}}{2}$ **e** $\frac{2 \pm \sqrt{2}}{2}$ **f** $-\frac{1}{3}, 2$
g $-1 \pm \sqrt{1 - k}$ **h** $\frac{-1 \pm \sqrt{1 - k^2}}{k}$
i $\frac{3k \pm \sqrt{9k^2 - 4}}{2}$

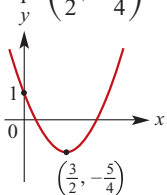
4 a $y = (x - 1)^2 + 2$
t. pt (1, 2)



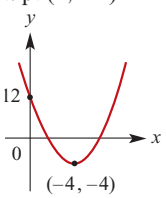
b $y = (x + 2)^2 - 3$
t. pt (−2, −3)



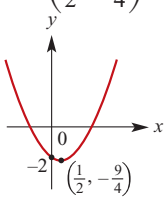
c $y = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$
t. pt $\left(\frac{3}{2}, -\frac{5}{4}\right)$



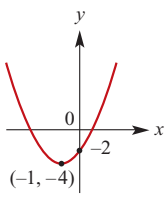
d $y = (x - 4)^2 - 4$
t. pt (4, −4)



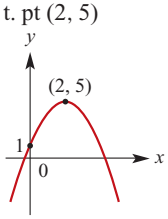
e $y = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$
t. pt $\left(\frac{1}{2}, -\frac{9}{4}\right)$



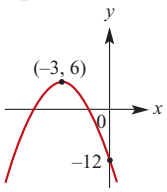
f $y = 2(x + 1)^2 - 4$
t. pt (−1, −4)



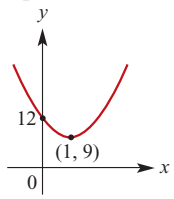
g $y = -(x - 2)^2 + 5$
t. pt (2, 5)



h $y = -2(x + 3)^2 + 6$
t. pt (−3, 6)

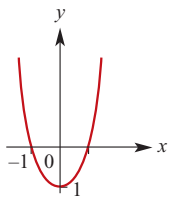


i $y = 3(x - 1)^2 + 9$
t. pt (1, 9)

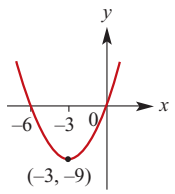


Exercise 3F

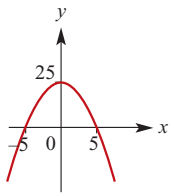
1 a 7 **b** 7 **c** 1
2 a −2 **b** 8 **c** −4
3 a



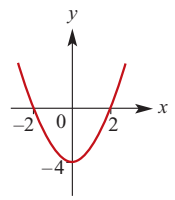
b



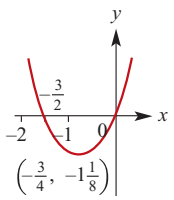
c



d



e



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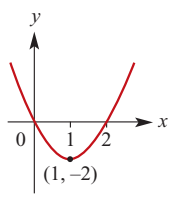
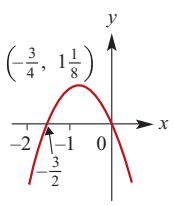
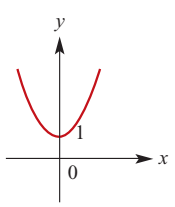


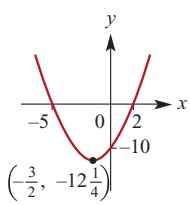
g



h



4 a



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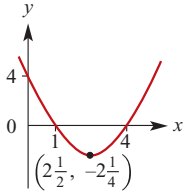

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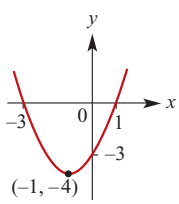
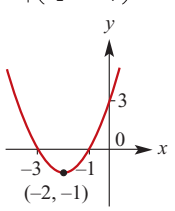
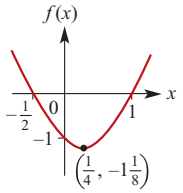


d



e



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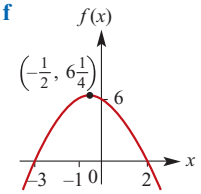

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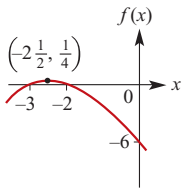

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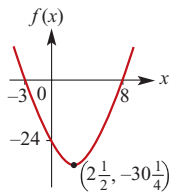


Exercise 3G

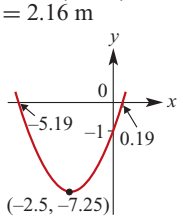
1 a i 40 **ii** $2\sqrt{10}$ **b i** 28 **ii** $2\sqrt{7}$
c i 172 **ii** $2\sqrt{43}$ **d i** 96 **ii** $4\sqrt{6}$
e i 189 **ii** $3\sqrt{21}$

2 a $1+\sqrt{5}$ **b** $\dfrac{3-\sqrt{5}}{2}$
c $\dfrac{1+\sqrt{5}}{2}$ **d** $1+2\sqrt{2}$

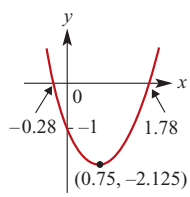
3 a $-3\pm\sqrt{13}$ **b** $\dfrac{7\pm\sqrt{61}}{2}$ **c** $\dfrac{1}{2}, 2$
d $-1\pm\dfrac{3}{2}\sqrt{2}$ **e** $-2\pm\dfrac{3}{2}\sqrt{2}$ **f** $1\pm\dfrac{\sqrt{30}}{5}$
g $1\pm\dfrac{\sqrt{2}}{2}$ **h** $1, \dfrac{-3}{2}$ **i** $\dfrac{-3\pm\sqrt{6}}{5}$
j $\dfrac{-13\pm\sqrt{145}}{12}$ **k** $\dfrac{2\pm\sqrt{4-2k^2}}{2k}$
l $\dfrac{2k\pm\sqrt{6k^2-2k}}{2(1-k)}$

4 $r = 2.16$ m

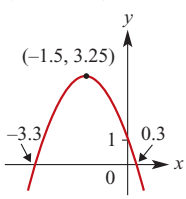
5 a



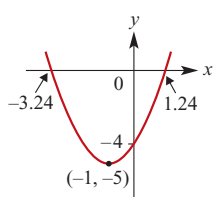
b



c



d



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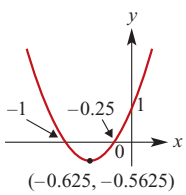

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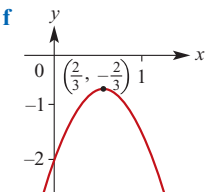


Exercise 3H

a 1.5311 **b** −1.1926 **c** 1.8284 **d** 1.4495

Exercise 3I

1 a 20 **b** −12 **c** 25 **d** 41 **e** 41
2 a Crosses the x-axis **b** Does not cross
c Just touches the x-axis
d Crosses the x-axis **e** Does not cross
f Does not cross
3 a 2 real roots **b** No real roots **c** 2 real roots
d 2 real roots **e** 2 real roots **f** No real roots
4 a $\Delta = 0$, one rational root
b $\Delta = 1$, two rational roots
c $\Delta = 17$, two irrational roots
d $\Delta = 0$, one rational root
e $\Delta = 57$, two irrational roots
f $\Delta = 1$, two rational roots
5 The discriminant $= (m+4)^2 \geq 0$ for all m, therefore rational solution(s).

Exercise 3J

1 a $\{x : x \geq 2\} \cup \{x : x \leq -4\}$
b $\{x : -3 < x < 8\}$ **c** $\{x : -2 \leq x \leq 6\}$
d $\{x : x > 3\} \cup \left\{x : x < -\dfrac{3}{2}\right\}$
e $\left\{x : -\dfrac{3}{2} < x < -\dfrac{2}{3}\right\}$ **f** $\{x : -3 \leq x \leq -2\}$
g $\left\{x : x > \dfrac{2}{3}\right\} \cup \left\{x : x < -\dfrac{3}{4}\right\}$
h $\left\{x : \dfrac{1}{2} \leq x \leq \dfrac{3}{5}\right\}$ **i** $\{x : -4 \leq x \leq 5\}$
j $\left\{p : \dfrac{1}{2}(5-\sqrt{41}) \leq p \leq \dfrac{1}{2}(5+\sqrt{41})\right\}$
k $\{y : y < -1\} \cup \{y : y > 3\}$
l $\{x : x \leq -2\} \cup \{x : x \geq -1\}$

2 a i $-\sqrt{5} < m < \sqrt{5}$ **ii** $m = \pm\sqrt{5}$
iii $m > \sqrt{5}$ or $m < -\sqrt{5}$
b i $0 < m < \dfrac{4}{3}$ **ii** $m = \dfrac{4}{3}$
iii $m > \dfrac{4}{3}$ or $m < 0$
c i $-\dfrac{4}{5} < m < 0$ **ii** $m = 0$ or $m = -\dfrac{4}{5}$
iii $m < -\dfrac{4}{5}$ or $m > 0$
d i $-2 < m < 1$ **ii** $m = -2$ or 1
iii $m > 1$ or $m < -2$

3 $p > \dfrac{4}{3}$ **4** $p = \dfrac{-1}{2}$ **5** $-2 < p < 8$

Exercise 3K

1 a (2, 0), (–5, 7) **b** (1, –3), (4, 9)
c (1, –3), (–3, 1) **d** (–1, 1), (–3, 3)

e $\left(\frac{1+\sqrt{33}}{2}, -3-\sqrt{33}\right)$, $\left(\frac{1-\sqrt{33}}{2}, -3+\sqrt{33}\right)$

f $\left(\frac{5+\sqrt{33}}{2}, 23+3\sqrt{33}\right)$, $\left(\frac{5-\sqrt{33}}{2}, 23-3\sqrt{33}\right)$

2 a Touch at (2, 0) **b** Touch at (3, 9)
c Touch at (−2, −4) **d** Touch at (−4, −8)

3 a $x = 8, y = 16$ and $x = -1, y = 7$

b $x = -\frac{16}{3}, y = 37\frac{1}{3}$ and $x = 2, y = 30$

c $x = \frac{4}{5}, y = 10\frac{2}{5}$ and $x = -3, y = 18$

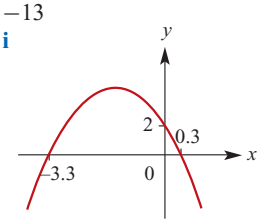
d $x = 10\frac{2}{3}, y = 0$ and $x = 1, y = 29$

e $x = 0, y = -12$ and $x = \frac{3}{2}, y = -7\frac{1}{2}$

f $x = 1.14, y = 14.19$ and $x = -1.68, y = 31.09$

4 a -13

b i



ii $m = -6 \pm \sqrt{32} = -6 \pm 4\sqrt{2}$

5 a $c = \frac{-1}{4}$ **b** $c > -\frac{1}{4}$

6 $a = 3$ or $a = -1$ **7** $b = 1$

8 $y = (2+2\sqrt{3})x - 4 - 2\sqrt{3}$ and $y = (2-2\sqrt{3})x - 4 + 2\sqrt{3}$

Exercise 3L

1 2 **2** $a = -4, c = 8$

3 $a = \frac{4}{7}, b = \frac{-24}{7}$ **4** $a = -2, b = 1, c = 6$

5 a $y = -\frac{5}{16}x^2 + 5$ **b** $y = x^2$

c $y = \frac{1}{11}x^2 + \frac{7}{11}x$ **d** $y = x^2 - 4x + 3$

e $y = -\frac{5}{4}x^2 - \frac{5}{2}x + 3\frac{3}{4}$

f $y = x^2 - 4x + 6$

6 $y = \frac{5}{16}(x+1)^2 + 3$

7 $y = -\frac{1}{2}(x^2 - 3x - 18)$

8 $y = (x+1)^2 + 3$ **9** $y = \frac{1}{180}x^2 - x + 75$

10 a C **b** B **c** D **d** A

11 $y = 2x^2 - 4x$ **12** $y = x^2 - 2x - 1$

13 $y = -2x^2 + 8x - 6$

14 a $y = ax(x-10), a > 0$

b $y = a(x+4)(x-10), a < 0$

c $y = \frac{1}{18}(x-6)^2 + 6$

d $y = a(x-8)^2, a < 0$

15 a $y = -\frac{1}{4}x^2 + x + 2$

b $y = x^2 + x - 5$

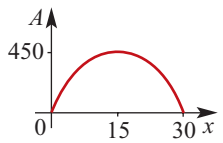
16 $r = -\frac{1}{8}t^2 + 2\frac{1}{2}t - 6\frac{3}{8}$

17 a B **b** D

Exercise 3M

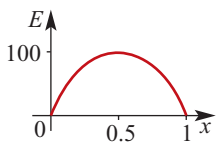
1 a $A = 60x - 2x^2$

b



c Maximum area = 450 m²

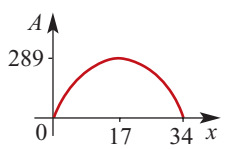
2 a



b 0 and 1 **c** 0.5 **d** 0.23 and 0.77

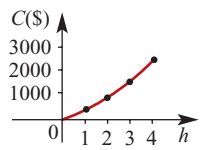
3 a $A = 34x - x^2$

b



c 289 cm²

4 a



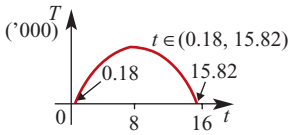
The domain depends on the height of the alpine area. For example in Victoria the highest mountain is approx. 2 km high and the minimum alpine height would be approx. 1 km, thus for Victoria, Domain = [1, 2].

b Theoretically no, but of course there is a practical maximum

c $ 1225

Answers

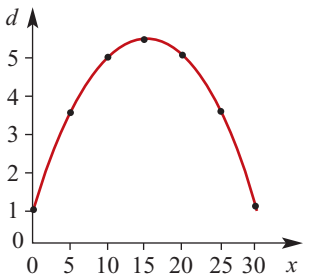
5 a



b 8874 units

6 a

x	0	5	10	15	20	25	30
d	1	3.5	5	5.5	5	3.5	1



b i 5.5 m

ii $15 - 5\sqrt{7}$ m or $15 + 5\sqrt{7}$ m from the bat

iii 1 m above the ground.

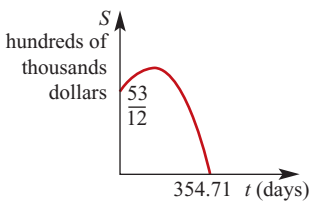
7 a $y = -2x^2 - x + 5$ **b** $y = 2x^2 - x - 5$

c $y = 2x^2 + \frac{5}{2}x - \frac{11}{2}$

8 $a = -\frac{16}{15}$, $b = \frac{8}{5}$, $c = 0$

9 a $a = -\frac{7}{21600}$, $b = \frac{41}{400}$, $c = \frac{53}{12}$

b



c i $S =$ \$1 236 666 **ii** $S =$ \$59 259

Multiple-choice questions

1 A **2** C **3** C **4** E **5** B
6 C **7** E **8** E **9** D **10** A

Short-answer questions (technology-free)

1 a $\left(x + \frac{9}{2}\right)^2$ **b** $(x + 9)^2$ **c** $\left(x - \frac{2}{5}\right)^2$

d $(x + b)^2$ **e** $(3x - 1)^2$ **f** $(5x + 2)^2$

2 a $-3x + 6$ **b** $-ax + a^2$ **c** $49a^2 - b^2$

d $x^2 - x - 12$ **e** $2x^2 - 5x - 12$ **f** $x^2 - y^2$

g $a^3 - b^3$ **h** $6x^2 + 8xy + 2y^2$ **i** $3a^2 - 5a - 2$

j $4xy$ **k** $2u + 2v - uv$ **l** $-3x^2 + 15x - 12$

3 a $4(x - 2)$ **b** $x(3x + 8)$ **c** $3x(8a - 1)$

d $(2 - x)(2 + x)$ **e** $a(u + 2v + 3w)$

f $a^2(2b - 3a)(2b + 3a)$ **g** $(1 - 6ax)(1 + 6ax)$

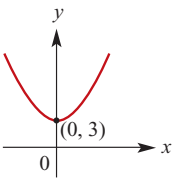
h $(x + 4)(x - 3)$ **i** $(x + 2)(x - 1)$

j $(2x - 1)(x + 2)$ **k** $(3x + 2)(2x + 1)$

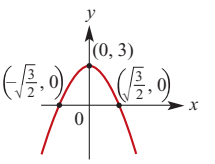
l $(3x + 1)(x - 3)$ **m** $(3x - 2)(x + 1)$

n $(3a - 2)(2a + 1)$ **o** $(3x - 2)(2x - 1)$

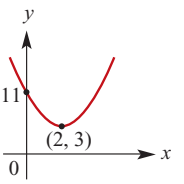
4 a



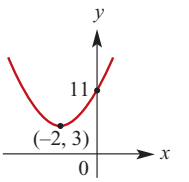
b



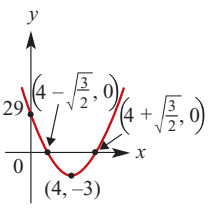
c



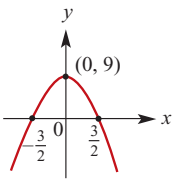
d



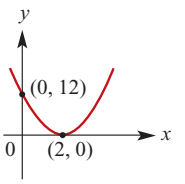
e



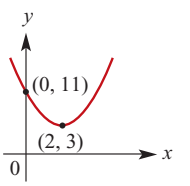
f



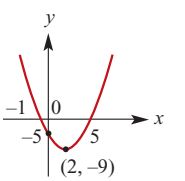
g



h



5 a



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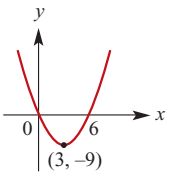

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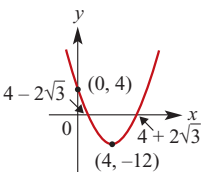

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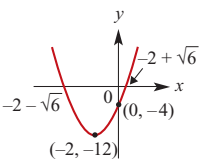

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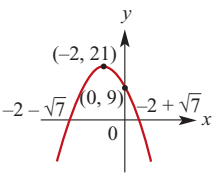
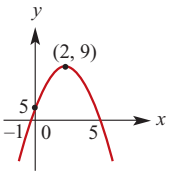
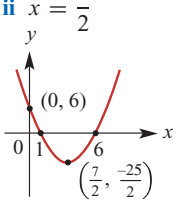


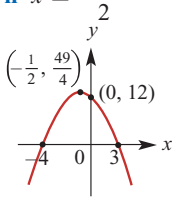
f



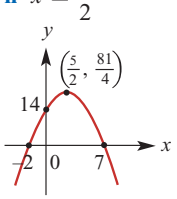
6 a ii $x = \frac{7}{2}$



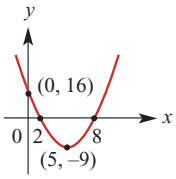
b ii $x = -\frac{1}{2}$



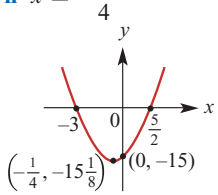
c ii $x = \frac{5}{2}$



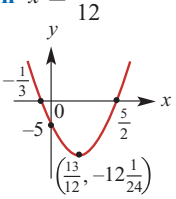
d ii $x = 5$



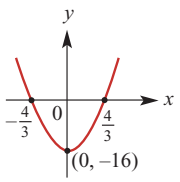
e ii $x = -\frac{1}{4}$



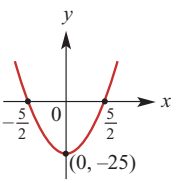
f ii $x = \frac{13}{12}$



g ii $x = 0$



h ii $x = 0$



7 a −0.55, −5.45 **b** −1.63, −7.37
c 3.414, 0.586 **d** −0.314, −3.186
e −0.719, −2.781 **f** 0.107, −3.107

8 $y = \frac{5}{3}x(x - 5)$

9 $y = 3(x - 5)^2 + 2$

10 $y = 5(x - 1)^2 + 5$

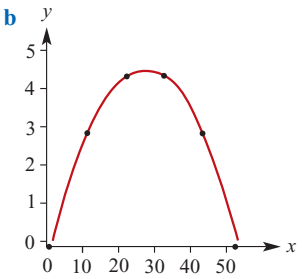
11 a (3, 9), (−1, 1)
b (−1.08, 2.34), (5.08, 51.66)
c (0.26, 2), (−2.6, 2)
d $\left(\frac{1}{2}, \frac{1}{2}\right)$, (−2, 8)

12 a $m = \pm\sqrt{8} = \pm 2\sqrt{2}$
b $m \leq -\sqrt{5}$ or $m \geq \sqrt{5}$
c $b^2 - 4ac = 16 > 0$

Extended-response questions

1 a $y = -0.0072x(x - 50)$

b



c 10.57 m and 39.43 m $\left(25 - \frac{25\sqrt{3}}{3}\text{ m} \quad \text{and} \quad 25 + \frac{25\sqrt{3}}{3}\text{ m}\right)$
d 3.2832 m
e 3.736 m (correct to 3 decimal places)

2 a Width of rectangle $= \frac{12 - 4x}{6}$ m, length of rectangle $= \frac{12 - 4x}{3}$ m

b $A = \frac{17}{9}x^2 - \frac{16}{3}x + 8$

c Length for square $= \frac{96}{17}$ m and length for rectangle $= \frac{108}{17}$ m ($\approx 5.65 \times 6.35$ m)

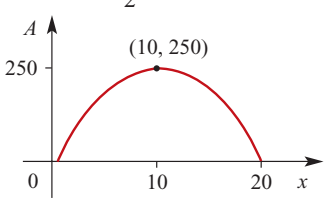
3 a $V = 0.72x^2 - 1.2x$ **b** 22 hours

4 a $V = 10\,800x + 120x^2$
b $V = 46.6x^2 + 5000x$ **c** $l = 55.18$ m

5 a $l = 50 - \frac{5x}{2}$

b $A = 50x - \frac{5}{2}x^2$

c



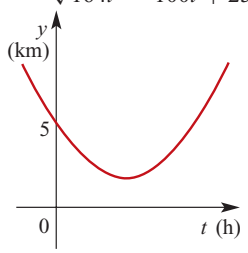
d Maximum area = 250 m^2 when $x = 10$ m

6 $x = \frac{-1 + \sqrt{5}}{2}$

7 a $\sqrt{25 + x^2}$
b i $16 - x$ **ii** $\sqrt{x^2 - 32x + 265}$
c 7.5 **d** 10.840 **e** 12.615

8 a i $y = \sqrt{64t^2 + 100(t - 0.5)^2}$
$= \sqrt{164t^2 - 100t + 25}$

ii



iii $t = \frac{1}{2}$; 1.30 pm $t = \frac{9}{82}$; 1.07 pm

iv 0.305; 1.18 pm; distance 3.123 km

b i 0, $\frac{25}{41}$ **ii** $\frac{25 \pm 2\sqrt{269}}{82}$

9 b $2x + 2y = b$

c $8x^2 - 4bx + b^2 - 16a^2 = 0$

e i $x = 6 \pm \sqrt{14}$, $y = 6 \mp \sqrt{14}$

ii $x = y = \sqrt{2}a$

f $x = \frac{(5 \pm \sqrt{7})a}{4}$, $y = \frac{(5 \mp \sqrt{7})a}{4}$

10 a $b = -2, c = 4, h = 1$

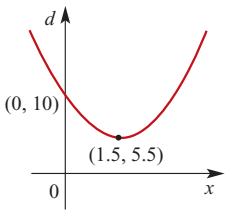
b i $(x, -6 + 4x - x^2)$ **ii** $(x, x - 1)$

iii (0, –1) (1, 0) (2, 1) (3, 2) (4, 3)

iv $y = x - 1$

c i $d = 2x^2 - 6x + 10$

ii



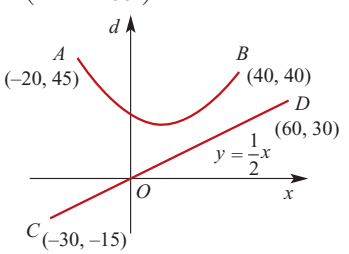
iii min value of $d = 5.5$ occurs when $x = 1.5$

11 a $45\sqrt{5}$

b i $y = \frac{1}{600}(7x^2 - 190x + 20\,400)$

ii $\left(\frac{190}{14}, \frac{5351}{168}\right)$

c



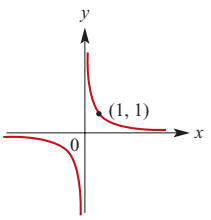
d i The distance (measured parallel to the *y*-axis) between path and pond.

ii minimum value $= \frac{473}{24}$ when $x = 35$

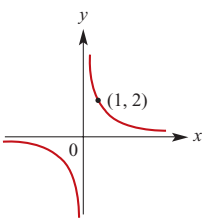
Chapter 4

Exercise 4A

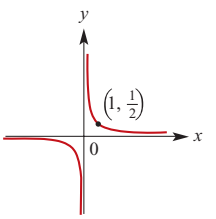
1 a



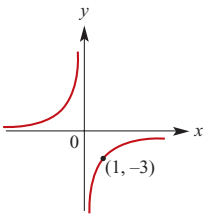
b



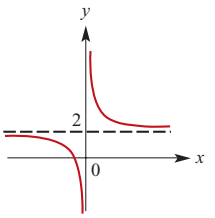
c



d



e



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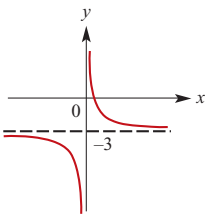
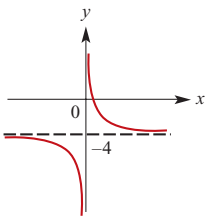


g



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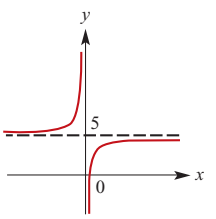
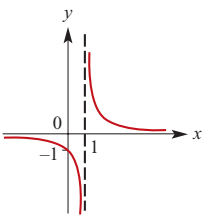


i



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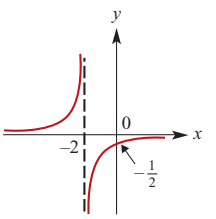

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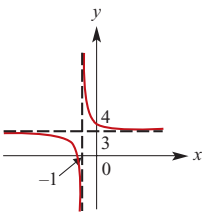
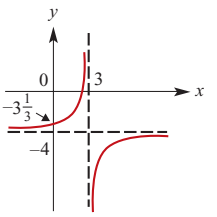


l



2 a $y = 0, x = 0$ **b** $y = 0, x = 0$

c $y = 0, x = 0$ **d** $y = 0, x = 0$

e $y = 2, x = 0$ **f** $y = -3, x = 0$

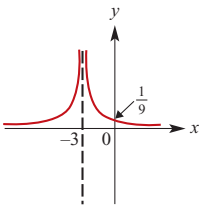
g $y = -4, x = 0$ **h** $y = 5, x = 0$

i $y = 0, x = 1$ **j** $y = 0, x = -2$

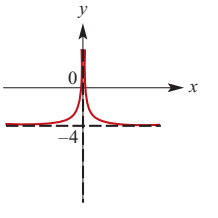
k $y = 3, x = -1$ **l** $y = -4, x = 3$

Exercise 4B

1 a



b



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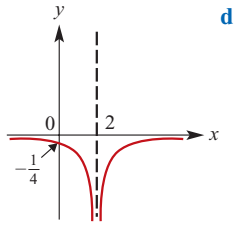
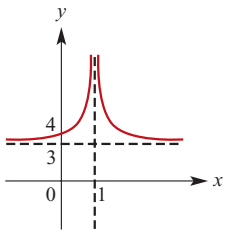
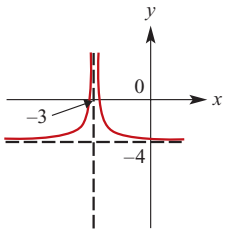


d



e



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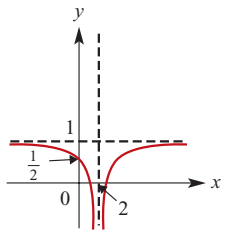

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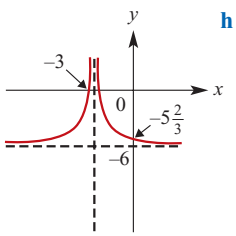

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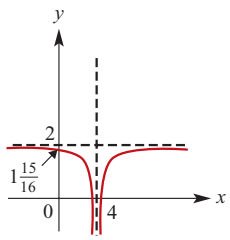
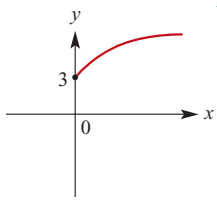


2 a $y = 0, x = -3$ **b** $y = -4, x = 0$
c $y = 0, x = 2$ **d** $y = 3, x = 1$
e $y = -4, x = -3$ **f** $y = 1, x = 2$
g $y = -6, x = -3$ **h** $y = 2, x = 4$

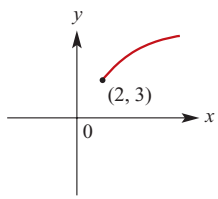
Exercise 4C

a



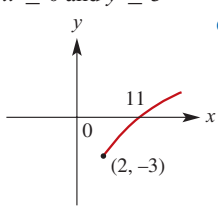
$x \geq 0$ and $y \geq 3$

b



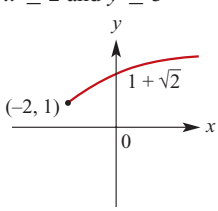
$x \geq 2$ and $y \geq 3$

c



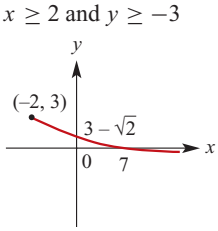
$x \geq 2$ and $y \geq -3$

d



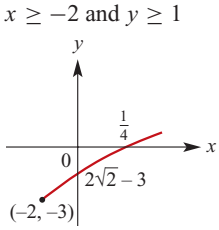
$x \geq -2$ and $y \geq 1$

e



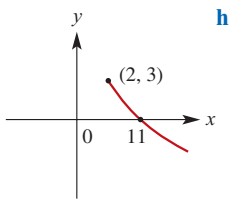
$x \geq -2$ and $y \leq 3$

f



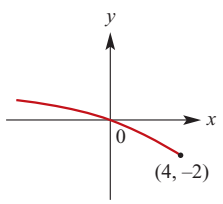
$x \geq -2$ and $y \geq -3$

g



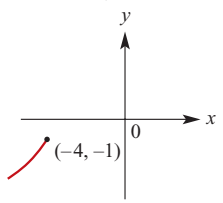
$x \geq 2$ and $y \leq 3$

h



$x \leq 4$ and $y \geq -2$

i



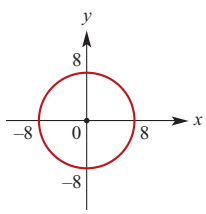
$x \leq -4$ and $y \leq -1$

Exercise 4D

1 a $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 16$
c $(x-1)^2 + (y-3)^2 = 25$
d $(x-2)^2 + (y+4)^2 = 9$
e $(x+3)^2 + (y-4)^2 = \frac{25}{4}$
f $(x+5)^2 + (y+6)^2 = (4.6)^2$

2 a $C(1, 3), r = 2$ **b** $C(2, -4), r = \sqrt{5}$
c $C(-3, 2), r = 3$ **d** $C(0, 3), r = 5$
e $C(-3, -2), r = 6$ **f** $C(3, -2), r = 2$
g $C(-2, 3), r = 5$ **h** $C(4, -2), r = \sqrt{19}$

3 a



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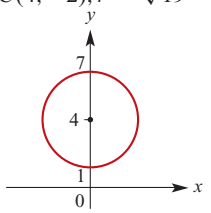

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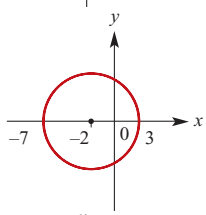

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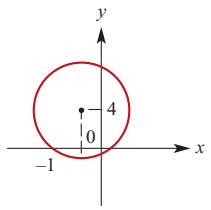

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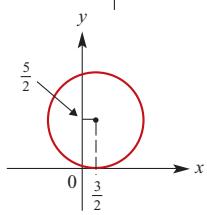

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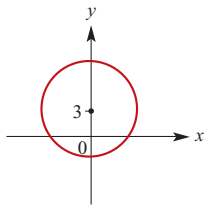

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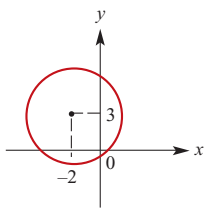

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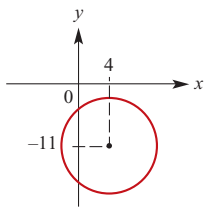

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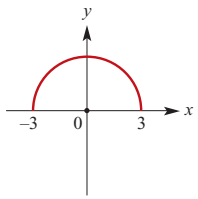

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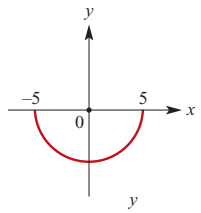

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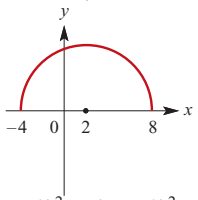

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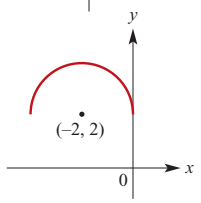


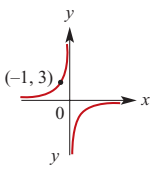
4 $(x-2)^2+(y+3)^2=9$
5 $(x-2)^2+(y-1)^2=20$
6 $(x-4)^2+(y-4)^2=20$
7 Centre $(-2, 3)$, radius $= 6$
8 $2\sqrt{21}$ (x-axis), $4\sqrt{6}$ (y-axis)

Multiple-choice questions

1 E **2** B **3** E **4** A **5** A
6 D **7** D **8** C **9** E **10** B

Short-answer questions (technology-free)

1 a



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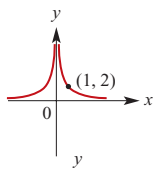

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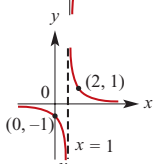

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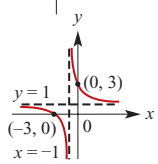

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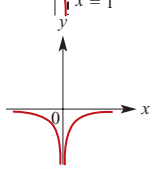

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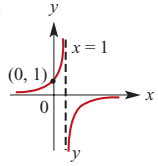

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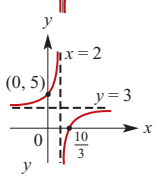

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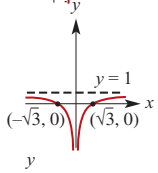

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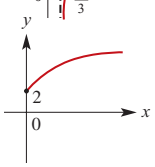

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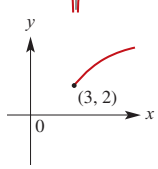

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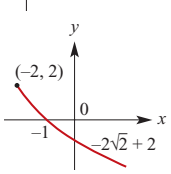


2 a $(x-3)^2+(y+2)^2=25$

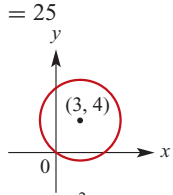
b $\left(x-\frac{3}{2}\right)^2+\left(y+\frac{5}{2}\right)^2=\frac{50}{4}$

c $\left(x-\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2=\frac{17}{8}$

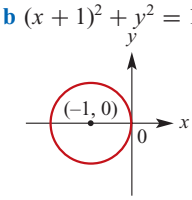
d $(x+2)^2+(y-3)^2=13$
e $(x-3)^2+(y-3)^2=18$
f $(x-2)^2+(y+3)^2=13$

3 $2y+3x=0$ **4** $2x+2y=1$ or $y=x-\frac{5}{2}$

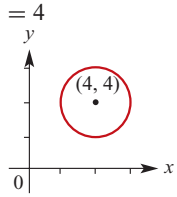
5 a $(x-3)^2+(y-4)^2=25$



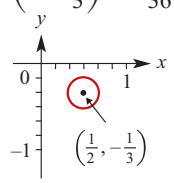
b $(x+1)^2+y^2=1$



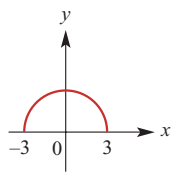
c $(x-4)^2+(y-4)^2=4$



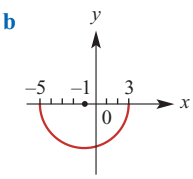
d $\left(x-\frac{1}{2}\right)^2+\left(y+\frac{1}{3}\right)^2=\frac{1}{36}$



6 a



b



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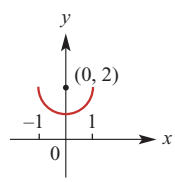

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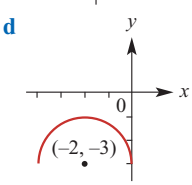


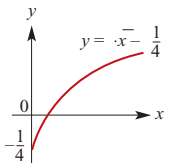
Extended-response questions

1 a $(x-10)^2+y^2=25$ **c** $m=\pm\frac{\sqrt{3}}{3}$
d $P\left(\frac{15}{2}, \frac{\pm5\sqrt{3}}{2}\right)$ **e** $5\sqrt{3}$
2 a $x^2+y^2=16$
b ii $m=\pm\frac{\sqrt{3}}{3}$; $y=\frac{\sqrt{3}}{3}x-\frac{8\sqrt{3}}{3}$, $y=-\frac{\sqrt{3}}{3}x+\frac{8\sqrt{3}}{3}$
3 a $\frac{4}{3}$ **b** $\frac{-3}{4}$ **c** $4y+3x=25$ **d** $\frac{125}{12}$
4 a i $\frac{y_1}{x_1}$ **ii** $\frac{-x_1}{y_1}$
c $\sqrt{2}x+\sqrt{2}y=8$ or $\sqrt{2}x+\sqrt{2}y=-8$

5 a $y = \dfrac{-\sqrt{3}}{3}x + \dfrac{2\sqrt{3}}{3}a,\ y = \dfrac{\sqrt{3}}{3}x - \dfrac{2\sqrt{3}}{3}a$

b $x^2 + y^2 = 4a^2$

6 bii



$\left(\frac{1}{4}, \frac{1}{4}\right)$

c i $\dfrac{-1}{4} < k < 0$ **ii** $k = 0$ or $k < \dfrac{-1}{4}$

iii $k > 0$

7 a $0 < k < \dfrac{1}{4}$ **b** $k = \dfrac{1}{4}$ or $k \leq 0$

Chapter 5

Exercise 5A

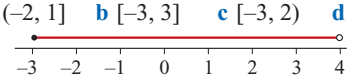
1 a {7, 11} **b** {7, 11}

c {1, 2, 3, 5, 7, 11, 15, 25, 30, 32}

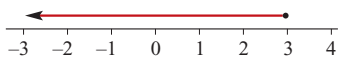
d {1, 2, 3, 5, 15} **e** {1} **f** {1, 7, 11}

2 a (–2, 1] **b** [–3, 3] **c** [–3, 2) **d** (–1, 2)

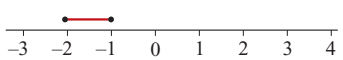
3 a



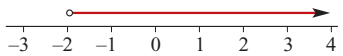
b



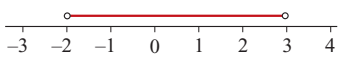
c



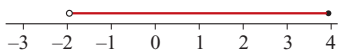
d



e



f



4 a [–1, 2] **b** (–4, 2] **c** $(0, \sqrt{2})$

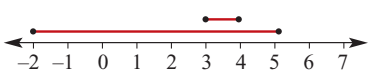
d $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{\sqrt{2}}\right]$ **e** (–1, ∞) **f** (–∞, –2]

g (–∞, ∞) **h** [0, ∞) **i** (–∞, 0]

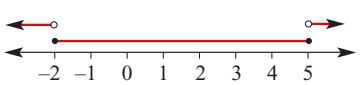
5 a {1, 7} **b** {7}

c B, i.e. {7, 11, 25, 30, 32} **d** (2, ∞)

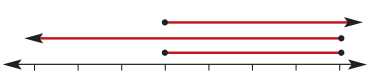
6 a



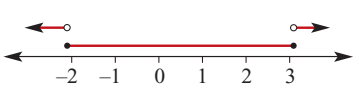
b



c



d



Exercise 5B

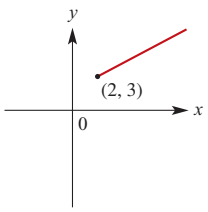
1 a Domain = [–2, 2], Range = [–1, 2]

b Domain = [–2, 2], Range = [–2, 2]

c Domain = R, Range = [–1, ∞)

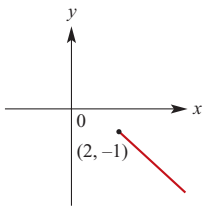
d Domain = R, Range = (–∞, 4]

2 a



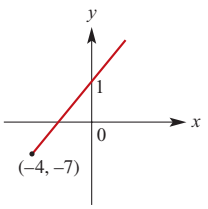
Range = [3, ∞)

b



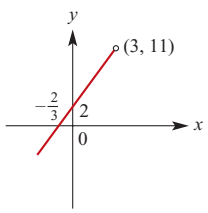
Range = (–∞, –1]

c



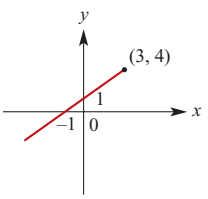
Range = [–7, ∞)

d



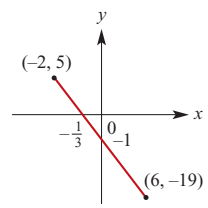
Range = (–∞, 11)

e



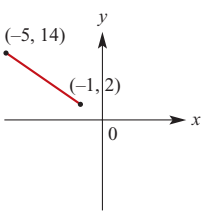
Range = (–∞, 4]

f



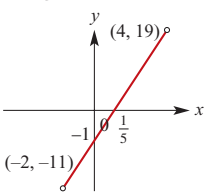
Range = [–19, 5]

g



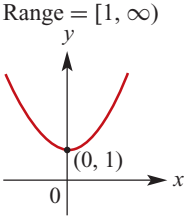
Range = [2, 14]

h

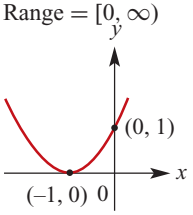


Range = (–11, 19)

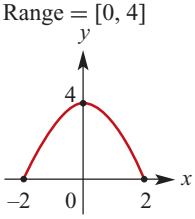
3 a Range = [1, ∞)



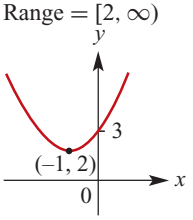
b Range = [0, ∞)



c Range = [0, 4]

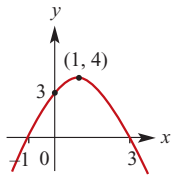


d Range = [2, ∞)

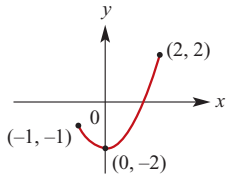


Answers

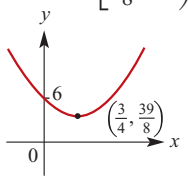
e Range $= (-\infty, 4]$



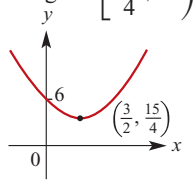
f Range $= [-2, 2]$



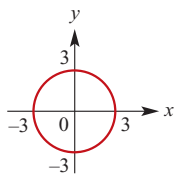
g Range $= \left[\frac{39}{8}, \infty\right)$



h Range $= \left[\frac{15}{4}, \infty\right)$

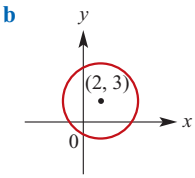


4 a



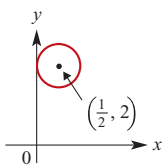
Domain $-3 \le x \le 3$
Range $-3 \le y \le 3$

b



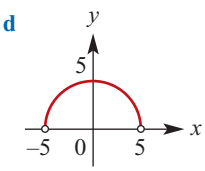
Domain $-2 \le x \le 6$
Range $-1 \le y \le 7$

c



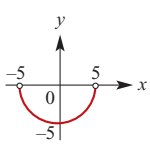
Domain $0 \le x \le 1$
Range $1\frac{1}{2} \le y \le 2\frac{1}{2}$

d



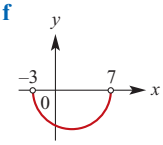
Domain $-5 \le x \le 5$
Range $0 \le y \le 5$

e



Domain $-5 \le x \le 5$
Range $-5 \le y \le 0$

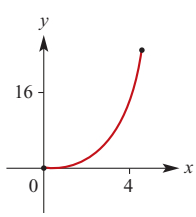
f



Domain $-3 \le x \le 7$
Range $-5 \le y \le 0$

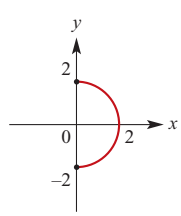
Exercise 5C

1 a



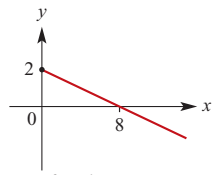
a function
Domain = [0, 4]
Range = [0, 16]

b



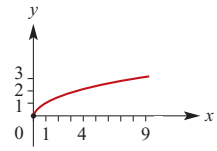
not a function
Domain = [0, 2]
Range = [–2, 2]

c



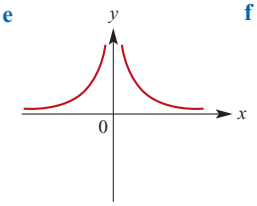
a function
Domain = [0, ∞)
Range = (–∞, 2]

d



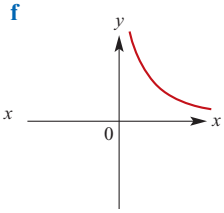
a function
Domain = $\{x : x \ge 0\}$
Range = $\{y : y \ge 0\}$

e



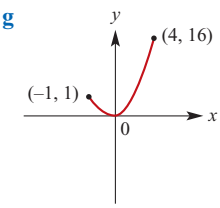
a function
Domain = R\{0}
Range = R+

f



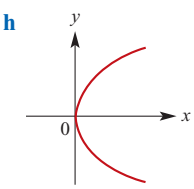
a function
Domain = $R+$
Range = $R+$

g



a function
Domain = [–1, 4]
Range = [0, 16]

h



not a function
Domain = [0, ∞)
Range = R

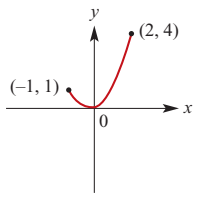
2 a Not a function, Domain $= \{0, 1, 2, 3\}$; Range $= \{1, 2, 3, 4\}$
b A function, Domain $= \{-2, -1, 0, 1, 2\}$; Range $= \{-5, -2, -1, 2, 4\}$
c Not a function, Domain $= \{-1, 0, 3, 5\}$; Range $= \{1, 2, 4, 6\}$
d A function, Domain $= \{1, 2, 4, 5, 6\}$; Range $= \{3\}$
e A function, Domain $= R$; Range $= \{-2\}$
f Not a function, Domain $= \{3\}$; Range $= Z$
g A function, Domain $= R$; Range $= R$
h A function, Domain $= R$; Range $= [5, \infty)$
i Not a function, Domain $= [-3, 3]$; Range $= [-3, 3]$

3 a **i** -3 **ii** 5 **iii** -5 **iv** 9
b **i** 4 **ii** -4 **iii** $\frac{4}{3}$ **iv** 2
c **i** 4 **ii** 36 **iii** 36 **iv** $(a-2)^2$
d **i** 0 **ii** $\frac{a}{1+a}$ **iii** $\frac{-a}{1-a}$ **iv** $1-a$

4 a 1 **b** $\frac{1}{6}$ **c** ± 3 **d** $-1, 4$ **e** $-1, 3$ **f** $-2, 3$

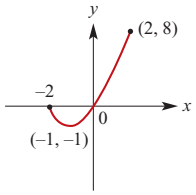
5 a $g(-1) = -1$, $g(2) = 8$, $g(-2) = 0$
b $h(-1) = 3$, $h(2) = 18$, $h(-2) = -14$
c **i** $g(-3x) = 9x^2 - 6x$
ii $g(x-5) = x^2 - 8x + 15$
iii $h(-2x) = -16x^3 - 4x^2 + 6$
iv $g(x+2) = x^2 + 6x + 8$
v $h(x^2) = 2x^6 - x^4 + 6$

6 a $f(2) = 5$; $f(-4) = 29$ **b** Range $= [-3, \infty)$
7 a $f(2) = 7$ **b** $x = 2$ **c** $x = -1$
8 a 2 **b** ± 1 **c** $x = \pm\sqrt{3}$
9 a $x = -1$ **b** $x > -1$ **c** $x = -\frac{6}{7}$
10 a $f: R \to R,\ f(x) = 3x + 2$
b $f: R \to R,\ f(x) = -\frac{3}{2}x + 6$
c $f: [0, \infty) \to R,\ f(x) = 2x + 3$
d $f: [-1, 2] \to R,\ f(x) = 5x + 6$
e $f: [-5, 5] \to R,\ f(x) = -x^2 + 25$
f $f: [0, 1] \to R,\ f(x) = 5x - 7$
11 a



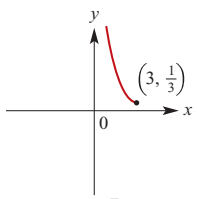
Range $= [0, 4]$

b



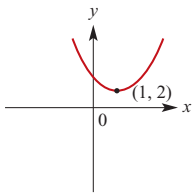
Range $= [-1, 8]$

c



Range $= \left[\frac{1}{3}, \infty\right)$

d

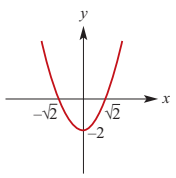


Range $= [2, \infty)$

Exercise 5D

1 One-to-one functions are **b**, **d**, **e** and **g**
2 Functions are **a**, **c**, **d**, **f** and **g**. One-to-one functions are **c** and **g**.
3 a Domain $= R$, Range $= R$
b Domain $= R^+ \cup \{0\}$. Range $= R^+ \cup \{0\}$
c Domain $= R$, Range $= [1, \infty)$
d Domain $= [-3, 3]$, Range $= [-3, 0]$
e Domain $= R^+$, Range $= R^+$
f Domain $= R$, Range $= (-\infty, 3]$
g Domain $= [2, \infty)$, Range $= R^+ \cup \{0\}$
h Domain $= \left[\frac{1}{2}, \infty\right)$, Range $= [0, \infty)$
i Domain $= \left(-\infty, \frac{3}{2}\right]$, Range $= [0, \infty)$
j Domain $= R \setminus \left\{\frac{1}{2}\right\}$, Range $= R \setminus \{0\}$
k Domain $= R \setminus \left\{\frac{1}{2}\right\}$, Range $= (-3, \infty)$
l Domain $= R \setminus \left\{\frac{1}{2}\right\}$, Range $= R \setminus \{2\}$
4 a Domain $= R$, Range $= R$
b Domain $= R$, Range $= [2, \infty)$
c Domain $= [-4, 4]$, Range $= [-4, 0]$
d Domain $= R \setminus \{-2\}$, Range $= R \setminus \{0\}$
5 $y = \sqrt{2 - x}$, Domain $= (-\infty, 2]$, Range $= R^+ \cup \{0\}$
$y = -\sqrt{2 - x}$, Domain $= (-\infty, 2]$, Range $= (-\infty, 0]$

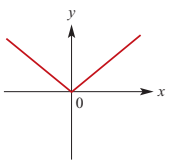
6 a



b $f_1: [0, \infty) \to R,\ f_1(x) = x^2 - 2$,
$f_2: (-\infty, 0] \to R,\ f_2(x) = x^2 - 2$

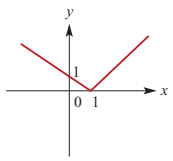
Exercise 5E

1 a



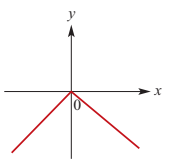
Range $= [0, \infty)$

b



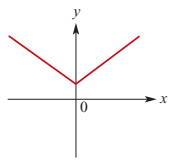
Range $= [0, \infty)$

c



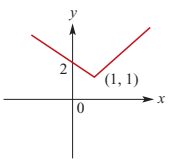
Range $= (-\infty, 0]$

d



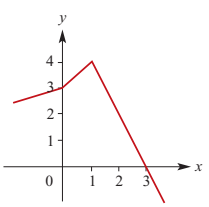
Range $= [1, \infty)$

e



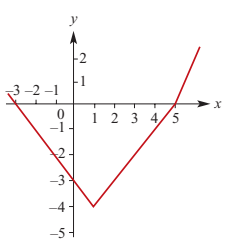
Range $= [1, \infty)$

2 a

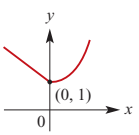


b Range $= (-\infty, 4]$

3

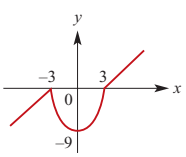


4 a



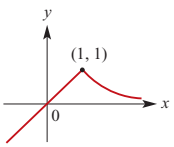
b Range $= [1, \infty)$

5 a



b Range $= R$

6 a



b Range $= (-\infty, 1]$

7 $f(x) = \begin{cases} x + 3, & -3 \le x \le -1 \\ -x + 1, & -1 < x \le 2 \\ -\frac{1}{2}x, & 2 \le x \le 4 \end{cases}$

Exercise 5F

1 a $a = -3, b = \frac{1}{2}$ **b** 6

2 $f(x) = 7 - 5x$

3 a i $f(0) = -\frac{9}{2}$ **ii** $f(1) = -3$

b 3

4 a $f(p) = 2p + 5$ **b** $f(p + h) = 2p + 2h + 5$

c $2h$ **d** 2

5 –2

6 b i 25.06 **ii** 25.032 **iii** 25.2 **iv** 26

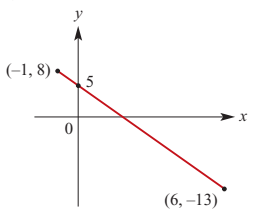
7 $f(x) = -7(x - 2)(x - 4)$

8 $f(x) = (x - 3)^2 + 7$, Range $= [7, \infty)$

9 a $\left(-\infty, -\frac{15}{8}\right]$ **b** $\left[3\frac{7}{8}, \infty\right)$

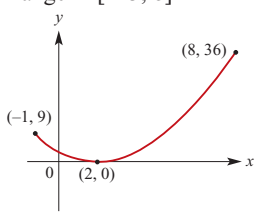
c $(-\infty, 20]$ **d** $(-\infty, 3]$

10 a



b Range $= [-13, 8]$

11 a



b Range $= [0, 36]$

12 a Domain $-3 \le x \le 3$ **b** Domain $1 \le x \le 3$
Range $-3 \le y \le 3$ Range $-1 \le y \le 1$
c Domain $0 \le x \le 1$ **d** Domain $-1 \le x \le 9$
Range $0 \le y \le 1$ Range $-5 \le y \le 5$
e Domain $-4 \le x \le 4$
Range $-2 \le y \le 6$

13 a $\{2, 4, 6, 8\}$ **b** $\{4, 3, 2, 1\}$
c $\{-3, 0, 5, 12\}$ **d** $\{1, \sqrt{2}, \sqrt{3}, 2\}$

14 $f(x) = \frac{1}{10}(x - 4)(x - 5)$; $a = \frac{1}{10}, b = -\frac{9}{10}$, $c = 2$

15 $f(x) = -2(x - 1)(x + 5)$

$g(x) = -50(x - 1)\left(x + \frac{1}{5}\right)$

16 a $k < \frac{-37}{12}$ **b** $k = -\frac{25}{12}$

Exercise 5G

1 a $\{(3, 1)(6, -2)(5, 4)(1, 7)\}$; domain $=$ $\{3, 6, 5, 1\}$; range $= \{1, -2, 4, 7\}$.

b $f^{-1}(x) = \frac{6 - x}{2}$; domain $= R$, range $= R$

c $f^{-1}(x) = 3 - x$ domain $= [-2, 2]$, range $= [1, 5]$

d $f^{-1}(x) = x - 4$ domain $= [4, \infty)$, range $= R^+$

e $f^{-1}(x) = x - 4$ domain $= (-\infty, 8]$, range $= (-\infty, 4]$

f $f^{-1}(x) = \sqrt{x}$; domain $= R^+ \cup \{0\}$, range $= R^+ \cup \{0\}$

g $f^{-1}(x) = 2 + \sqrt{x - 3}$; domain $= [3, \infty)$, range $= [2, \infty)$

h $f^{-1}(x) = 4 - \sqrt{x - 6}$; domain $= [6, \infty)$, range $= (-\infty, 4]$

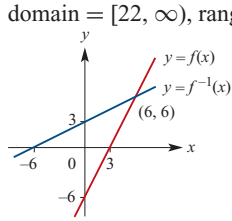
i $f^{-1}(x) = 1 - x^2$; domain $= [0, 1]$, range $= [0, 1]$

j $f^{-1}(x) = \sqrt{16 - x^2}$; domain $= [0, 4]$, range $= [0, 4]$

k $f^{-1}(x) = \frac{16 - x}{2}$; domain $= [2, 18]$, range $= [-1, 7]$

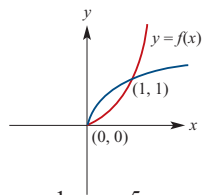
l $f^{-1}(x) = -4 + \sqrt{x - 6}$; domain $= [22, \infty)$, range $= [0, \infty)$

2 a



b (6, 6)

3 a



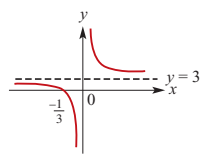
b (0, 0) and (1, 1)

4 $a = \frac{-1}{2}, b = \frac{5}{2}$

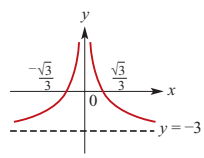
5 a $f^{-1}(x) = a - x^2$ **b** $a = 1$ or $a = 2$

Exercise 5H

1 a



b



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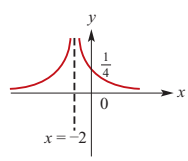

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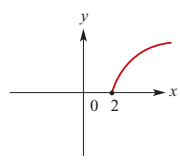

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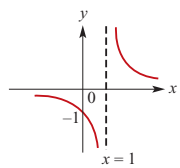

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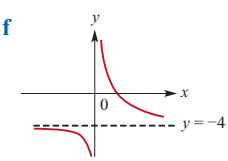

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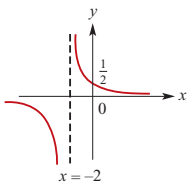

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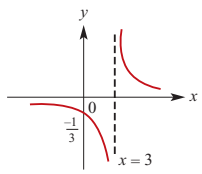

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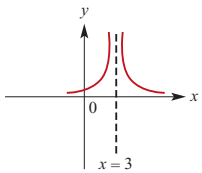

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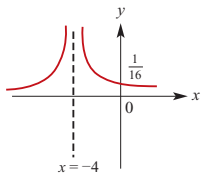

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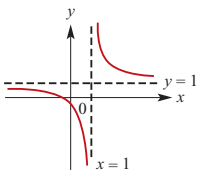

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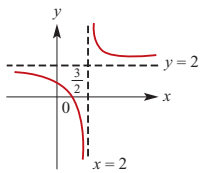
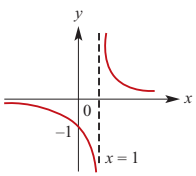
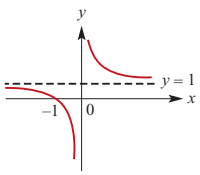


2 a



b



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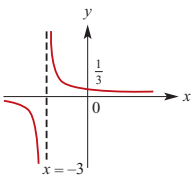

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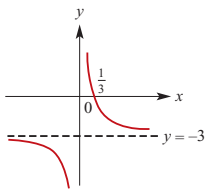

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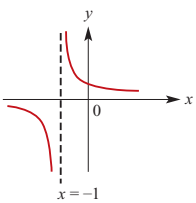

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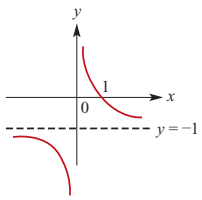
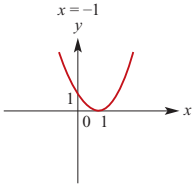


3 a



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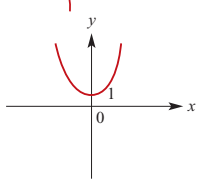

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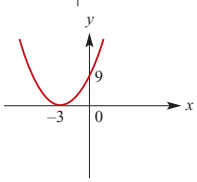

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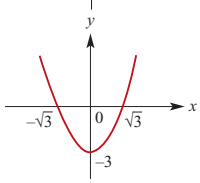

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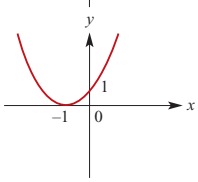

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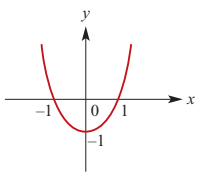
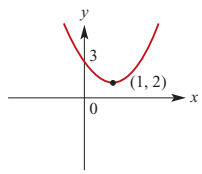
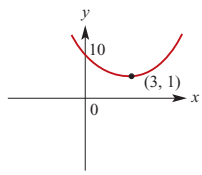


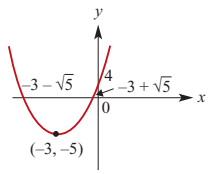
4 a



b



c



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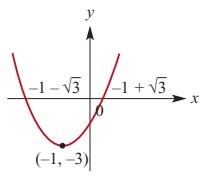

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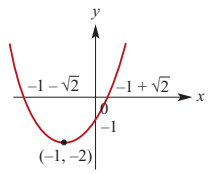

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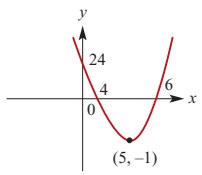
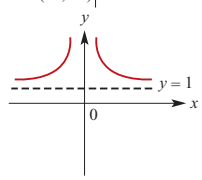
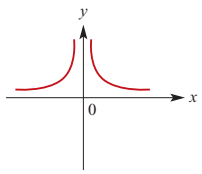


5 a



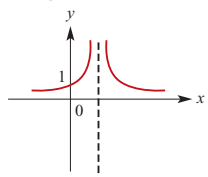
Range $= (1, \infty)$

b



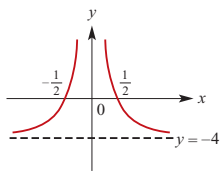
Range $= (0, \infty)$

c



Range $= (0, \infty)$

d



Range $= (-4, \infty)$

Exercise 5I

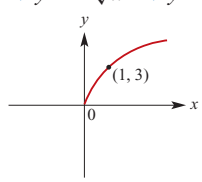
1 a i $y = 4x^2$ ii $y = \frac{x^2}{25}$ iii $y = \frac{2x^2}{3}$
iv $y = 4x^2$ v $y = -x^2$ vi $y = x^2$

b i $y = \frac{1}{4x^2}$ ii $y = \frac{25}{x^2}$ iii $y = \frac{2}{3x^2}$
iv $y = \frac{4}{x^2}$ v $y = \frac{-1}{x^2}$ vi $y = \frac{1}{x^2}$

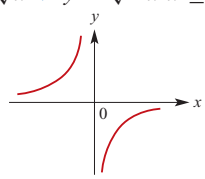
c i $y = \frac{1}{2x}$ ii $y = \frac{5}{x}$ iii $y = \frac{2}{3x}$
iv $y = \frac{4}{x}$ v $y = \frac{-1}{x}$ vi $y = \frac{-1}{x}$

d i $y = \sqrt{2x}$ ii $y = \sqrt{\frac{x}{5}}$ iii $y = \frac{2\sqrt{x}}{3}$
iv $y = 4\sqrt{x}$ v $y = -\sqrt{x}$ vi $y = \sqrt{-x}$ $x \leq 0$

2 a

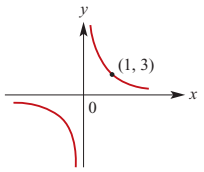


b



Answers

c



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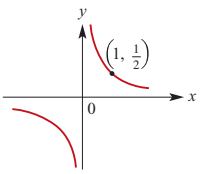
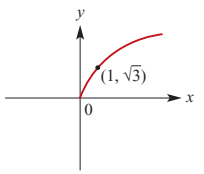


e



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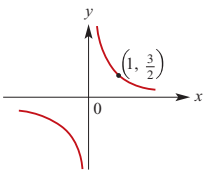


Exercise 5J

1 a $y = 3\sqrt{x-2}$ **b** $y = -\sqrt{x+3}$

c $y = -3\sqrt{x}$ **d** $y = -\sqrt{\frac{x}{2}}$

e $y = 2\sqrt{x-2} - 3$ **f** $y = \sqrt{\frac{x+2}{2}} - 3$

2 a $y = \frac{3}{x-2}$ **b** $y = \frac{-1}{x+3}$

c $y = -\frac{3}{x}$ **d** $y = -\frac{2}{x}$

e $y = \frac{2}{x-2} - 3$ **f** $y = \frac{2}{x+2} - 3$

3 a i A dilation of factor 2 from the x-axis followed by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

ii A reflection in the x-axis followed by a translation of 1 unit in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

iii A dilation of factor $\frac{1}{2}$ from the y-axis followed by a translation of $\frac{1}{2}$ unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

b i A dilation of factor 2 from the x-axis followed by a translation of 3 units in the negative direction of the x-axis

ii A translation of 3 units in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

iii A translation of 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

c i A translation of 3 units in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

ii A dilation of factor $\frac{1}{3}$ from the y-axis followed by a dilation of factor 2 from the x-axis

iii A reflection in the x-axis followed by a translation of 2 units in the positive direction of the y-axis

Exercise 5K

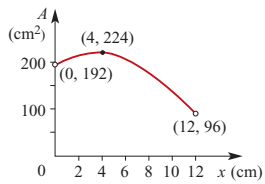
1 a i $A = (8 + x)y - x^2$

ii $P = 2x + 2y + 16$

b i $A = 192 + 16x - 2x^2$

ii $0 < x < 12$

iii



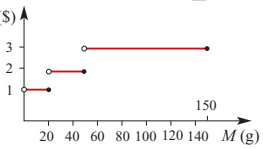
iv 224 cm²

2 a $C = 1.20$ for $0 < m \leq 20$

$= 2.00$ for $20 < m \leq 50$

$= 3.00$ for $50 < m \leq 150$

b



Domain = (0, 150]
Range = {1.20, 2.00, 3.00}

3 a $C = 0.30$ for $0 \leq d < 25$

$= 0.40$ for $25 \leq d < 50$

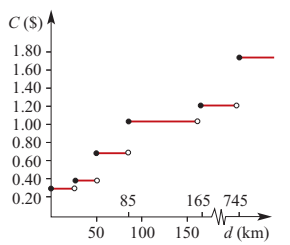
$= 0.70$ for $50 \leq d < 85$

$= 1.05$ for $85 \leq d < 165$

$= 1.22$ for $165 \leq d < 745$

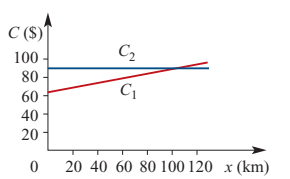
$= 1.77$ for $d \geq 745$

b



4 a i $C_1 = 64 + 0.25x$ **ii** $C_2 = 89$

b



c $x > 100$ km

Multiple-choice questions

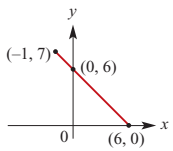
1 B **2** E **3** D **4** C **5** E

6 B **7** D **8** E **9** C **10** D

Short-answer questions (technology-free)

1 a -16 **b** 26 **c** $-\frac{2}{3}$

2 a



b Range $= [0, 7]$

3 a Range $= R$ **b** Range $= [-5, 4]$

c Range $= [0, 4]$ **d** Range $= (-\infty, 9]$

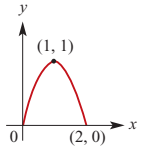
e Range $= (2, \infty)$ **f** $\{-6, 2, 4\}$

g Range $= [0, \infty)$ **h** $R \setminus \{2\}$

i Range $= [-5, 1]$ **j** Range $= [-1, 3]$

4 a $a = -15, b = \frac{33}{2}$ **b** Domain $= R \setminus \{0\}$

5 a



b $[0, 1]$

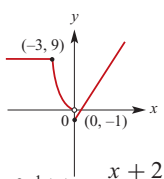
6 $a = 3, b = -5$ **7** $a = -\frac{1}{2}, b = 2, c = 0$

8 a $R \setminus \{2\}$ **b** $[2, \infty)$ **c** $[-5, 5]$

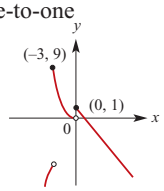
d $R \setminus \left\{\frac{1}{2}\right\}$ **e** $[-10, 10]$ **f** $(-\infty, 4]$

9 **b**, **c**, **d**, **e**, **f**, **g**, and **j** are one-to-one

10 a



b



11 a $f^{-1}(x) = \frac{x+2}{3}$, Domain $= [-5, 13]$

b $f^{-1}(x) = (x-2)^2 - 2$, Domain $= [2, \infty)$

c $f^{-1}(x) = \sqrt{\frac{x}{3}} - 1$, Domain $= [0, \infty)$

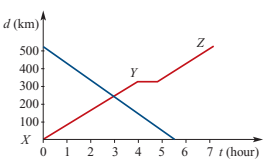
d $f^{-1}(x) = -\sqrt{x} + 1$, Domain $= [0, \infty)$

12 a $y = \sqrt{x-2} + 3$ **b** $y = 2\sqrt{x}$ **c** $y = -\sqrt{x}$

d $y = \sqrt{-x}$ **e** $y = \sqrt{\frac{x}{3}}$

Extended-response questions

1 a



Coach starting from X:

$d = 80t$ for $0 \le t \le 4$

$d = 320$ for $4 < t \le 4\frac{3}{4}$

$d = 80t - 60$ for $4\frac{3}{4} < t \le 7\frac{1}{4}$

Range $= [0, 520]$

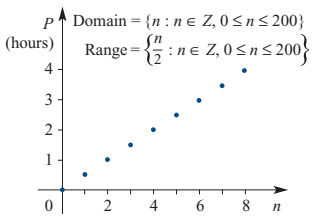
Coach starting from Z:

$d = 520 - \frac{1040}{11}t \quad 0 \le t \le 5\frac{1}{2}$

Range $= [0, 520]$

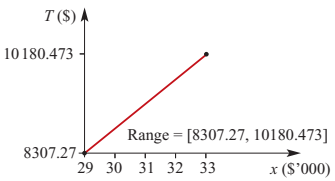
b The coaches pass $238\frac{1}{3}$ km from X.

2 a $P = \frac{1}{2}n$ **b**



3 a $T = 0.4683x - 5273.4266$

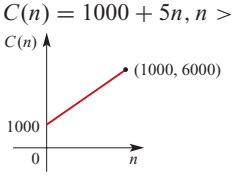
b



c \$8775.57 (to nearest cent)

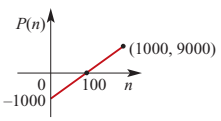
4 a **i** $C(n) = 1000 + 5n, n > 0$

ii



b **i** $P(n) = 15n - (1000 + 5n)$
$= 10n - 1000$

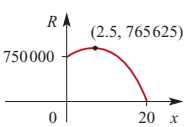
ii



5 $V = 8000(1 - 0.05n) = 8000 - 400n$

6 a $R = (50000 - 2500x)(15 + x)$
$= 2500(x + 15)(20 - x)$

b



c Price for max $=$ \$17.50

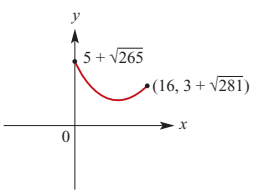
7 a $A(x) = \frac{x}{4}(2a - (6 - \sqrt{3})x)$

b $0 < x < \frac{a}{3}$ **c** $\frac{a^2}{4(6-\sqrt{3})}$ cm^2

8 a **i** $d(x) = \sqrt{x^2 + 25} + \sqrt{(16 - x)^2 + 9}$

ii $0 \le x \le 16$

b **i**



Answers

ii 1.54 **iii** 3.40 or 15.04

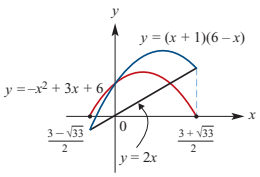
c **i** minimum at $x = 10$
minimum of $d(x) = 8\sqrt{5}$

ii range $= [8\sqrt{5}, 5 + \sqrt{265}]$

9 a $A\left(\frac{3+\sqrt{33}}{2}, 3+\sqrt{33}\right)$,
$B\left(\frac{3-\sqrt{33}}{2}, 3-\sqrt{33}\right)$

b **i** $d(x) = -x^2 + 3x + 6$

ii



c **i** maximum value of $d(x)$ is 8.25 **ii** $[0, 8.25]$

d **i** $A(2.45, 12.25)$ $B(-2.45, -12.25)$

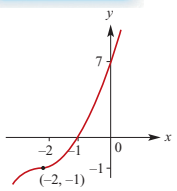
ii $d(x) = -x^2 + 6$ **iii** Range $= [0, 6]$

iv maximum value of $d(x)$ is 6

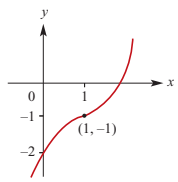
Chapter 6

Exercise 6A

1 a



b



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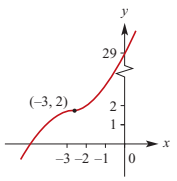

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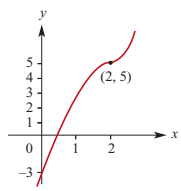

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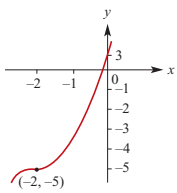
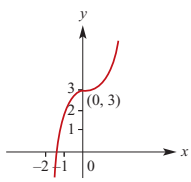


2 a



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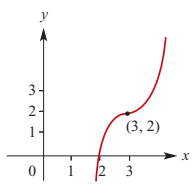

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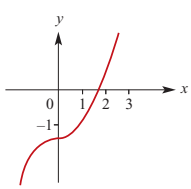

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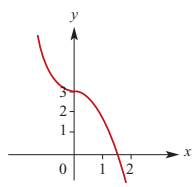

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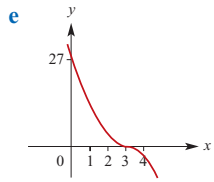

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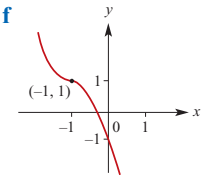

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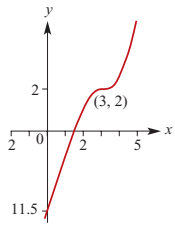

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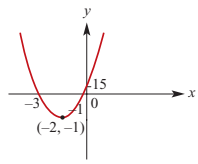

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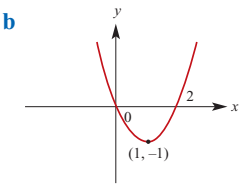

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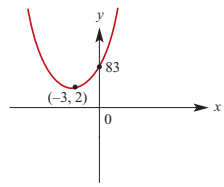

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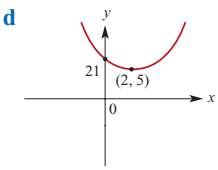

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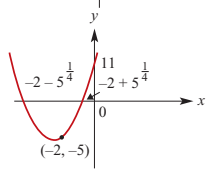

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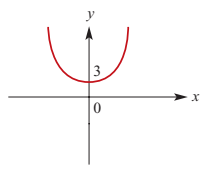

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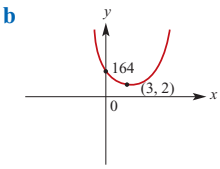

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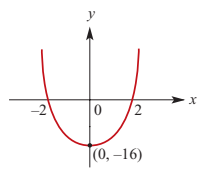

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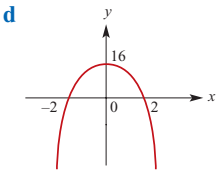

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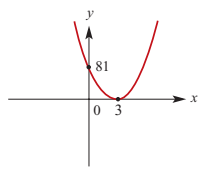

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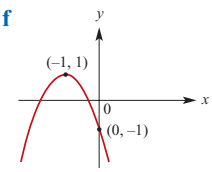


Exercise 6B

1 a $x^2 + 2x + \frac{3}{x-1}$ **b** $2x^2 - x - 3 + \frac{6}{x+1}$

c $3x^2 - 10x + 22 - \dfrac{43}{x+2}$

d $x^2 - x + 4 - \dfrac{8}{x+1}$

e $2x^2 + 3x + 10 + \dfrac{28}{x-3}$

f $2x^2 - 5x + 37 - \dfrac{133}{x+4}$ **g** $x^2 + x + \dfrac{2}{x+3}$

2 a $\dfrac{1}{2}x^2 + \dfrac{7}{4}x - \dfrac{3}{8} + \dfrac{103}{8(2x+5)}$

b $x^2 + 2x - 3 - \dfrac{2}{2x+1}$

c $\dfrac{1}{3}x^2 - \dfrac{8}{9}x - \dfrac{8}{27} + \dfrac{19}{27(3x-1)}$

d $x^2 - x + 4 + \dfrac{13}{x-2}$

e $x^2 + 2x - 15$

f $\dfrac{1}{2}x^2 + \dfrac{3}{4}x - \dfrac{3}{8} - \dfrac{5}{8(2x+1)}$

3 a $x^2 + 3x + 8 + \dfrac{9}{x-1}$

b $x^2 - \dfrac{x}{2} + \dfrac{9}{4} + \dfrac{21}{4(2x-1)}$

Exercise 6C

1 a -2 **b** -29 **c** 15 **d** 4 **e** 7
f -12 **g** 0 **h** -5 **i** -8

2 a $a = -3$ **b** $a = 2$
c $a = 4$ **d** $a = -10$

Exercise 6D

2 a 6 **b** 28 **c** $-\dfrac{1}{3}$

3 a $(x-1)(x+1)(2x+1)$ **b** $(x+1)^3$
c $(x-1)(6x^2 - 7x + 6)$
d $(x-1)(x+5)(x-4)$
e $(x+1)^2(2x-1)$ **f** $(x+1)(x-1)^2$
g $(x-2)(4x^2 + 8x + 19)$
h $(x+2)(2x+1)(2x-3)$

4 a $(x-1)(x^2 + x + 1)$
b $(x+4)(x^2 - 4x + 16)$
c $(3x-1)(9x^2 + 3x + 1)$
d $(4x-5)(16x^2 + 20x + 25)$
e $(1-5x)(1 + 5x + 25x^2)$
f $(3x+2)(9x^2 - 6x + 4)$
g $(4m - 3n)(16m^2 + 12mn + 9n^2)$
h $(3b + 2a)(9b^2 - 6ab + 4a^2)$

5 a $(x+2)(x^2 - x + 1)$
b $(3x+2)(x-1)(x-2)$
c $(x-3)(x+1)(x-2)$
d $(3x+1)(x+3)(2x-1)$

6 $a = 3, b = -3, P(x) = (x-1)(x+3)(x+1)$
7 b i n odd **ii** n even
8 a $a = 1, b = 1$ **b i** $P(x) = x^3 - 2x^2 + 3$

Exercise 6E

1 a 1, –2, 4 **b** 4, 6 **c** $\dfrac{1}{2}, 3, -\dfrac{2}{3}$

d 1, –1, –2 **e** 2, 3, –5 **f** $-1, -\dfrac{2}{3}, 3$

g $1, -\sqrt{2}, \sqrt{2}$ **h** $-\dfrac{2}{5}, -4, 2$

i $-\dfrac{1}{2}, \dfrac{1}{3}, 1$ **j** $-2, -\dfrac{3}{2}, 5$

2 a –6, 2, 3 **b** $-2, -\dfrac{2}{3}, \dfrac{1}{2}$ **c** 3

d -1 **e** $-1, 3$ **f** $3, -2 \pm \sqrt{3}$

3 a $-2, 0, 4$ **b** $0, -1 \pm 2\sqrt{3}$
c $-5, 0, 8$ **d** $0, -1 \pm \sqrt{17}$

4 a $0, \pm 2\sqrt{2}$ **b** $1 + 2\sqrt[3]{2}$ **c** -2

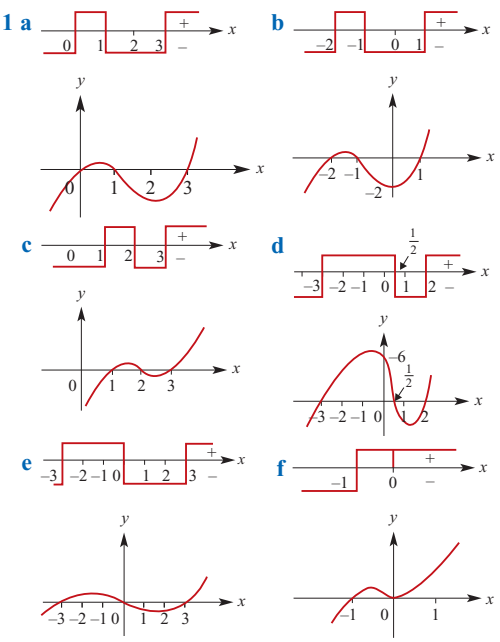
d -5 **e** $\dfrac{1}{10}$

5 a 1 **b** -1 **c** $5, \pm\sqrt{10}$ **d** $\pm 4, a$

6 a $2(x-9)(x-13)(x+11)$
b $(x+11)(x+3)(2x-1)$
c $(x+11)(2x-9)(x-11)$
d $(2x-1)(x+11)(x+15)$

Exercise 6F

1 a



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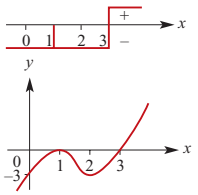

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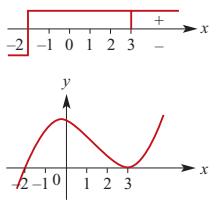

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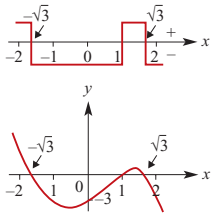

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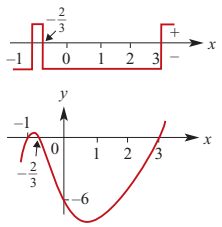

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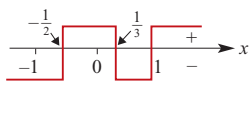

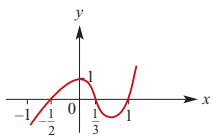

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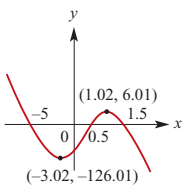

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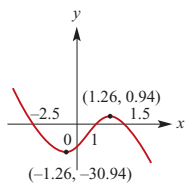

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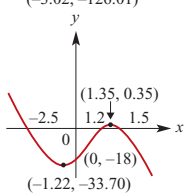

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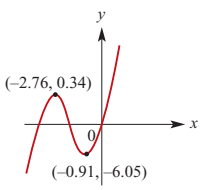

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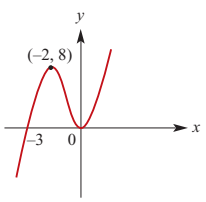

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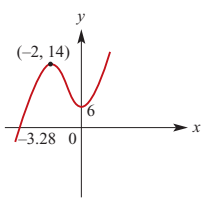


3 $(x+1)(x+1)(x-3)=0$, ∴ Graph just touches the x-axis at $x=-1$ and cuts it at $x=3$.

Exercise 6G

a $\{x : x \leq -2\} \cup \{x : 1 \leq x \leq 3\}$
b $\{x : x \geq 4\} \cup \{x : -2 \leq x \leq -1\}$
c $\{x : x < 1\}$
d $\{x : -2 < x < 0\} \cup \{x : x > 3\}$
e $x \leq -1$ **f** $x \geq 1$ **g** $x > 4$ **h** $x \leq -3$

Exercise 6H

1 a $y = -\frac{1}{8}(x+2)^3$ **b** $y - 2 = -\frac{1}{4}(x-3)^3$

2 $y = 2x(x-2)^2$
3 $y = -2x(x+4)^2$
4 a $y = (x-3)^3 + 2$
b $y = \frac{23}{18}x^3 + \frac{67}{18}x^2$ **c** $y = 5x^3$

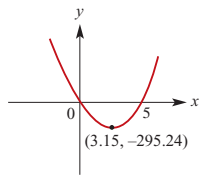
5 a $y = -\frac{1}{3}x^3 + \frac{4}{3}x$ **b** $y = \frac{1}{4}x(x^2+2)$

6 a $y = -4x^3 - 50x^2 + 96x + 270$
b $y = 4x^3 - 60x^2 + 80x + 26$
c $y = x^3 - 2x^2 + 6x - 4$
d $y = 2x^3 - 3x$
e $y = 2x^3 - 3x^2 - 2x + 1$
f $y = x^3 - 3x^2 - 2x + 1$
g $y = -x^3 - 3x^2 - 2x + 1$

Exercise 6I

1 a $x = 0$ or $x = 3$
b $x = 2$ or $x = -1$ or $x = 5$ or $x = -3$
c $x = 0$ or $x = -2$ **d** $x = 0$ or $x = 6$
e $x = 0$ or $x = 3$ or $x = -3$
f $x = 3$ or $x = -3$
g $x = 0$ or $x = 4$ or $x = -4$
h $x = 0$ or $x = 4$ or $x = 3$
i $x = 0$ or $x = 4$ or $x = 5$
j $x = 2$ or $x = -2$ or $x = 3$ or $x = -3$
k $x = 4$ **l** $x = -4$ or $x = 2$

2 a



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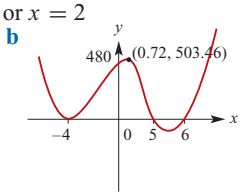

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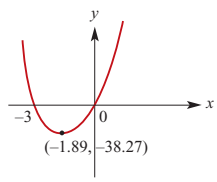

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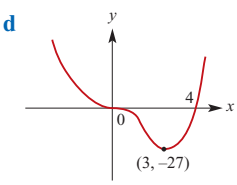

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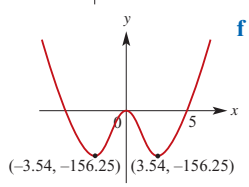

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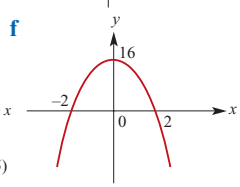

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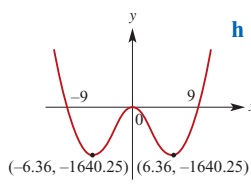

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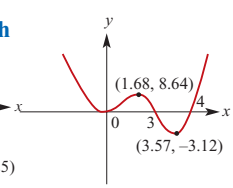

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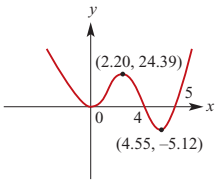

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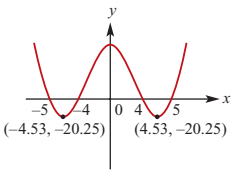

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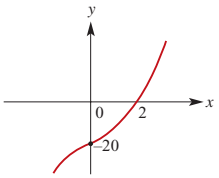

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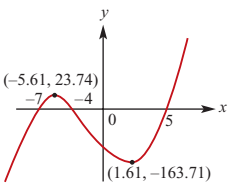


Exercise 6J

1 a $f(n) = n^2 + 3$ **b** $f(n) = n^2 - 3n + 5$

c $f(n) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$

d $f(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

e $f(n) = 2n^3 - 5$

2 a $f(n) = n^2$ **b** $f(n) = n(n+1)$

c $f(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

d $f(n) = \frac{4}{3}n^3 - \frac{1}{3}n$

e $f(n) = \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{7}{6}n$

f $f(n) = \frac{4}{3}n^3 + 3n^2 + \frac{5}{3}n$

3 $f(n) = \frac{1}{2}n^2 - \frac{1}{2}n$

4 $f(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

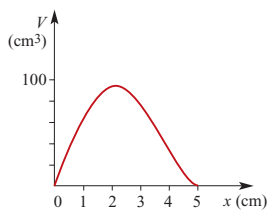
5 $f(n) = \frac{1}{4}n^2(n+1)^2$

Exercise 6K

1 a $l = 12 - 2x, w = 10 - 2x$

b $V = 4x(6 - x)(5 - x)$

c



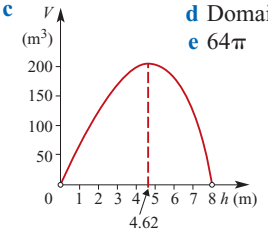
d $V = 80$

e $x = 3.56$ or $x = 0.51$

f V max $= 96.8$ cm^3 when $x = 1.81$

2 a $x = \sqrt{64 - h^2}$ **b** $V = \frac{\pi h}{3}(64 - h^2)$

c



d Domain $= \{h : 0 < h < 8\}$

e 64π

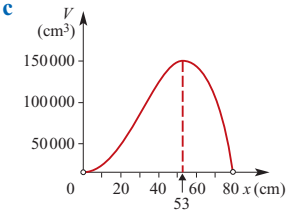
f $h = 2.48$ or $h = 6.47$

g V max ≈ 206.37 m^3, $h = 4.62$

3 a $h = 160 - 2x$

b $V = x^2(160 - 2x)$, Domain $= (0, 80)$

c



d $x = 20.498$ or $x = 75.63$

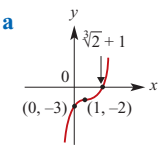
e V max $\approx 151\,703.7$ cm^3 when $x \approx 53$

Multiple-choice questions

1 B	**2** D	**3** A	**4** D	**5** A
6 C	**7** B	**8** B	**9** D	**10** B

Short-answer questions (technology-free)

1 a



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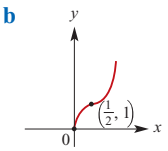

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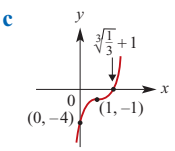
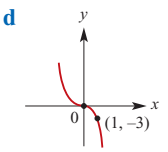
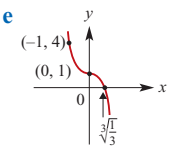


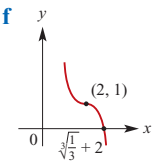
d



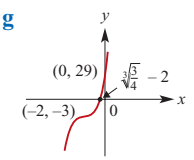
e



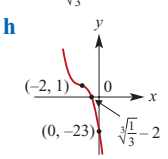
f



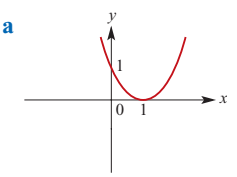
g



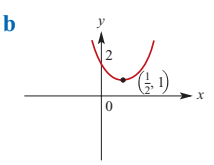
h



2 a

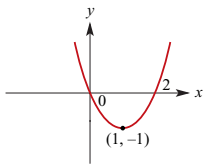


b



Answers

c



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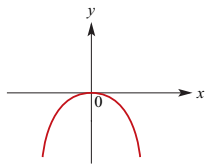

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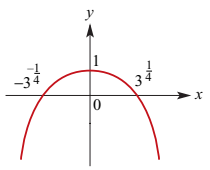

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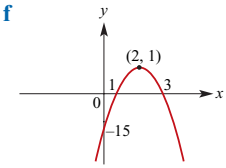
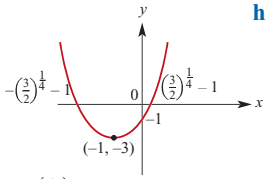
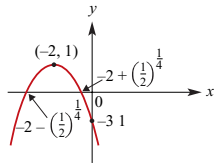


g



h



3 a $P\left(\frac{3}{2}\right) = 0$ and $P(-2) = 0, (3x + 1)$

b $x = -2,\ \frac{1}{2},\ 3$

c $x = -1,\ -\sqrt{11}, +\sqrt{11}$

d **i** $P\left(\frac{1}{3}\right) = 0$

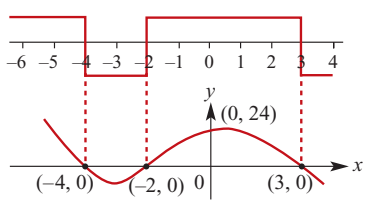
ii $(3x - 1)(x + 3)(x - 2)$

4 a $f(1) = 0$

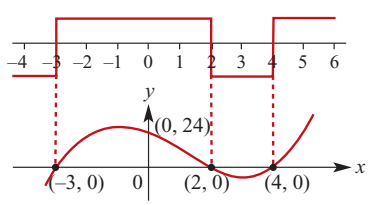
b $(x - 1)(x^2 + (1 - k)x + k + 1)$

5 $a = 3, b = -24$

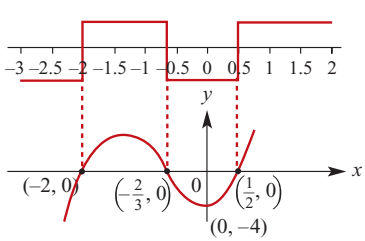
6 a



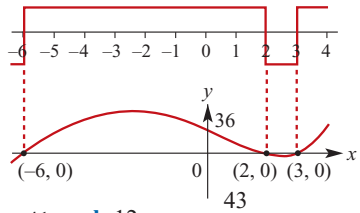
b



c



d



7 a -41 **b** 12 **c** $\frac{43}{9}$

8 $y = -\frac{2}{5}(x + 2)(x - 1)(x - 5)$

9 $y = \frac{2}{81}x(x + 4)^2$

10 a $a = 3, b = 8$ **b** $(x + 3)(2x - 1)(x - 1)$

11 a $y = (x - 2)^3 + 3$ **b** $y = 2x^3$

c $y = -x^3$ **d** $y = (-x)^3 = -x^3$

e $y = \left(\frac{x}{3}\right)^3 = \frac{x^3}{27}$

12 a $y = -(x - 2)^4 + 3$ **b** $y = 2x^4$

c $y = -(x + 2)^4 + 3$

13 a Dilation of factor 2 from the x-axis, translation of 1 unit in the positive direction of the x-axis, then translation of 3 units in the positive direction of the y-axis

b Reflection in the x-axis, translation of 1 unit in the negative direction of the x-axis, then translation of 2 units in the positive direction of the y-axis

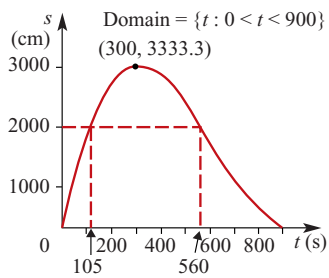
c Dilation of factor $\frac{1}{2}$ from the y-axis, translation of $\frac{1}{2}$ unit in the negative direction of the x-axis and translation of 2 units in the negative direction of the y-axis

Extended-response questions

1 a $v = \frac{1}{32\,400}(t - 900)^2$

b $s = \frac{t}{32\,400}(t - 900)^2$

c



d No, it is not feasible since the maximum range of the taxi is less than 3.5 km (~3–33 km).

e Maximum speed $\approx \frac{2000}{105} = 19$ m/s

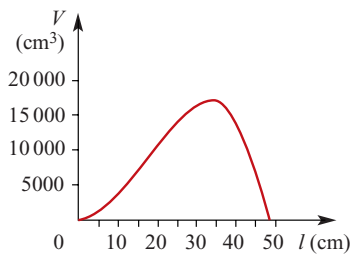
Minimum speed $\approx \frac{2000}{560} = 3.6$ m/s

2 a $R - 10 = a(x - 5)^3$

b $a = \frac{2}{25}$ **c** $R - 12 = \frac{12}{343}(x - 7)^3$

3 a 4730 cm^2 **b** $V = l^2(\sqrt{2365} - l)$

c



d **i** $l = 23.69$ or $l = 39.79$

ii $l = 18.1$ or $l = 43.3$

e V max $\approx 17\,039$ cm^3, $l \approx 32.42$ cm

4 a $a = \frac{-43}{15\,000}$, $b = 0.095$, $c = \frac{-119}{150}$, $d = 15.8$

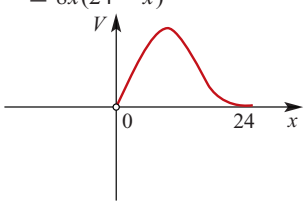
b **i** Closest to the ground (5.59, 13.83),

ii furthest from the ground (0, 15.8)

5 a $V = (96 - 4x)(48 - 2x)x$

$= 8x(24 - x)^2$

b



i $0 < x < 24$

ii $V_{max} = 16\,384$ cm^3 when $x = 8.00$

c 15 680 cm^3 **d** 14 440 cm^3

e 9720 cm^3

Chapter 7

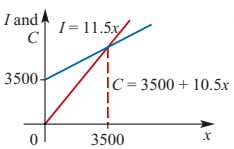
7.1 Multiple-choice questions

1 A	**2** D	**3** D	**4** C	**5** B
6 C	**7** A	**8** E	**9** B	**10** A
11 E	**12** B	**13** D	**14** D	**15** E
16 B	**17** D	**18** E	**19** D	**20** B
21 D	**22** D	**23** A	**24** B	**25** D
26 D	**27** B	**28** C	**29** A	**30** C
31 A	**32** B	**33** C	**34** D	**35** E
36 E	**37** C	**38** C	**39** C	**40** A

7.2 Extended-response questions

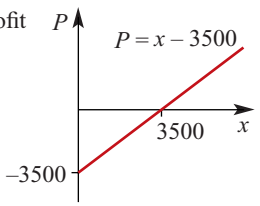
1 a $C = 3500 + 10.5x$ **b** $I = 11.5x$

c



d 3500

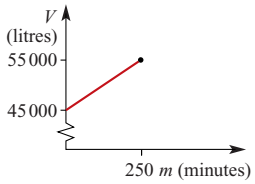
e Profit



f 5500

2 a $V = 45\,000 + 40m$ **b** 4 hours 10 minutes

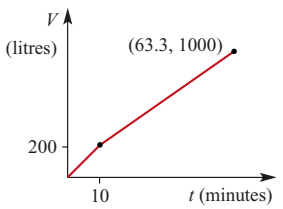
c



3 a 200 L

b $V = \begin{cases} 20t & 0 \le t \le 10 \\ 15t + 50 & 10 < t \le \frac{190}{3} \end{cases}$

c

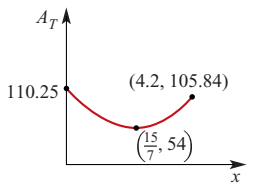


4 a $A_r = 6x^2$ **b** $A_s = (10.5 - 2.5x)^2$

c $0 \le x \le 4.2$

d $A_T = 12.25x^2 - 52.5x + 110.25$

e



f 110.25 cm^2 (area of rectangle = 0)

g rectangle: 9×6, square: 3×3, ($x = 3$) or rectangle: $\frac{27}{7} \times \frac{18}{7}$; square: $\frac{51}{7} \times \frac{51}{7}$

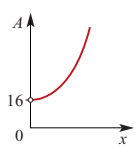
5 a 20 m **b** 20 m **c** 22.5 m

6 a $A = 10x^2 + 28x + 16$

b **i** 54 cm^2 **ii** 112 cm^2

c 3 cm

d



e $V = 2x^3 + 8x^2 + 8x$

f $x = 3$ **g** $x = 6.66$

7 a **i** $A = (10 + x)y - x^2$

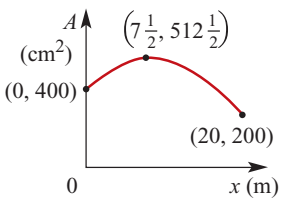
ii $P = 2(y + x + 10)$

b **i** $A = 400 + 30x - 2x^2$

ii $512\frac{1}{2}$ m^2 **iii** $0 \le x \le 20$

Answers

iv

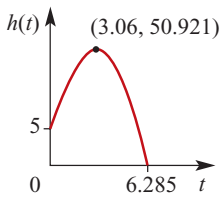


8 a $A = 6x^2 + 7xy + 2y^2$

c **i** $x = 0.5$ m **ii** $y = 0.25$ m

9 a 50.9 m **b** $t = 6.12$ seconds

c

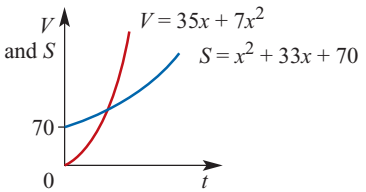


d 6.285 seconds

10 a $x + 5$ **b** $V = 35x + 7x^2$

c $S = x^2 + 33x + 70$

d



e 3.25 m **f** 10 cm

11 a $2y + 3x = 22$

b **i** B(0, 11) **ii** D(8, −l)

c 52 units² **d** 6.45 units

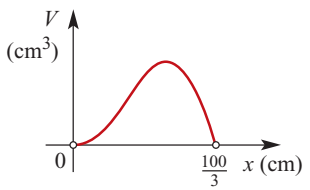
12 a 25 km/h **b** tap A 60 min; tap B 75 min

c 4 cm

13 a $h = 100 - 3x$ **b** $V = 2x^2(100 - 3x)$

c $0 < x < \frac{100}{3}$

d



e **i** $x = 18.142$ or $x = 25.852$

ii $x = 12.715$ or $x = 29.504$

f V max $= 32\,921.811$ cm³ where $x = 22.222$

g **i** $S = 600x - 14x^2$

ii $S\text{max} = \frac{45\,000}{7}$ cm², where $x = \frac{150}{7}$

h $x = 3.068$ or $x = 32.599$

14 a $y = (7.6 \times 10^{-5})x^3 - 0.0276x^2 + 2.33x$

b $y = (7.6 \times 10^{-5})x^3 - 0.0276x^2 + 2.33x + 5$

c 57.31 m

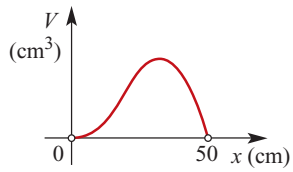
15 a $y = \frac{3}{4}x - 4$ **b** $y = -\frac{4}{3}x + \frac{38}{3}$

c $D(8, 2)$ **d** 5 units **e** 50 units²

16 a **i** $y = 250 - 5x$

ii $V = x^2(250 - 5x) = 5x^2(50 - x)$

b



c (0, 50) **d** $x = 11.378$ or $x = 47.813$

e V max $= 92\,592.59$ cm³, where $x = 33.33$ and $y = 83.33$

Chapter 8

Exercise 8A

1 $\{H, T\}$ **2** $\{1, 2, 3, 4, 5, 6\}$

3 a 52 **b** 4

c clubs ♣, hearts ♥, spades ♠, diamonds ♦

d clubs and spades are black, diamonds and hearts are red

e 13 **f** ace, king, queen, jack

g 4 **h** 16

4 a $\{BB, BR, RB, RR\}$

b $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

c $\{MMM, MMF, MFM, FMM, MFF, FMF, FFM, FFF\}$

5 a $\{0, 1, 2, 3, 4, 5\}$

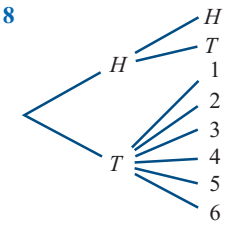
b $\{0, 1, 2, 3, 4, 5, 6\}$ **c** $\{0, 1, 2, 3\}$

6 a $\{0, 1, 2, 3, \ldots\}$ **b** $\{0, 1, 2, 3, \ldots, 41\}$

c $\{1, 2, 3, \ldots\}$

7 a $\{2, 4, 6\}$ **b** $\{FFF\}$ **c** Ø

8



$\{HH, HT, T1, T2, T3, T4, T5, T6\}$

9 a $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

b $\{(2, 4), (3, 3), (4, 2)\}$

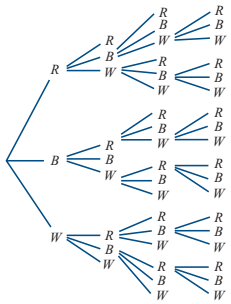
10 a $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}$

b $\{(l, 1), (2, 2), (3, 3)\}$

11 a $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6) (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6) (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

b $\{(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6), (5, 2), (5, 4), (5, 6)\}$

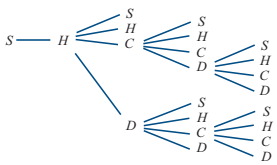
12 a



{(*RR*), (*RBR*), (*RBB*), (*RBWR*), (*RBWB*), (*RBWW*), (*RWR*), (*RWBR*), (*RWBB*), (*RWBW*), (*RWW*), (*BRR*), (*BRB*), (*BRWR*), (*BRWB*), (*BRWW*), (*BB*), (*BWRR*), (*BWRB*), (*BWRW*), (*BWB*), (*BWW*), (*WRR*), (*WRBR*), (*WRBB*), (*WRBW*), (*WRW*), (*WBRR*), (*WBRB*), (*WBRW*), (*WBB*), (*WBW*), (*WW*)}

b 4

13 a



{(*SHS*), (*SHH*), (*SHCS*), (*SHCH*), (*SHCC*), (*SHCDS*), (*SHCDH*), (*SHCDC*), (*SHCDD*), (*SHDS*), (*SHDH*), (*SCDCS*), (*SHDCH*), (*SHDCC*), (*SHDCD*), (*SHDD*)}

b 5

Exercise 8B

1 a $\frac{17}{50}$ **b** $\frac{1}{10}$ **c** $\frac{4}{15}$ **d** $\frac{1}{200}$

2 a No **b** answers will vary

c answers will vary **d** Yes

e As the number of trials approaches infinity the relative frequency approaches the value of the probability.

3 a No **b** answers will vary

c answers will vary **d** Yes

e As the number of trials approaches infinity the relative frequency approaches the value of the probability.

4 $\Pr(\text{a 6 from first die}) \approx \frac{78}{500} = 0.156$

$\Pr(\text{a 6 from second die}) \approx \frac{102}{700} \approx 0.146$

$\therefore$ choose the first die.

5 a 0.702 **b** 0.722

c The above estimates for the probability should be recalculated.

d 0.706

6 $\Pr(4) = \frac{1}{3}$

7 $\Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \frac{2}{13}$,

$\Pr(6) = \frac{4}{13}$, $\Pr(1) = \frac{1}{13}$

8 $\Pr(A) = 0.225$ **9** $\Pr(A') = 0.775$

Exercise 8C

1 a $\frac{1}{3}$ **b** $\frac{1}{8}$ **c** $\frac{1}{4}$ **d** $\frac{5}{18}$

2 a 0.141 **b** 0.628 **c** 0.769

3 a $\frac{1}{365}$ **b** $\frac{30}{365}$ **c** $\frac{30}{365}$ **d** $\frac{90}{365}$

e $\frac{364}{365}$ **f** $\frac{334}{365}$

4 a $\frac{1}{4}$ **b** $\frac{1}{2}$ **c** $\frac{4}{13}$ **d** $\frac{3}{4}$

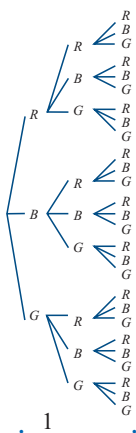
5 a $\frac{8}{13}$ **b** $\frac{9}{13}$ **c** $\frac{5}{13}$ **d** $\frac{1}{13}$

6 a $\frac{1}{2}$ **b** $\frac{1}{18}$ **c** $\frac{5}{18}$

7 a $\frac{1}{12}$ **b** $\frac{1}{2}$ **c** $\frac{7}{12}$

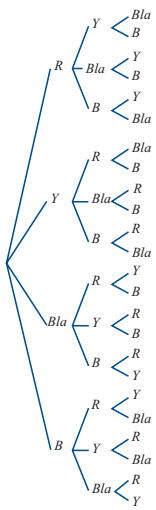
8 $\frac{1}{4}$

9 a



b i $\frac{1}{27}$ **ii** $\frac{2}{9}$ **iii** $\frac{1}{3}$ **iv** $\frac{2}{9}$

10 a



b i 0.25 **ii** $\frac{12}{24} = 0.5$ **iii** $\frac{1}{12}$

Answers

Answers

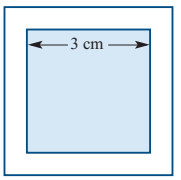
11 a $\frac{5}{13}$ **b** $\frac{11}{13}$

12 a

1st ball \ 2nd ball	1	2	3	4	5
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)

b i $\frac{4}{25}$ **ii** $\frac{4}{5}$ **iii** $\frac{3}{25}$

13 a



b $\frac{9}{25}$

14 $\frac{\pi}{4}$

15 a $\frac{4\pi}{25}$ **b** $1 - \frac{4\pi}{25}$

Exercise 8D

1 a {1, 2, 3, 4, 6} **b** {2, 4}
c {5, 6, 7, 8, 9, 10} **d** {1, 3}
e {1, 3, 5, 6, 7, 8, 9, 10} **f** {5, 7, 8, 9, 10}
2 a {1, 2, 3, 5, 6, 7, 9, 10, 11}
b {1, 3, 5, 7, 9, 11} **c** {2, 4, 6, 8, 10, 12}
d {1, 3, 5, 7, 9, 11} **e** {1, 3, 5, 7, 9, 11}
3 a {E, H, M, S} **b** {C, H, I, M}
c {A, C, E, I, S, T} **d** {H, M}
e {C, E, H, I, M, S} **f** {H, M}
4 a 20 **b** 45
5 a 6 **b** 1 **c** 18 **d** 2
6 a $\frac{2}{3}$ **b** 0 **c** $\frac{1}{2}$ **d** $\frac{5}{6}$
7 a $\frac{1}{2}$ **b** $\frac{1}{3}$ **c** $\frac{1}{6}$; $\frac{2}{3}$
8 a 1 **b** $\frac{4}{11}$ **c** $\frac{9}{11}$ **d** $\frac{6}{11}$ **e** $\frac{7}{11}$ **f** $\frac{4}{11}$

Exercise 8E

1 a 0.2 **b** 0.5 **c** 0.3 **d** 0.7
2 a 0.75 **b** 0.4 **c** 0.87 **d** 0.48
3 a 0.63 **b** 0.23 **c** 0.22 **d** 0.77
4 a 0.45 **b** 0.40 **c** 0.25 **d** 0.70
5 a 0.9 **b** 0.6 **c** 0.1 **d** 0.9
6 a 95% **b** 5%
7 a A = {J♥, Q♥, K♥, A♥, J♠, Q♠, K♠, A♠, J♦, Q♦, K♦, A♦, J♣, Q♣, K♣, A♣}
C = {2♥, 3♥, 4♥, 5♥, 6♥, 7♥, 8♥, 9♥, 10♥, J♥, Q♥, K♥, A♥}
b i Pr(a picture card) = $\frac{4}{13}$
ii Pr(a heart) = $\frac{1}{4}$
iii Pr(a heart picture card) = $\frac{1}{13}$
iv Pr(a picture card or a heart) = $\frac{25}{52}$
v Pr(a picture card or a club, diamond or spade) = $\frac{43}{52}$
8 a $\frac{8}{15}$ **b** $\frac{7}{10}$ **c** $\frac{2}{15}$ **d** $\frac{1}{3}$
9 a 0.8 **b** 0.57 **c** 0.28 **d** 0.08
10 a 0.81 **b** 0.69 **c** 0.74 **d** 0.86
11 a 0 **b** 1 **c** $\frac{1}{5}$ **d** $\frac{1}{3}$
12 a 0.88 **b** 0.58 **c** 0.30 **d** 0.12

Exercise 8F

1 $\frac{1}{4}$
2 a $\frac{65}{284}$ **b** $\frac{137}{568}$ **c** $\frac{21}{65}$ **d** $\frac{61}{246}$
3 a 0.06 **b** 0.2
4 a $\frac{4}{7}$ **b** 0.3 **c** $\frac{15}{22}$
5 a 0.2 **b** 0.5 **c** 0.4
6 a 0.2 **b** $\frac{10}{27}$ **c** $\frac{1}{3}$
7 a 0.3 **b** 0.75
8 a $\frac{1}{2}$ **b** $\frac{3}{4}$ **c** $\frac{1}{2}$ **d** 1 **e** $\frac{2}{3}$ **f** $\frac{1}{2}$
9 16% **10** $\frac{1}{5}$
11 a $\frac{1}{16}$ **b** $\frac{1}{169}$ **c** $\frac{1}{4}$ **d** $\frac{16}{169}$
12 a $\frac{1}{17}$ **b** $\frac{1}{221}$ **c** $\frac{25}{102}$ **d** $\frac{20}{221}$
13 a $0.230\,808 \approx 0.231$
14 a $\frac{15}{28}$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$ **d** $\frac{2}{5}$
e $\frac{3}{7}$ **f** $\frac{8}{13}$ **g** $\frac{5}{28}$ **h** $\frac{3}{14}$
15 a 0.85 **b** 0.6 **c** 0.51 **d** 0.51
16 0.4; 68%
17 a i 0.444 **ii** 0.4 **iii** 0.35 **iv** 0.178 **v** 0.194
b 0.372 **c i** 0.478 **ii** 0.425
18 a i 0.564 **ii** 0.05 **iii** 0.12 **iv** 0.0282 **v** 0.052
b 0.081 **c** 0.35
19 a $\frac{1}{6}$ **b** $\frac{53}{90}$ **c** $\frac{15}{53}$
20 a $B \subseteq A$ **b** $A \cap B = \emptyset$ **c** $A \subseteq B$

Exercise 8G

1 a Yes **b** Yes **c** No
2 0.6 **4** No
5 a 0.6 **b** 0.42 **c** 0.88
6 a 0.35 **b** 0.035 **c** 0.1225 **d** 0.025
7 a $\frac{4}{15}$ **b** $\frac{1}{15}$ **c** $\frac{133}{165}$ **d** $\frac{6}{11}$ **e** $\frac{4}{15}$; No
9 a 0.35 **b** 0.875

10 a $\frac{18}{65}$ **b** $\frac{12}{65}$ **c** $\frac{23}{65}$ **d** $\frac{21}{65}$
e $\frac{4}{65}$ **f** $\frac{8}{65}$ **g** $\frac{2}{15}$ **h** $\frac{8}{21}$; No
11 a **i** 0.75 **ii** 0.32 **iii** 0.59
b No **c** No
12 b **i** $\frac{1}{8}$ **ii** $\frac{3}{8}$ **iii** $\frac{7}{8}$
13 b **i** 0.09 **ii** 0.38 **iii** 0.29 **iv** 0.31
14 a $\frac{1}{216}$ **b** $\frac{1}{8}$ **c** $\frac{1}{2}$ **d** $\frac{1}{36}$
15 a $\frac{1}{32}$ **b** $\frac{1}{32}$ **c** $\frac{1}{2}$ **d** $\frac{1}{16}$
16 a $\frac{1}{6}$ **b** $\frac{1}{30}$ **c** $\frac{1}{6}$ **d** $\frac{5}{6}$ **e** $\frac{1}{6}$
17 a $\frac{1}{2}$ **b** $\frac{1}{8}$ **c** $\frac{1}{2}$

Multiple-choice questions

1 B **2** C **3** A **4** C **5** D
6 E **7** E **8** C **9** A **10** B

Short-answer questions (technology-free)

1 a $\frac{1}{6}$ **b** $\frac{5}{6}$ **2** 0.007
3 a $\frac{1}{3}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$
4 a 0.36 **b** $\frac{87}{245}$ **5** $\frac{4}{15}$
6 a {156, 165, 516, 561, 615, 651}
b $\frac{2}{3}$ **c** $\frac{1}{3}$
7 a $\frac{5}{12}$ **b** $\frac{1}{4}$
8 a $\frac{2}{7}$ **b** $\frac{32}{63}$ **c** $\frac{9}{16}$
9 a 0.65 **b** No
10 0.999 89 **11 a** 0.2 **b** 0.4
12 No
13 a 0.036 **b** 0.027 **c** 0.189 **d** 0.729
14 a 0.7 **b** 0.3 **c** $\frac{1}{3}$ **d** $\frac{2}{3}$
15 a $\frac{1}{27}$ **b** $\frac{4}{27}$ **c** $\frac{4}{9}$ **d** $\frac{20}{27}$ **e** $\frac{2}{5}$

Extended-response questions

1 a $\beta - \gamma$ **b** $\alpha - \gamma$ **c** $1 - \alpha - \beta + \gamma$
d $1 - \gamma$ **e** $1 - \alpha - \beta + \gamma$ **f** $1 - \gamma$
2 a A: $\frac{3}{28}$ B: $\frac{3}{4}$ **b** A: $\frac{9}{64}$ B: $\frac{49}{64}$
c 0.125 **d** 0.155
3 a B is a subset of A
b A and B are mutually exclusive
c A and B are independent
4 a $\frac{1}{4}$ **b** $\frac{1}{3}$ **c** **i** $\frac{1}{16}$ **ii** $\frac{1}{4^n}$ **d** $\frac{1}{4}$
5 a 0.15 **b** 0.148
c

T	18	19	20	21	22	23	24
Pr	0.072	0.180	0.288	0.258	0.148	0.046	0.008

6 a 0.143 **b** $0.071 \leq p \leq 0.214$ **c** 0.022
7 a 0.6 **b** $\frac{1}{3}$ **c** $\frac{2}{7}$ **d** 0.108
e **i** $\frac{3}{20}$ **ii** $\frac{4}{7}$

Chapter 9

Exercise 9A

1 a 11 **b** 12 **c** 37 **d** 29
2 a 60 **b** 500 **c** 350 **d** 512
3 a 128 **b** 160
4 20 **5** 63 **6** 26 **7** 240
8 260 000 **9** 17 576 000 **10** 30

Exercise 9B

1 a 6 **b** 120 **c** 5040 **d** 2 **e** 1 **f** 1
2 a 20 **b** 72 **c** 6 **d** 56 **e** 120 **f** 720
3 120 **4** 5040 **5** 24 **6** 720
7 720 **8** 336
9 a 5040 **b** 210 **10 a** 120 **b** 120
11 a 840 **b** 2401 **12 a** 480 **b** 1512
13 a 60 **b** 24 **c** 252
14 a 150 **b** 360 **c** 1560
15 a 720 **b** 48

Exercise 9C

1 a 3 **b** 3 **c** 6 **d** 4
2 a 10 **b** 10 **c** 35 **d** 35
3 a 190 **b** 100 **c** 4950 **d** 31125
4 a 20 **b** 7 **c** 28 **d** 1225
5 1716 **6** 2300 **7** 133 784 560
8 8 145 060 **9** 18
10 a 5 852 925 **b** 1 744 200
11 100 386 **12 a** 792 **b** 336
13 a 150 **b** 75 **c** 6 **d** 462 **e** 81
14 a 8 436 285 **b** 3003 **c** 66 **d** 2 378 376
15 186 **16** 32 **17** 256 **18** 31 **19** 57
20 **a** 20 **b** 21

Exercise 9D

1 a 0.5 **b** 0.5 **2** 0.375
3 a 0.2 **b** 0.6 **c** 0.3
4 0.2 **5** $\frac{329}{858}$
6 a $\frac{2^7}{2^8 - 1} \approx 0.502$ **b** $\frac{56}{255}$ **c** $\frac{73}{85}$
7 a $\frac{5}{204}$ **b** $\frac{35}{136}$
8 a $\frac{25}{49}$ **b** $\frac{24}{49}$ **c** $\frac{3}{7}$ **d** 0.2
9 a $\frac{1}{6}$ **b** $\frac{5}{6}$ **c** $\frac{17}{21}$ **d** $\frac{34}{35}$

Answers

Answers

10 a 0.659 **b** 0.341 **c** 0.096 **d** 0.282

11 a $\frac{5}{42}$ **b** $\frac{20}{21}$ **c** $\frac{15}{37}$

Multiple-choice questions

1 E **2** D **3** A **4** D **5** C
6 B **7** C **8** A **9** E **10** E

Short-answer questions (technology-free)

1 a 499 500 **b** 1 000 000 **c** 1 000 000
2 648 **3** 120 **4** $8n$ **5** 5416
6 36 750 **7** 11 025
8 a 10 **b** 32 **9** 1200
10 a $\frac{1}{8}$ **b** $\frac{3}{8}$ **c** $\frac{3}{28}$

Extended-response questions

1 a 2880 **b** 80 640
2 a 720 **b** 48 **c** 336
3 a 60 **b** 36
4 a 210 **b** 100 **c** 80
5 a 1365 **b** 210 **c** 1155
6 a 3060 **b** 330 **c** 1155
7 Division 1: 1.228×10^{-7}
Division 2: 1.473×10^{-6}
Division 3: 2.726×10^{-5}
Division 4: 1.365×10^{-3}
Division 5: 3.362×10^{-3}
8 a 1.290×10^{-4} **b** 6.449×10^{-4}

Chapter 10

Exercise 10A

1 a no **b** no **c** yes **d** no **e** no
2 a $\Pr(X = 2)$ **b** $\Pr(X > 2)$ **c** $\Pr(X \geq 2)$
d $\Pr(X < 2)$ **e** $\Pr(X \geq 2)$ **f** $\Pr(X > 2)$
g $\Pr(X \leq 2)$ **h** $\Pr(X \geq 2)$ **i** $\Pr(X \leq 2)$
j $\Pr(X \geq 2)$ **k** $\Pr(2 < X < 5)$
3 a {2} **b** {3, 4, 5} **c** {2, 3, 4, 5}
d {0, 1} **e** {0, 1, 2} **f** {2, 3, 4, 5}
g {3, 4, 5} **h** {2, 3, 4} **i** {3, 4}
4 a $\frac{1}{15}$ **b** $\frac{3}{5}$ **5 a** 0.09 **b** 0.69
6 a 0.49 **b** 0.51 **c** 0.74
7 a 0.6 **b** 0.47 **c** $\frac{2}{3}$
8 a {*HHH*, *HTH*, *HHT*, *HTT*, *THH*, *TTH*, *THT*, *TTT*}
b $\frac{3}{8}$ **c**

x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

d $\frac{7}{8}$ **e** $\frac{4}{7}$
9 a {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} **b** $\frac{1}{6}$
c

y	2	3	4	5	6	7	8	9	10	11	12
$p(y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

10 a {1, 2, 3, 4, 5, 6} **b** $\frac{7}{36}$
c

1	2	3	4	5	6
$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

11 a 0.09 **b** 0.4 **c** 0.51
12 a

y	−3	−2	1	3
$p(y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b $\frac{7}{8}$

Exercise 10B

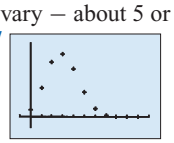
1 0.378 **2** $\frac{28}{57} \approx 0.491$ **3** $\frac{12}{13} \approx 0.923$
4 $\frac{60}{253} \approx 0.237$ **5** 0.930 **6** 0.109

Exercise 10C

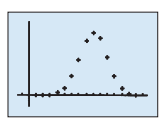
1 a 0.185 **b** 0.060 **2 a** 0.194 **b** 0.930
3 a 0.137 **b** 0.446 **c** 0.554
4 a 0.008 **b** 0.268 **c** 0.468
5 a 0.056 **b** 0.391 **6** 0.018
7 a $\Pr(X = x) = \binom{5}{x}(0.1)^x(0.9)^{5-x}$
$x = 0, 1, 2, 3, 4, 5$ or

x	0	1	2	3	4	5
$p(x)$	0.591	0.328	0.073	0.008	0.000	0.000

b Most probable number is 0
8 0.749 **9** 0.021 **10** 0.5398 **11** $\frac{175}{256}$
12 a 0.988 **b** 0.9999 **c** 8.1×10^{-11}
13 a 0.151 **b** 0.302 **14** 5.8%
15 a **i** 0.474 **ii** 0.224 **iii** 0.078
b Answers will vary – about 5 or more.
16 0.014 **17**



18



19 a 5 **b** 8
20 a 13 **b** 22
21 a 16 **b** 29
22 a 45 **b** 59

23 a 0.3087 **b** $\dfrac{0.3087}{1-(0.3)^5} \approx 0.3095$

24 a 0.3020 **b** 0.6242 **c** 0.3225

Exercise 10D

1 Exact answer 0.172
2 a About 50 : 50
b One set of simulations gave the answer 1.9

Exercise 10E

2 Exact answer 29.29
3 a One set of simulations gave the answer 8.3.
b One set of simulations gave the answer 10.7.
4 Exact answer is 0.0009.
5 a One set of simulations gave the answer 3.5.

Multiple-choice question

1 B **2** A **3** C **4** A **5** E
6 C **7** A **8** D **9** B **10** E

Short-answer questions (technology-free)

1 a 0.92 **b** 0.63 **c** 0.8

2

x	1	2	3	4
$p(x)$	0.25	0.28	0.30	0.17

3

x	2	3	4
$p(x)$	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$

4 a

1st choice / 2nd choice	1	2	3	6	7	9
1	2	3	4	7	8	10
2	3	4	5	8	9	11
3	4	5	6	9	10	12
6	7	8	9	12	13	15
7	8	9	10	13	14	16
9	10	11	12	15	16	18

b { 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18 }

c

x	2	3	4	5	6	7
$\Pr(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$
x	8	9	10	11	12	13
$\Pr(X = x)$	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$
x	14	15	16	18		
$\Pr(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$		

5 a 0.051 **b** 0.996 **c** $\dfrac{243}{256} \approx 0.949$

6 a $\dfrac{9}{64}$ **b** $\dfrac{37}{64}$

7 a 0.282 **b** 0.377 **c** 0.341

8 a 0.173 **b** 0.756 **c** 0.071

9 a $\left(\dfrac{p}{100}\right)^{15}$ **b** $15\left(\dfrac{p}{100}\right)^{14}\left(1-\dfrac{p}{100}\right)$

c $\left(\dfrac{p}{100}\right)^{15} + 15\left(\dfrac{p}{100}\right)^{14}\left(1-\dfrac{p}{100}\right) + 105\left(1-\dfrac{p}{100}\right)^{2}\left(\dfrac{p}{100}\right)^{13}$

10 a $\dfrac{117}{125}$ **b** $m = 5$

Extended-response questions

1 a

x	1	2	3	4
$p(x)$	0.54	0.16	0.06	0.24

b 0.46

2 a **i** 0.1 **ii** 0.6 **iii** $\dfrac{2}{3}$

b **i** 0.0012 **ii** 0.2508

3 a $\dfrac{3}{5}$

b **i** $\dfrac{7}{40}$ **ii** $\dfrac{3}{10}$

c **i** $\dfrac{11}{40}$ **ii** $\dfrac{11}{17}$

4 a 0.003 **b** 5.319×10^{-6}

5 0.8 **6** 0.969

7 a 0.401 **b** $n \geq 45$

8 a $1 - q^2$ **b** $1 - 4q^3 + 3q^4$ **c** $\dfrac{1}{3} < q < 1$

9 0.966 (exact answer)

10 a 0.734 (exact answer)
b About 7 (by simulation)

11 a $\dfrac{13}{8}$ **b** 3.7

12 b $\Pr(A) = 0.375$, $\Pr(B) = 0.375$, $\Pr(C) = 0.125$, $\Pr(D) = 0.125$ (exact answer)

Chapter 11

11.1 Multiple-choice questions

1 E **2** C **3** E **4** B **5** E
6 E **7** C **8** C **9** B **10** D
11 D **12** D **13** E **14** A **15** E
16 E **17** B **18** C **19** C **20** A
21 E **22** E **23** C **24** D **25** D

11.2 Extended-response questions

1 a **i** $\dfrac{15}{28}$ **ii** $\dfrac{37}{56}$ **iii** $\dfrac{43}{49}$

b **i** $\dfrac{9}{14}$ **ii** $\dfrac{135}{392}$

Answers

2 a $\frac{1}{2}$ **b** $\frac{13}{36}$

3 a $\frac{3}{8}$ **b** $\frac{1}{56}$ **c** $\frac{3}{28}$ **d** $\frac{6}{7}$

4 a 0.0027 **b** 0.12 **c** 0.17 **d** 0.72

5 a $\frac{59}{120}$ **b** $\frac{45}{59}$

6 a $\frac{167}{360}$ **b i** $\frac{108}{193}$ **ii** $\frac{45}{193}$

7 a i $\frac{1}{9}$ **ii** $\frac{5}{18}$

b i $\frac{1}{81}$ **ii** $\frac{13}{324}$

8 a i $m = 30, q = 35, s = 25$

ii $m + q = 65$

b $\frac{3}{10}$ **c** $\frac{7}{12}$

9 a 0.084 **b** 0.52 **c** 0.68

10 a 60 **b** 8 **c** 0.1

11 a $\frac{1}{60}$ **b** $\frac{1}{5}$ **c** $\frac{3}{5}$ **d** $\frac{6}{13}$

12 a i 10 000 cm^2 **ii** 400 cm^2 **iii** 6400 cm^2

b i 0.04 **ii** 0.12 **iii** 0.64

c i 0.0016 **ii** 0.000 64

13 a $\frac{7}{18}$ **b** $\frac{13}{36}$ **c** $\frac{23}{108}$

14 a i 0.328 **ii** 0.205 **iii** 0.672

b i 11 **ii** 18

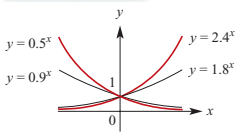
15 a i 0.121 **ii** 0.851 **iii** 0.383

b i 9 **ii** 14

Chapter 12

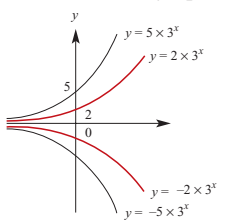
Exercise 12A

1



- All pass through (0, 1)
- base > 1, increasing
- base < 1, decreasing
- horizontal asymptote, $y = 0$

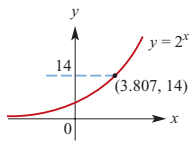
2



For $y = a \times b^x$

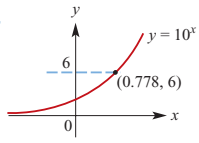
- y-axis intercept $(0, a)$
- c and d are reflections of a and b in the x-axis
- horizontal asymptote, $y = 0$

3



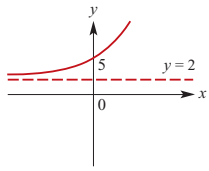
$x = 3.807$

4

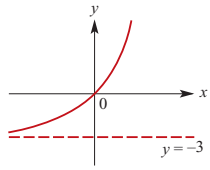


$x = 0.778$

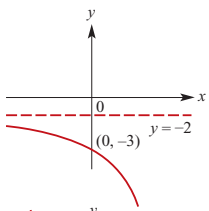
5 a



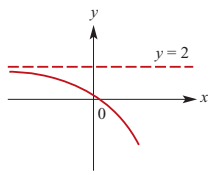
b



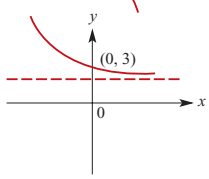
c



d



e



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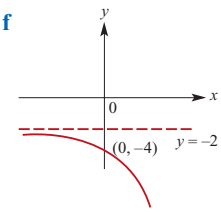
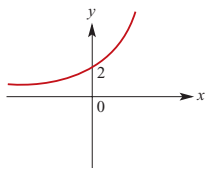
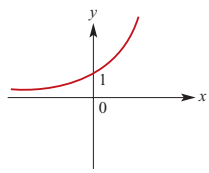


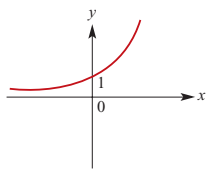
6 a



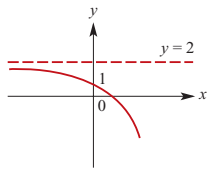
b



c



d



Exercise 12B

1 a x^5 **b** $8x^7$ **c** x^2 **d** $2x^3$ **e** a^6 **f** 2^6

g x^2y^2 **h** x^4y^6 **i** $\frac{x^3}{y^3}$ **j** $\frac{x^6}{y^4}$

2 a x^9 **b** 2^{16} **c** 3^{17} **d** q^8p^9

e $a^{11}b^3$ **f** 2^8x^{18} **g** $m^{11}n^{12}p^{-2}$ **h** $2a^5b^{-2}$

3 a x^2y^3 **b** $8a^8b^3$ **c** x^5y^2 **d** $\frac{9}{2}x^2y^3$

4 a $\frac{1}{n^4p^5}$ **b** $\frac{2x^8z}{y^4}$ **c** $\frac{b^5}{a^5}$ **d** $\frac{a^3b}{c}$

e $a^{n+2}\ b^{n+1}\ c^{n-1}$

5 a 3^{17n} **b** 2^{3-n} **c** $\frac{3^{4n-11}}{2^2}$

d $2^{n+1}3^{3n-1}$ **e** 5^{3n-2} **f** $2^{3x-3} \times 3^{-4}$

g $3^{6-n} \times 2^{-5n}$ **h** $3^3 = 27$ **i** 6

6 a $2^{12} = 4096$ **b** $5^5 = 3125$ **c** $3^3 = 27$

Exercise 12C

1 a 25 **b** 27 **c** $\frac{1}{9}$ **d** 16 **e** $\frac{1}{2}$ **f** $\frac{1}{4}$ **g** $\frac{1}{25}$
h 16 **i** $\frac{1}{10\,000}$ **j** 1000 **k** 27 **l** $\frac{3}{5}$

2 a $a^{\frac{1}{6}}b^{-\frac{7}{6}}$ **b** $a^{-6}b^{\frac{9}{2}}$ **c** $3^{-\frac{7}{3}} \times 5^{-\frac{7}{6}}$
d $\frac{1}{4}$ **e** x^6y^{-8} **f** $a^{\frac{14}{15}}$

3 a $(2x-1)^{3/2}$ **b** $(x-1)^{5/2}$ **c** $(x^2+1)^{3/2}$
d $(x-1)^{4/3}$ **e** $x(x-1)^{-\frac{1}{2}}$ **f** $(5x^2+1)^{4/3}$

Exercise 12D

1 a 3 **b** 3 **c** $\frac{1}{2}$ **d** $\frac{3}{4}$ **e** $\frac{1}{3}$ **f** 4 **g** 2 **h** 3 **i** 3

2 a 1 **b** 2 **c** $-\frac{3}{2}$ **d** $\frac{4}{3}$ **e** -1 **f** 8 **g** 3
h -4 **i** 8 **j** 4 **k** $3\frac{1}{2}$ **l** 6 **m** $7\frac{1}{2}$

3 a $\frac{4}{5}$ **b** $\frac{3}{2}$ **c** $5\frac{1}{2}$

4 a 0 **b** 0, -2 **c** 1, 2 **d** 0, 1

5 a 2.32 **b** 1.29 **c** 1.26 **d** 1.75

6 a $x > 2$ **b** $x > \frac{1}{3}$ **c** $x \leq \frac{1}{2}$ **d** $x < 3$
e $x < \frac{3}{4}$ **f** $x > 1$ **g** $x \leq 3$

Exercise 12E

1 a $\log_2(10a)$ **b** 1 **c** $\log_2\left(\frac{9}{4}\right)$ **d** 1
e $-\log_5 6$ **f** -2 **g** $3\log_2 a$ **h** 9

2 a 3 **b** 4 **c** -7 **d** -3 **e** 4 **f** -3 **g** 4
h -6 **i** -9 **j** -1 **k** 4 **l** -2

3 a 2 **b** 7 **c** 9 **d** 1 **e** $\frac{5}{2}$
f $\log_x a^5$ **g** 3 **h** 1

4 a 2 **b** 27 **c** $\frac{1}{125}$ **d** 8 **e** 30
f $\frac{2}{3}$ **g** 8 **h** 64 **i** 4 **j** 10

5 a 5 **b** 32.5 **c** 22 **d** 20
e $\frac{3+\sqrt{17}}{2}$ **f** 3 *or* 0

6 $2+3a-\frac{5c}{2}$ **8** 10

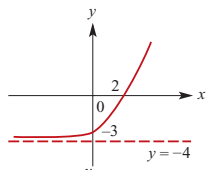
9 a 4 **b** $\frac{6}{5}$ **c** 3 **d** 10 **e** 9 **f** 2

Exercise 12F

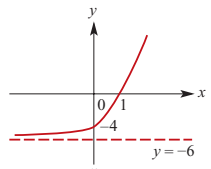
1 a 2.81 **b** -1.32 **c** 2.40 **d** 0.79 **e** -2.58
f -0.58 **g** -4.30 **h** -1.38 **i** 3.10 **j** -0.68

2 a $x > 3$ **b** $x < 1.46$ **c** $x < -1.15$
d $x \leq 2.77$ **e** $x \geq 1.31$

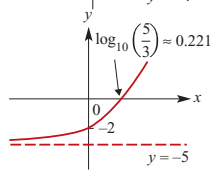
3 a



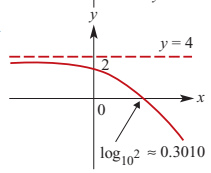
b



c



d



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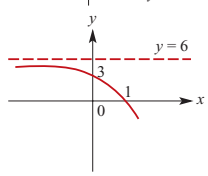

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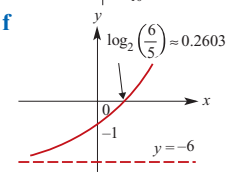
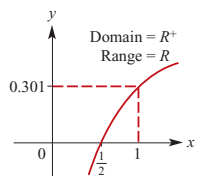


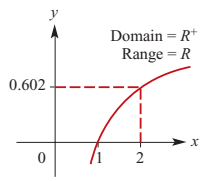
4 $d_0 = 41.88$, $m = 0.094$

Exercise 12G

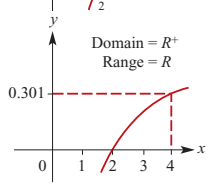
1 a



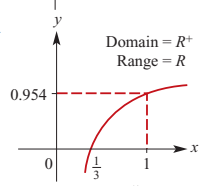
b



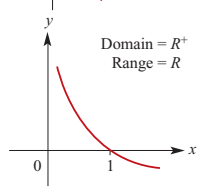
c



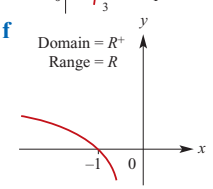
d



e



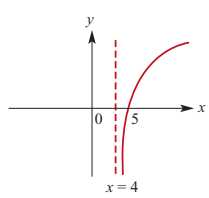
f



2 a $y = 2\log_{10} x$ **b** $y = 10^{\frac{1}{3}x}$
c $y = \frac{1}{3}\log_{10} x$ **d** $y = \frac{1}{3}10^{\frac{1}{2}x}$

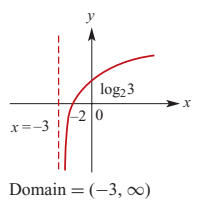
3 a $y = \log_3(x-2)$ **b** $y = 2^x + 3$
c $y = \log_3\left(\frac{x-2}{4}\right)$ **d** $y = \log_5(x+2)$
e $y = \frac{1}{3} \times 2^x$ **f** $y = 3 \times 2^x$
g $y = 2^x - 3$ **h** $y = \log_3\left(\frac{x+2}{5}\right)$

4 a



Domain $= (4, \infty)$

b

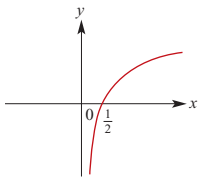


Domain $= (-3, \infty)$

Answers

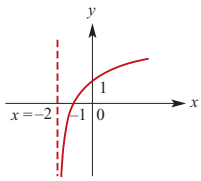
Answers

c



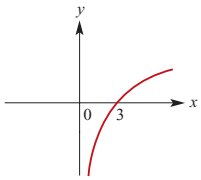
Domain = $(0, \infty)$

d



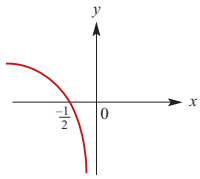
Domain = $(-2, \infty)$

e



Domain = $(0, \infty)$

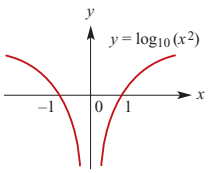
f

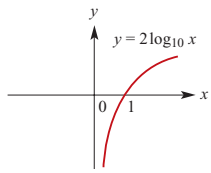


Domain = $(-\infty, 0)$

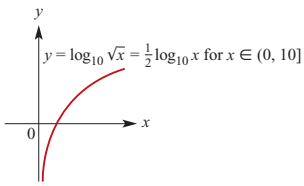
5 a 0.64 **b** 0.40

6

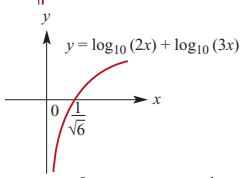


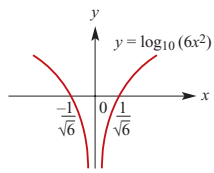


7



8





9 $a = \dfrac{6}{\left(\frac{10}{3}\right)^{\frac{2}{3}}}$ and $k = \dfrac{1}{3}\log_{10}\left(\frac{10}{3}\right)$

Exercise 12H

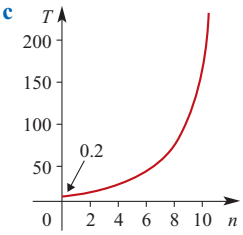
1 $y = 1.5 \times 0.575^x$ **2** $p = 2.5 \times 1.35^t$

3 a

Cuts, n	Sheets	Total thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	1.6
4	16	3.2
5	32	6.4
6	64	12.8
7	128	25.6
8	256	51.2
9	512	102.4
10	1024	204.8

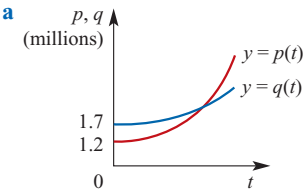
b $T = 0.2(2)^n$

c



d 214 748.4 m

4 a



b i $t = 12.56\ldots$ (mid 1962)
ii $t = 37.56\ldots$ (mid 1987)

Multiple-choice questions

1 C **2** A **3** C **4** C **5** A
6 B **7** A **8** A **9** A **10** A

Short-answer questions (technology-free)

1 a a^4 **b** $\dfrac{1}{b^2}$ **c** $\dfrac{1}{m^2n^2}$
d $\dfrac{1}{ab^6}$ **e** $\dfrac{3a^6}{2}$ **f** $\dfrac{5}{3a^2}$
g a^3 **h** $\dfrac{n^8}{m^4}$ **i** $\dfrac{1}{p^2q^4}$
j $\dfrac{8}{5a^{11}}$ **k** $2a$ **l** $a^2 + a^6$

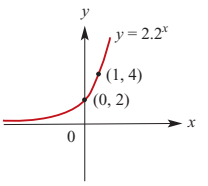
2 a $\log_2 7$ **b** $\dfrac{1}{2}\log_2 7$ **c** $\log_{10} 2$
d $\log_{10}\left(\dfrac{7}{2}\right)$ **e** $1 + \log_{10} 11$ **f** $1 + \log_{10} 101$
g $\dfrac{1}{5}\log_2 100$ **h** $-\log_2 10$

3 a 6 **b** 7 **c** 2 **d** 0
e 3 **f** −2 **g** −3 **h** 4

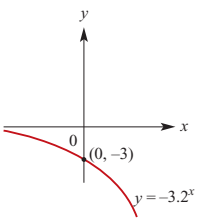
4 a $\log_{10} 6$ **b** $\log_{10} 6$ **c** $\log_{10}\left(\dfrac{a^2}{b}\right)$
d $\log_{10}\left(\dfrac{a^2}{25\,000}\right)$ **e** $\log_{10} y$ **f** $\log_{10}\left(\dfrac{a^2b^3}{c}\right)$

5 a $x = 3$ **b** $x = 3$ or $x = 0$
c $x = 1$ **d** $x = 2$ or $x = 3$

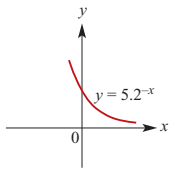
6 a



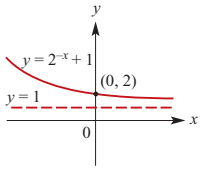
b



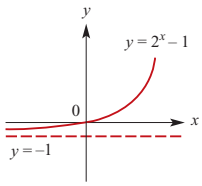
c



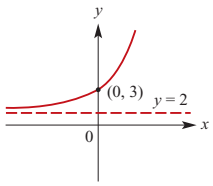
d



e



f



7 a $x = 1$

9 $x = 3$ **10 a** $k = \frac{1}{7}$ **b** $q = \frac{3}{2}$

11 a $a = \frac{1}{2}$ **b** $y = -4$ or $y = 20$

Extended-response questions

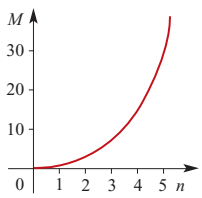
1 a

n	0	1	2	3	4
M	0	1	3	7	15

b $M = 2^n - 1$

n	5	6	7
M	31	63	127

c



d

Three discs	1	2	3
Times moved	4	2	1

Four discs	1	2	3	4
Times moved	8	4	2	1

2 $n = 2$

3 a $\left(\frac{1}{2}\right)^{3n}$ **b** $\left(\frac{1}{2}\right)^{5n-2}$ **c** $n = 3$

4 a $729\left(\frac{1}{4}\right)^n$ **b** $128\left(\frac{1}{2}\right)^n$ **c** 4 times

5 a Batch 1 $= 15(0.95)^n$ Batch 2 $= 20(0.94)^n$

b 32 years

6 a X \$1.82 Y \$1.51 Z \$2.62

b X \$4.37 Y \$4.27 Z \$3.47

c Intersect at $t = 21.784\ldots$ and $t = 2.090\ldots$ therefore February 1997 until September 1998

d February 1998 until September 1998, approximately 8 months.

7 a 13.81 years **b** 7.38 years

8 a temperature $= 87.065 \times 0.94^t$

b i 87.1° **ii** 18.56°

c temperature $= 85.724 \times 0.94^t$

d i 85.72° **ii** 40.82°

e 28.19 minutes

9 a $a = 0.2$ and $b = 5$

b i $z = x \log_{10} b$ **ii** $a = 0.2$ and $k = \log_{10} 5$

10 a $y = 2 \times 1.585^x$ **b** $y = 2 \times 10^{0.2x}$

c $x = 5 \log_{10}\left(\frac{y}{2}\right)$

Chapter 13

Exercise 13A

1 a $\frac{\pi}{3}$ **b** $\frac{4\pi}{5}$ **c** $\frac{4\pi}{3}$ **d** $\frac{11\pi}{6}$ **e** $\frac{7\pi}{3}$ **f** $\frac{8\pi}{3}$

2 a 120° **b** 150° **c** 210° **d** 162°
e 100° **f** 324° **g** 220° **h** 324°

3 a 34.38° **b** 108.29° **c** 166.16° **d** 246.94°
e 213.14° **f** 296.79° **g** 271.01° **h** 343.77°

4 a 0.66 **b** 1.27 **c** 1.87 **d** 2.81
e 1.47 **f** 3.98 **g** 2.39 **h** 5.74

5 a −60° **b** −720° **c** −540° **d** −180°
e 300° **f** −330° **g** 690° **h** −690°

6 a -2π **b** -3π **c** $-\frac{4\pi}{3}$
d -4π **e** $-\frac{11\pi}{6}$ **f** $-\frac{7\pi}{6}$

Exercise 13B

1 a 0, 1 **b** −1, 0 **c** 1, 0 **d** 1, 0
e 0, −1 **f** 1, 0 **g** −1, 0 **h** 0, 1

2 a 0.95 **b** 0.75 **c** −0.82 **d** 0.96
e −0.5 **f** −0.03 **g** −0.86 **h** 0.61

3 a 0; −1 **b** −1; 0 **c** −1; 0 **d** −1; 0
e −1; 0 **f** 0; −1 **g** 0; −1 **h** 0; −1

Exercise 13C

1 a 0 **b** 0 **c** undefined
d 0 **e** undefined **f** undefined

2 a −34.23 **b** −2.57 **c** −0.97
d −1.38 **e** 0.95 **f** 0.75 **g** 1.66

3 a 0 **b** 0 **c** 0 **d** 0 **e** 0 **f** 0

Exercise 13D

1 a 67°59′ **b** 4.5315 **c** 2.5357
d 6.4279 **e** 50°12′ **f** 3.4202
g 2.3315 **h** 6.5778 **i** 6.5270

2 a $a = 0.7660$, $b = 0.6428$

b $c = -0.7660$, $d = 0.6428$

c i $\cos 140° = -0.76604$, $\sin 140° = 0.6428$

ii $\cos 140° = -\cos 40°$

Exercise 13E

1 a -0.42 **b** -0.7 **c** -0.42 **d** -0.38
e 0.42 **f** -0.38 **g** -0.7 **h** 0.7

2 a $120°$ **b** $240°$ **c** $-60°$
d $120°$ **e** $240°$ **f** $300°$

3 a $\frac{5\pi}{6}$ **b** $\frac{7\pi}{6}$ **c** $\frac{11\pi}{6}$

4 a $a = -\frac{1}{2}$ **b** $b = \frac{\sqrt{3}}{2}$ **c** $c = \frac{1}{2}$
d $d = \frac{-\sqrt{3}}{2}$ **e** $\tan(\pi - \theta) = -\sqrt{3}$
f $\tan(-\theta) = -\sqrt{3}$

5 a $-\frac{\sqrt{3}}{2}$ **b** $\frac{1}{2}$ **c** $-\sqrt{3}$ **d** $-\frac{\sqrt{3}}{2}$ **e** $-\frac{1}{2}$

6 a -0.7 **b** -0.6 **c** -0.4 **d** -0.6
e -0.7 **f** -0.7 **g** 0.4 **h** 0.6

Exercise 13F

1 a $\sin = \frac{\sqrt{3}}{2}, \cos = -\frac{1}{2}, \tan = -\sqrt{3}$

b $\sin = \frac{1}{\sqrt{2}}, \cos = -\frac{1}{\sqrt{2}}, \tan = -1$

c $\sin = -\frac{1}{2}, \cos = -\frac{\sqrt{3}}{2}, \tan = \frac{1}{\sqrt{3}}$

d $\sin = -\frac{\sqrt{3}}{2}, \cos = -\frac{1}{2}, \tan = \sqrt{3}$

e $\sin = -\frac{1}{\sqrt{2}}, \cos = \frac{1}{\sqrt{2}}, \tan = -1$

f $\sin = \frac{1}{2}, \cos = \frac{\sqrt{3}}{2}, \tan = \frac{1}{\sqrt{3}}$

g $\sin = \frac{\sqrt{3}}{2}, \cos = \frac{1}{2}, \tan = \sqrt{3}$

h $\sin = -\frac{1}{\sqrt{2}}, \cos = -\frac{1}{\sqrt{2}}, \tan = 1$

i $\sin = \frac{\sqrt{3}}{2}, \cos = \frac{1}{2}, \tan = \sqrt{3}$

j $\sin = -\frac{\sqrt{3}}{2}, \cos = \frac{1}{2}, \tan = -\sqrt{3}$

2 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $-\frac{1}{\sqrt{3}}$ **d** $-\frac{1}{2}$ **e** $-\frac{1}{\sqrt{2}}$
f $\sqrt{3}$ **g** $-\frac{\sqrt{3}}{2}$ **h** $\frac{1}{\sqrt{2}}$ **i** $-\frac{1}{\sqrt{3}}$

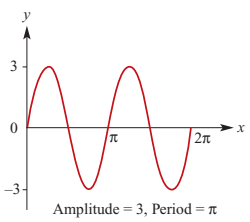
3 a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $\frac{1}{\sqrt{3}}$ **d** not defined
e 0 **f** $-\frac{1}{\sqrt{2}}$ **g** $\frac{1}{\sqrt{2}}$ **h** -1

Exercise 13G

1

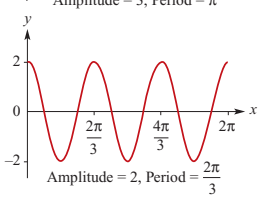
	Period	Amplitude
a	2π	2
b	π	3
c	$\frac{2\pi}{3}$	$\frac{1}{2}$
d	4π	3
e	$\frac{2\pi}{3}$	4
f	$\frac{\pi}{2}$	$\frac{1}{2}$
g	4π	2
h	2	2
i	4	3

2 a



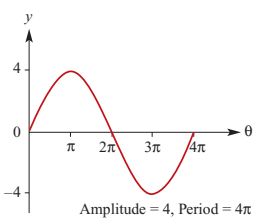
Amplitude = 3, Period = π

b



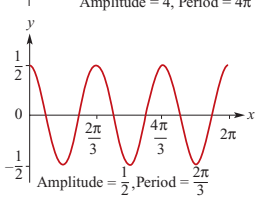
Amplitude = 2, Period = $\frac{2\pi}{3}$

c



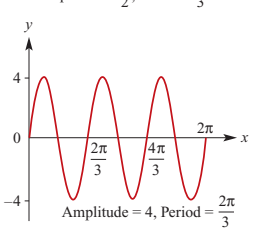
Amplitude = 4, Period = 4π

d



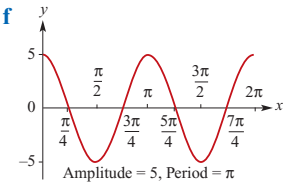
Amplitude = $\frac{1}{2}$, Period = $\frac{2\pi}{3}$

e



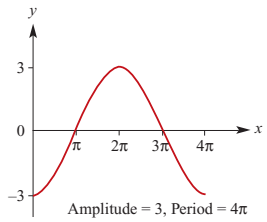
Amplitude = 4, Period = $\frac{2\pi}{3}$

f



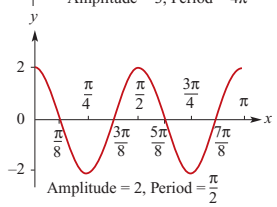
Amplitude = 5, Period = π

g



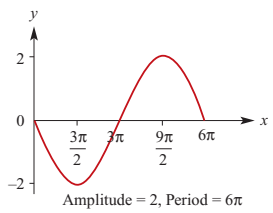
Amplitude = 3, Period = 4π

h



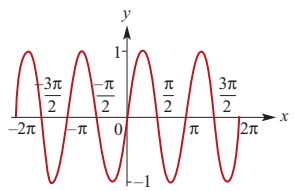
Amplitude = 2, Period = $\frac{\pi}{2}$

i

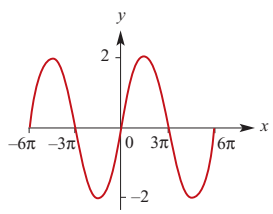


Amplitude = 2, Period = 6π

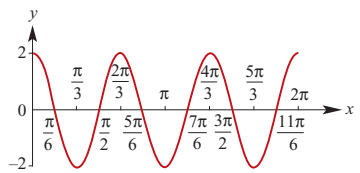
3 a



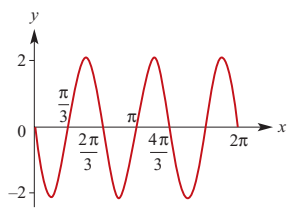
b



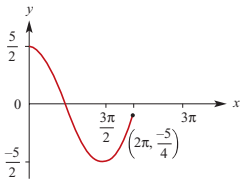
c



d



4



5 a dilation of factor 3 from the x-axis
amplitude $= 3$, period $= 2\pi$

b dilation of factor $\frac{1}{5}$ from the y-axis
amplitude $= 1$, period $= \dfrac{2\pi}{5}$

c dilation of factor 3 from the y-axis
amplitude $= 1$, period $= 6\pi$

d dilation of factor 2 from the x-axis
dilation of factor $\frac{1}{5}$ from the y-axis
amplitude $= 2$, period $= \dfrac{2\pi}{5}$

e dilation of factor $\frac{1}{5}$ from the y-axis
reflection in the x-axis
amplitude $= 1$, period $= \dfrac{2\pi}{5}$

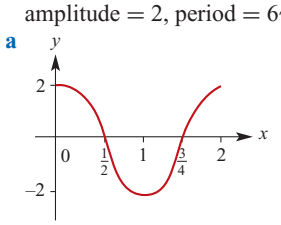
f reflection in the y-axis
amplitude $= 1$, period $= 2\pi$

g dilation of factor 3 from the y-axis
dilation of factor 2 from the x-axis
amplitude $= 2$, period $= 6\pi$

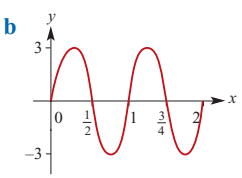
h dilation of factor 2 from the y-axis
dilation of factor 4 from the x-axis
reflection in the x-axis
amplitude $= 4$, period $= 4\pi$

i dilation of factor 3 from the y-axis
dilation of factor 2 from the x-axis
reflection in the y-axis
amplitude $= 2$, period $= 6\pi$

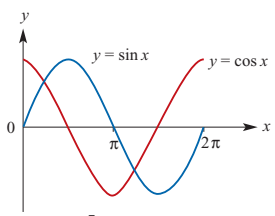
6 a



b



7

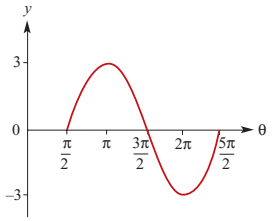


b $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$

Answers

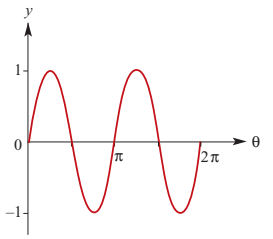
Exercise 13H

1 a



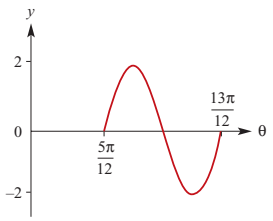
Period = 2π, Amplitude = 3, $y = \pm 3$

b



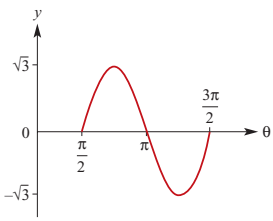
Period = π, Amplitude = 1, $y = \pm 1$

c



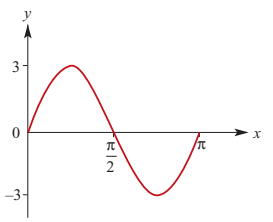
Period = $\frac{2\pi}{3}$, Amplitude = 2, $y = \pm 2$

d



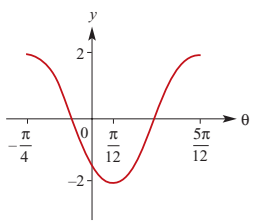
Period = π, Amplitude = $\sqrt{3}$, $y = \pm\sqrt{3}$

e



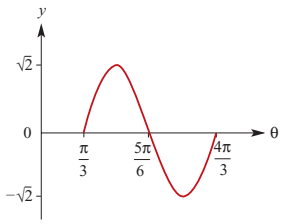
Period = π, Amplitude = 3, $y = -3, 3$

f



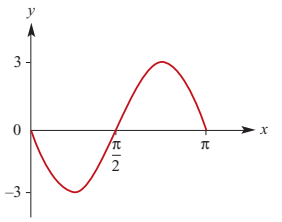
Period = $\frac{2\pi}{3}$, Amplitude = 2, $y = -2, 2$

g



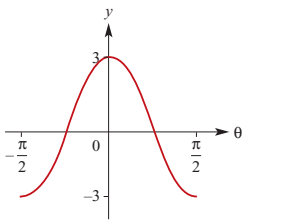
Period = π, Amplitude = $\sqrt{2}$, $y = -\sqrt{2}, \sqrt{2}$

h



Period = π, Amplitude = 3, $y = -3, 3$

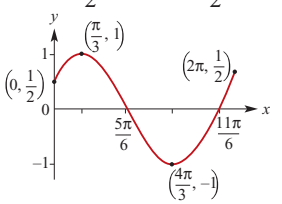
i



Period = π, Amplitude = 3, $y = 3, -3$

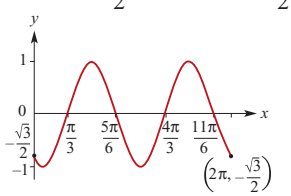
2 a $f(0) = \frac{1}{2}$ $f(2\pi) = \frac{1}{2}$

b



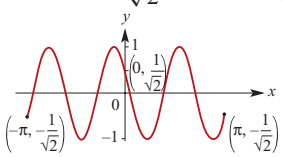
3 a $f(0) = -\frac{\sqrt{3}}{2}$ $f(2\pi) = -\frac{\sqrt{3}}{2}$

b



4 a $f(-\pi) = -\frac{1}{\sqrt{2}}$ $f(\pi) = -\frac{1}{\sqrt{2}}$

b



5 a $y = 3\sin\frac{x}{2}$ **b** $y = 3\sin 2x$

c $y = 2\sin\frac{x}{3}$ **d** $y = \sin 2\left(x - \frac{\pi}{3}\right)$

e $y = \sin\frac{1}{2}\left(x + \frac{\pi}{3}\right)$

Exercise 13I

1 a $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ **b** $\frac{\pi}{4}$ and $\frac{7\pi}{4}$

2 a 0.93 and 2.21 **b** 4.30 and 1.98
c 3.50 and 5.93 **d** 0.41 and 2.73
e 2.35 and 3.94 **f** 1.77 and 4.51

3 a 150 and 210 **b** 30 and 150 **c** 120 and 240
d 120 and 240 **e** 60 and 120 **f** 45 and 135

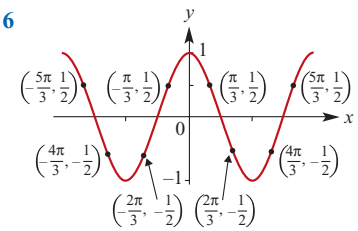
4 a 0.64, 2.498, 6.93, 8.781

b $\frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$

c $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$

5 a $\frac{3\pi}{4}, -\frac{3\pi}{4}$ **b** $\frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\frac{2\pi}{3}, -\frac{2\pi}{3}$

6



7 a $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

b $\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

c $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

d $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$

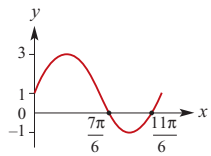
e $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

f $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

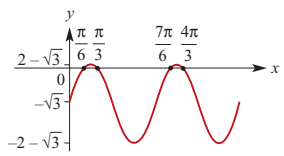
8 a 2.034, 2.678, 5.176, 5.820
b 1.892, 2.820, 5.034, 5.961
c 0.580, 2.562, 3.721, 5.704
d 0.309, 1.785, 2.403, 3.880, 4.498, 5.974

Exercise 13J

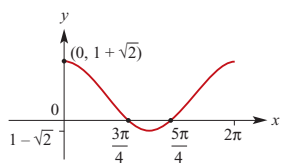
1 a



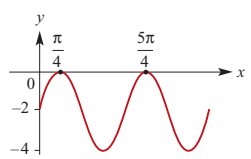
b



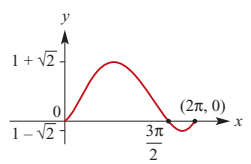
c



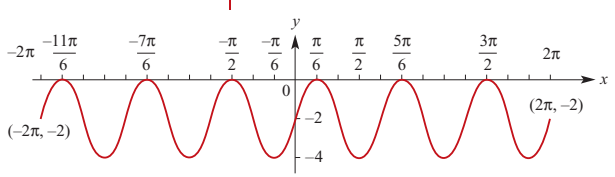
d



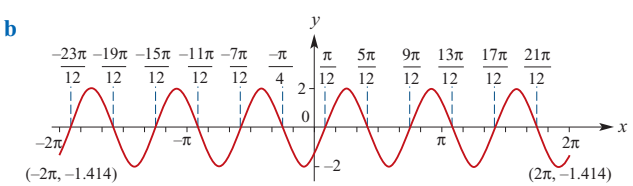
e



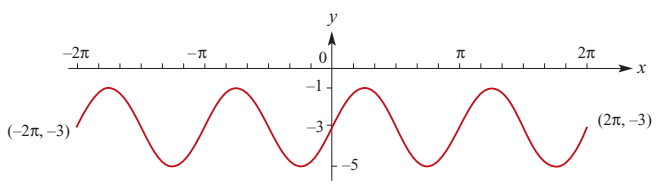
2 a



b

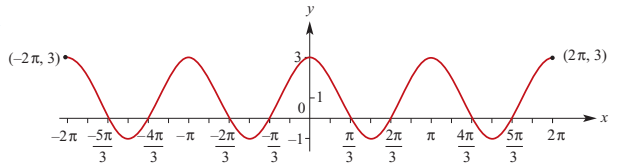


c

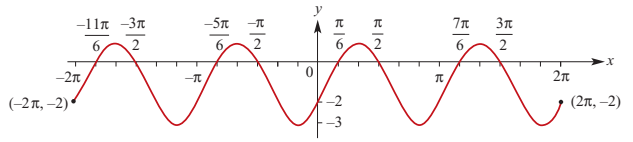


Answers

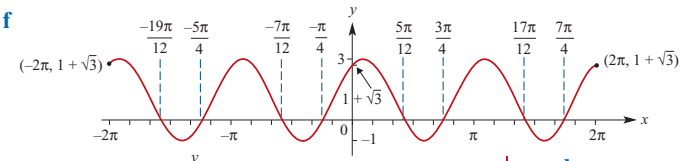
d



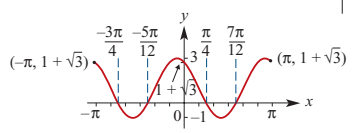
e



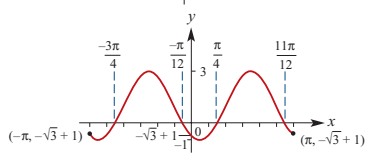
f



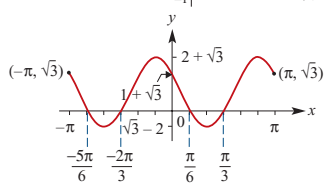
3 a



b



c



Exercise 13K

1 a 0.6 **b** 0.6 **c** −0.7 **d** 0.3 **e** −0.3
f $\frac{10}{7}(1.49)$ **g** −0.3 **h** 0.6 **i** −0.6 **j** −0.3

2 a $\frac{\pi}{3}$ **b** $\frac{\pi}{3}$ **c** $\frac{5\pi}{12}$ **d** $\frac{\pi}{14}$

3 $\sin x = \frac{-4}{5}$ and $\tan x = \frac{-4}{3}$

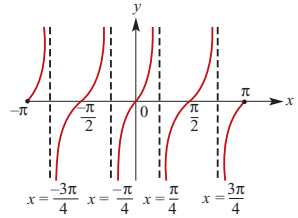
4 $\cos x = \frac{-12}{13}$ and $\tan x = \frac{-5}{12}$

5 $\sin x = \frac{-2\sqrt{6}}{5}$ and $\tan x = -2\sqrt{6}$

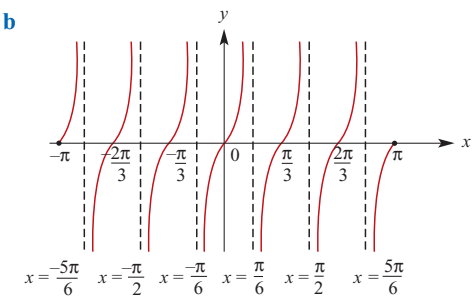
Exercise 13L

1 a $\frac{\pi}{4}$ **b** $\frac{3\pi}{2}$ **c** $\frac{\pi}{2}$

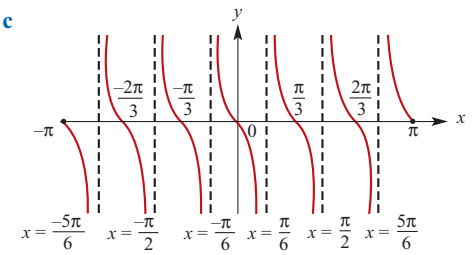
2 a



b



c



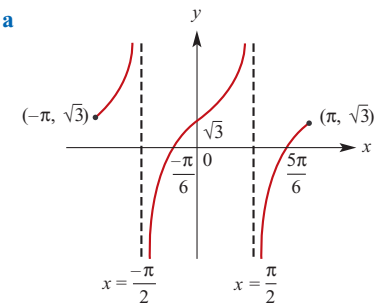
3 a $\frac{-7\pi}{8}, \frac{-3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$

b $\frac{-17\pi}{18}, \frac{-11\pi}{18}, \frac{-5\pi}{18}, \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$

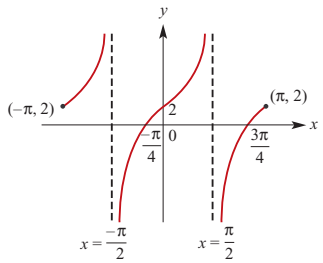
c $\frac{-5\pi}{6}, \frac{-\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

d $\frac{-13\pi}{18}, \frac{-7\pi}{18}, \frac{-\pi}{18}, \frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}$

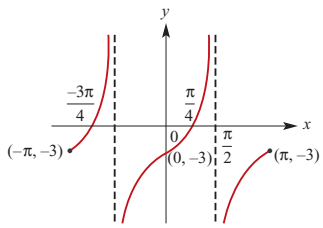
4 a



b



c



Exercise 13M

1 a 0.74 **b** 0.51
c 0.82 or −0.82 **d** 0 or 0.88

2 $y = a\sin(b\theta + c) + d$

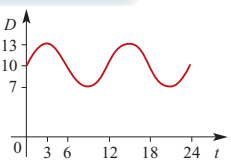
a $a = 1.993$ $b = 2.998$ $c = 0.003$
$d = 0.993$

b $a = 3.136$ $b = 3.051$ $c = 0.044$
$d = -0.140$

c $a = 4.971$ $b = 3.010$ $c = 3.136$
$d = 4.971$

Exercise 13N

1 a

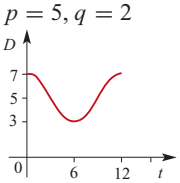


b $\{t : D(t) \geq 8.5\} = \{t : 0 \leq t \leq 7\} \cup \{t : 11 \leq t \leq 19\} \cup \{t : 23 \leq t \leq 24\}$

c 12.9 m

2 a $p = 5, q = 2$

b



c A ship can enter 2 hours after low tide.

3 a 5 **b** 1

c $t = 0.524$ s, 2.618 s, 4.712 s

d $t = 0$ s, 1.047 s, 2.094 s

e Particle oscillates about the point $x = 3$ from $x = 1$ to $x = 5$.

Multiple-choice questions

1 C **2** D **3** E **4** C **5** E
6 D **7** E **8** E **9** C **10** B

Short-answer questions (technology-free)

1 a $\frac{11\pi}{6}$ **b** $\frac{9\pi}{2}$ **c** 6π **d** $\frac{23\pi}{4}$ **e** $\frac{3\pi}{4}$
f $\frac{9\pi}{4}$ **g** $\frac{13\pi}{6}$ **h** $\frac{7\pi}{3}$ **i** $\frac{4\pi}{9}$

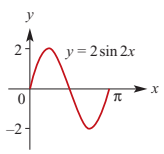
2 a 150° **b** 315° **c** 495° **d** 45° **e** 1350°
f −135° **g** −45° **h** −495° **i** −1035°

3 a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{2}}$ **c** $-\frac{1}{2}$ **d** $-\frac{\sqrt{3}}{2}$
e $\frac{\sqrt{3}}{2}$ **f** $-\frac{1}{2}$ **g** $\frac{1}{2}$ **h** $-\frac{1}{\sqrt{2}}$

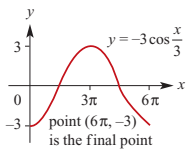
4

	Amplitude	Period
a	2	4π
b	3	$\frac{\pi}{2}$
c	$\frac{1}{2}$	$\frac{2\pi}{3}$
d	3	π
e	4	6π
f	$\frac{2}{3}$	3π

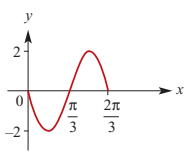
5 a



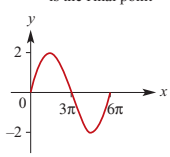
b



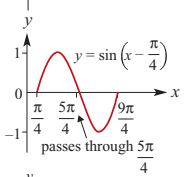
c



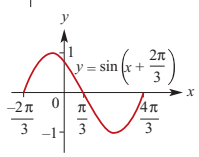
d



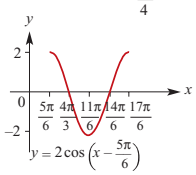
e



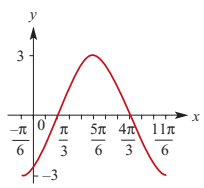
f



g



h



6 a $-\frac{2\pi}{3}, -\frac{\pi}{3}$

b $-\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$

c $\frac{\pi}{6}, \frac{3\pi}{2}$ **d** $\frac{7\pi}{6}$ **e** $\frac{\pi}{2}, \frac{7\pi}{6}$

Answers

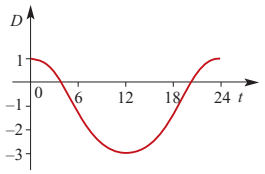
Answers

Extended-response questions

1 a **i** 1.83×10^{-3} hours
ii 11.79 hours
b 26 April ($t = 3.86$) 14 August ($t = 7.48$)

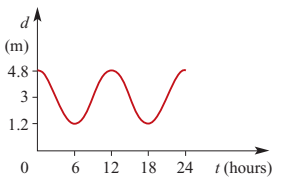
2 a 19.5°C **b** $D = -1 + 2\cos\left(\frac{\pi t}{12}\right)$

c



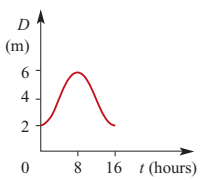
d $\{t : 4 < t < 20\}$

3 a



b 3.00 am 3.00 pm 3.00 am
c 9.00 am 9.00 pm **d** 10.03 am
e **i** 6.12 pm **ii** 5 trips

4 b



c $t = 16$ (8.00 pm)
d $t = 4$ and $t = 12$ (8.00 am and 4.00 pm) depth is 4 m
e **i** 1.5 m **ii** 2.086 m
f 9 hours 17 minutes

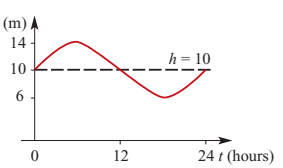
Chapter 14

14.1 Multiple-choice questions

1 B	**2** B	**3** B	**4** E	**5** D	**6** A
7 D	**8** C	**9** B	**10** A	**11** A	**12** D
13 A	**14** D	**15** D	**16** D	**17** A	**18** E
19 D	**20** D	**21** E	**22** A	**23** E	**24** B
25 D	**26** B				

14.2 Extended-response questions

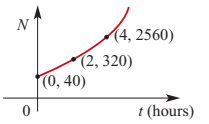
1 a



b $t = 3.2393$ and $t = 8.7606$
c The boat can leave the harbour for $t \in [0.9652, 11.0348]$

2 a 40 bacteria
b **i** 320 bacteria **ii** 2560 bacteria
iii 10 485 760 bacteria

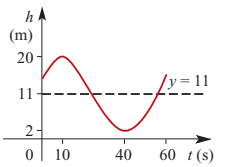
c



d 40 minutes, $\left(= \frac{2}{3} \text{ hours}\right)$

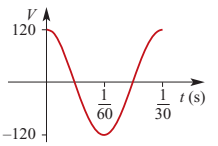
3 a 60 seconds

b



c [2, 20]
d After 40 seconds and they are at this height every 60 seconds after they first attain this height.
e At $t = 0$, $t = 20$ and $t = 60$ for $t \in [0, 60]$

4 a

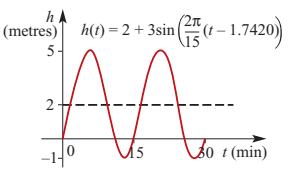


b $t = \frac{1}{180}$ s **c** $t = \frac{k}{30}$ s, $k = 0, 1, 2$

5 a **i** Period = 15 seconds
ii amplitude = 3 **iii** $c = \frac{2\pi}{15}$

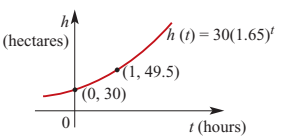
b $h = 1.74202$

c



6 a **i** 30 **ii** 49.5 **iii** 81.675
b 1.65 **c** 6.792

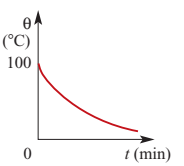
d



7 a

t	0	1	2	3	4	5
θ	100	60	40	30	25	22.5

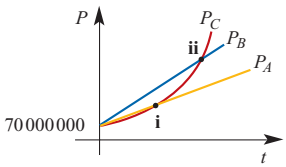
b



c 1 minute **d** 27.071

8 a $P_A = 70\,000\,000 + 3\,000\,000t$
$P_B = 70\,000\,000 + 5\,000\,000t$
$P_C = 70\,000\,000 \times 1.3^{\frac{t}{10}}$

b



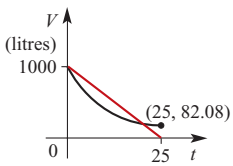
c **i** 35 years **ii** 67 years

9 a **i** 4 billion **ii** 5.944 billion **iii** 7.25 billion

b 2032

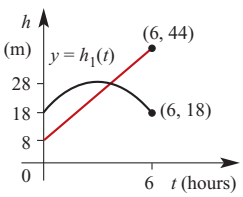
10 a $V_1(0) = V_2(0) = 1000$

b



c 82.08 litres **d** $t = 0$ and $t = 22.32$

11 a



b $t = 3.31$ Approximately 3.19 am (correct to nearest minute)

c **i** 9.00 am

d $8 + 6t$ metres

Chapter 15

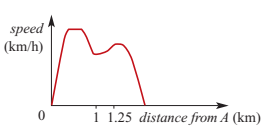
Exercise 15A

Note: For questions 1–4 there may not be a single correct answer.

1 C and D are the most likely. Scales should come into your discussion.

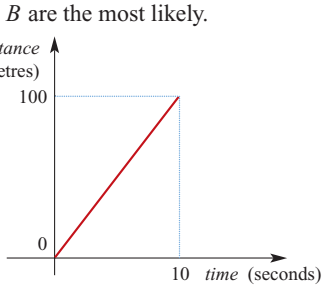
2

3

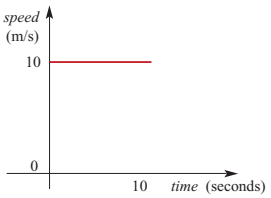


4 C or B are the most likely.

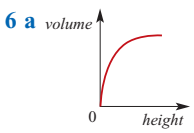
5 a



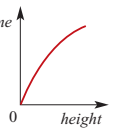
b



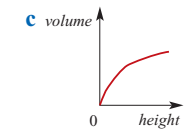
6 a



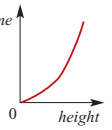
b



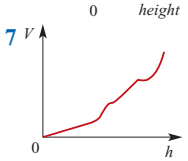
c



d



7



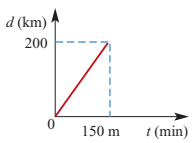
8 D **9** C

10 a $[-7, -4) \cup (0, 3]$ **b** $[-7, -4) \cup (0, 3]$

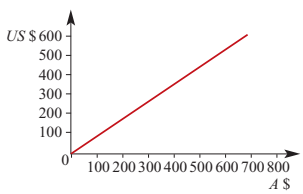
11 a $[-5, -3) \cup (0, 2]$ **c** $[-5, -3) \cup (0, 2]$

Exercise 15B

1 $\frac{4}{3}$ km/min = 80 km/h



2



3 a 60 km/h **b** 3 m/s

c 400 m/min = 24 km/h = $6\frac{2}{3}$ m/s

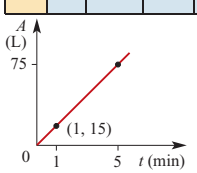
d 35.29 km/h **e** 20.44 m/s

4 a 8 litres/minute **b** 50 litres/minute

c $\frac{200}{17}$ litres/min **d** $\frac{135}{13}$ litres/min

5

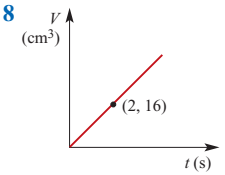
t	0	0.5	1	1.5	2	3	4	5
A	0	7.5	15	22.5	30	45	60	75



Answers

6 $\frac{\$200}{13}$ per hour = \$15.38 per hour

7 $208\frac{1}{3}$ m/s **8**



Exercise 15C

1 a 2 **b** 7 **c** $\frac{-1}{2}$ **d** $\frac{1-\sqrt{5}}{4}$

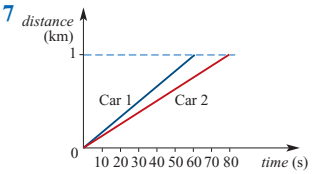
2 a $\frac{-25}{7}$ **b** $\frac{-18}{7}$ **c** 4 **d** $\frac{4b}{3a}$

3 a 4 m/s **b** 32 m/s

4 a \$2450.09 **b** \$150.03 per year

5 3.125 cm/min **6** C

7



Exercise 15D

1 a $\frac{1}{3}$ kg/month (answers will vary)

b $\frac{1}{2}$ kg/month (answers will vary)

c $\frac{1}{5}$ kg/month (answers will vary)

2 a ≈ 0.004 m³/s (answers will vary)

b ≈ 0.01 m³/s (answers will vary)

c ≈ 0.003 m³/s (answers will vary)

3 a $\frac{1}{80} = 0.0125$ litres/kg m

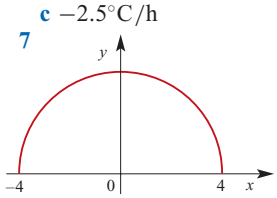
b $\frac{1}{60} \approx 0.0167$ litres/kg m

4 a $\approx$ 8 years **b** $\approx$ 7 cm/year

5 a 25°C at 1600 hours

b $\approx$ 3°C/h **c** −2.5°C/h

6 −0.5952 **7**



a 0 **b** −0.6 **c** −1.1

8 a 4

9 a 16 m³/min **b** 10 m³/min

10 a 18 million/min **b** 8.3 million/min

11 a 620 m³/min flowing out

b 4440 m³/min flowing out

c 284 000 m³/min flowing out

12 7.19

13 a 7 **b** 9 **c** 2 **d** 35

14 28 **b** 12

15 a 10 **b** 4

16 a i $\frac{2}{\pi} \approx 0.637$ **ii** $\frac{2\sqrt{2}}{\pi} \approx 0.9003$

iii 0.959 **iv** 0.998

b 1

17 a i 9 **ii** 4.3246 **iii** 2.5893 **iv** 2.3293

b 2.30

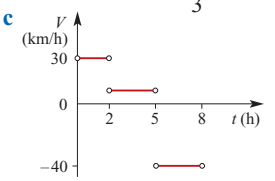
Exercise 15E

1 a 4 m/s **b** 1.12 m/s **c** 0 m/s

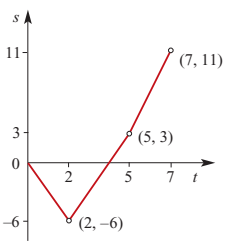
d $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ **e** (−1, 1)

2 a i 30 km/h **ii** $\frac{20}{3}$ km/h **iii** −40 km/h

c



3



4 a *C* **b** *A* **c** *B*

5 a +ve slowing down **b** +ve speeding up

c −ve slowing down **d** −ve speeding up

6 a gradually increasing speed

b constant speed (holds speed attained at **a**)

c final speeding up to finishing line

7 a $t = 6$ **b** 15 m/s **c** 17.5 m/s

d 20 m/s **e** −10 m/s **f** −20 m/s

8 a $t = 2.5$ **b** $0 \leq t < 2.5$

c 6 m **d** 5 seconds **e** 3 m/s

9 a 11 m/s **b** 15 m **c** 1 s **d** 2.8 s **e** 15 m/s

10 a $t = 2, t = 3, t = 8$ **b** $0 < t < 2.5$ and $t > 6$

c $t = 2.5$ and $t = 6$

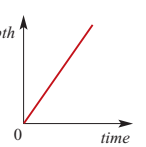
Multiple-choice questions

1 C **2** B **3** D **4** E **5** D

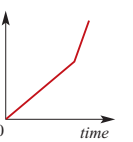
6 B **7** C **8** E **9** A **10** A

Short-answer questions (technology-free)

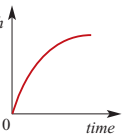
1 a



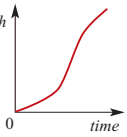
b



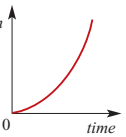
c



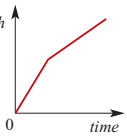
d



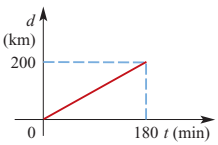
e



f

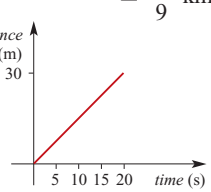


2 a

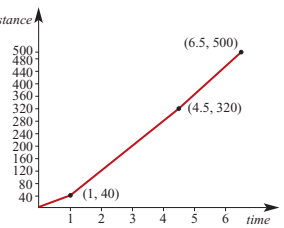


$$\text{constant speed} = \frac{200}{3}\text{ km/h} = \frac{200}{180}\text{ km/min} = \frac{10}{9}\text{ km/min}$$

b



c



3 36 cm^2/cm

4 a 1 **b** 13

5 a −2 m/s **b** −12.26 m/s **c** −14 m/s

Extended-response questions

1 a Yes, the relation is linear.

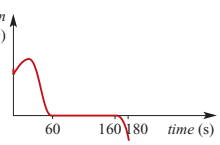
b 0.05 ohm/°C

2 a **i** 9.8 m/s **ii** 29.4 m/s

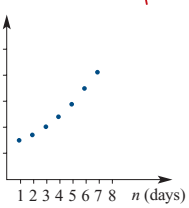
b **i** $4.9(8h - h^2)$ **ii** $4.9(8 - h)$ **iii** 39.2 m/s

3 a **i** $\frac{1}{4}$ m/s^2 **ii** 0.35 m/s^2

b



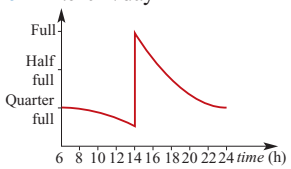
4 a



b gradient $= 5\frac{1}{4}$; Average rate of growth of the watermelon is $5\frac{1}{4}$ cm/day

c ≈ 4.5 cm/day

5



6 a $b + a$ $(a \neq b)$ **b** 3 **c** 4.01

7 a $2\frac{2}{3}$, $1\frac{3}{5}$; gradient $= -1\frac{1}{15}$

b 2.1053, 1.9048; gradient $= -1.003$

c −1.000 025 **d** −1.000 000 3 **e** gradient is −1

8 6

9 a $\approx 3\frac{1}{3}$ kg/year **b** ≈ 4.4 kg/year

c $\{t : 0 < t < 5\} \cup \{t : 10 < t < 12\}$

d $\{t : 5 < t < 7\} \cup \{t : 11 < t < 17\frac{1}{2}\}$

10 a **i** 2.5×10^8 **ii** 5×10^8

b 0.007 billion/year

c **i** 0.004 billion/year **ii** 0.015 billion/year

d 25 years after 2020

11 a **i** 1049.1 **ii** 1164.3 **iii** 1297.7 **iv** 1372.4

b 1452.8

12 a $a^2 + ab + b^2$ **b** 7 **c** 12.06 **d** $3b^2$

13 a B **b** A **c** 25 m **d** 45 s

e 0.98 m/s, 1.724 m/s, 1.136 m/s

14 a m **b** cm **c** $-c$ **d** results are the same

Chapter 16

Exercise 16A

1 2000 m/s **2** 7 per day

3 a 1 **b** $3x^2 + 1$ **c** 20 **d** $30x^2 + 1$ **e** 5

4 a $2x + 2$ **b** 13 **c** $3x^2 + 4x$

5 a $5 + 3h$ **b** 5.3 **c** 5

6 a $\frac{-1}{2+h}$ **b** −0.48 **c** $\frac{-1}{2}$

7 a $6 + h$ **b** 6.1 **c** 6

Exercise 16B

1 a $6x$ **b** 4 **c** 0 **d** $6x + 4$

e $6x^2$ **f** $8x - 5$ **g** $-2 + 2x$

2 a $2x + 4$ **b** 2 **c** $3x^2 - 1$

d $x - 3$ **e** $15x^2 + 6x$ **f** $-3x^2 + 4x$

3 a $12x^{11}$ **b** $21x^6$ **c** 5

d 5 **e** 0 **f** $10x - 3$

g $50x^4 + 12x^3$ **h** $8x^3 + x^2 - \frac{1}{2}x$

4 a −1 **b** 0 **c** $12x^2 - 3$ **d** $x^2 - 1$

e $2x + 3$ **f** $18x^2 - 8$ **g** $15x^2 + 3x$

5 a **i** 3 **ii** $3a^2$ **b** $3x^2$

Answers

6 a $\frac{dy}{dx} = 3(x-1)^2 \geq 0$ for all $x \in R$ and gradient of graph ≥ 0 for all x

b $\frac{dy}{dx} = 1$ for all $x \neq 0$ **c** $18x + 6$

7 a 1, gradient $= 2$ **b** 1, gradient $= 1$

c 3, gradient $= -4$ **d** -5, gradient $= 4$

e 28, gradient $= -36$ **f** 9, gradient $= -24$

8 a **i** $4x - 1$, 3, $\left(\frac{1}{2}, 0\right)$

ii $\frac{1}{2} + \frac{2}{3}x$, $\frac{7}{6}$, $\left(\frac{3}{4}, \frac{25}{16}\right)$

iii $3x^2 + 1$, 4, (0, 0)

iv $4x^3 - 31$, -27, $(2, -46)$

b coordinates of the point where gradient $= 1$

9 a $6t - 4$ **b** $-2x + 3x^2$ **c** $-4z - 4z^3$

d $6y - 3y^2$ **e** $6x^2 - 8x$ **f** $19.6t - 2$

10 a (4, 16) **b** (2, 8) and $(-2, -8)$

c (0, 0) **d** $\left(\frac{3}{2}, -\frac{5}{4}\right)$

e $(2, -12)$ **f** $\left(-\frac{1}{3}, \frac{4}{27}\right)$, (1, 0)

Exercise 16C

1 *b* and *d* **2** *a*, *b* and *e*

3 a $x = 1$ **b** $x = 1$ **c** $x > 1$

d $x < 1$ **e** $x = \frac{1}{2}$

4 a $(-\infty, -3) \cup \left(\frac{1}{2}, 4\right)$

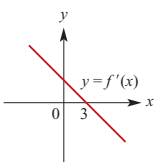
b $\left(-3, \frac{1}{2}\right) \cup (4, \infty)$ **c** $\left\{-3, \frac{1}{2}, 4\right\}$

5 a B **b** C **c** D **d** A **e** F **f** E

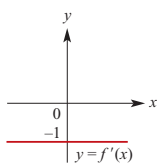
6 a $(-1, 1.5)$ **b** $(-\infty, -1) \cup (3, \infty)$

c $\{-1, 1.5\}$

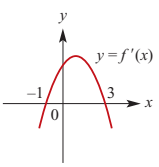
7 a



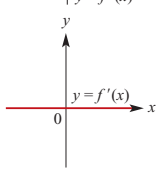
b



c

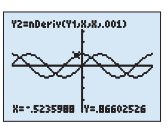


d

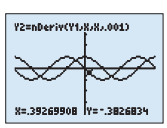


8 a (3, 0) **b** (4, 2) **9 a** $\left(\frac{1}{2}, -6\frac{1}{4}\right)$ **b** (0, −6)

10 a



b



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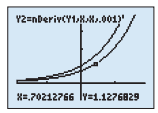


11 a $S = (0.6)t^2$ **b** 0.6 m/s, 5.4 m/s, 15 m/s

12 a height $= 450\,000$ m; speed $= 6000$ m/s

b $t = 25$ s

13 a $a = 2$, $b = -5$ **b** $\left(\frac{5}{4}, -\frac{25}{8}\right)$

Exercise 16D

1 a 15 **b** 1 **c** $-3\frac{1}{2}$ **d** $-2\frac{1}{2}$

e 0 **f** 4 **g** 2 **h** $2\sqrt{3}$

i -2 **j** 12 **k** $\frac{11}{9}$ **l** $\frac{1}{4}$

2 a 3, 4 **b** 7

3 a 0 as $f(0) = 0$, $\lim_{x \to 0^+} f(x) = 0$ but $\lim_{x \to 0^-} f(x) = 2$

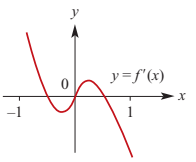
b 1 as $f(1) = 3$, $\lim_{x \to 1^+} f(x) = 3$ but $\lim_{x \to 1^-} f(x) = -1$

c 0 as $f(0) = 1$, $\lim_{x \to 0^+} f(x) = 1$ but $\lim_{x \to 0^-} f(x) = 0$

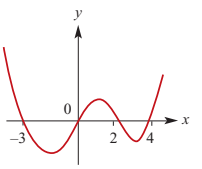
4 $x = 1$

Exercise 16E

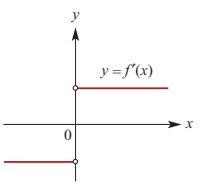
1 a



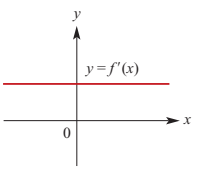
b



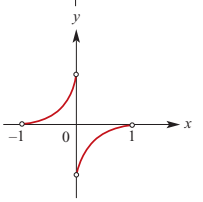
c



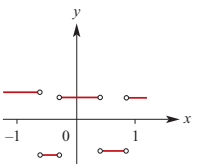
d



e



f



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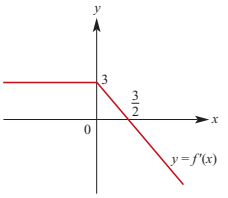
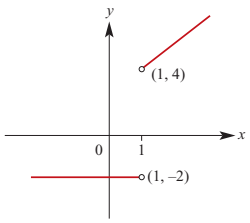


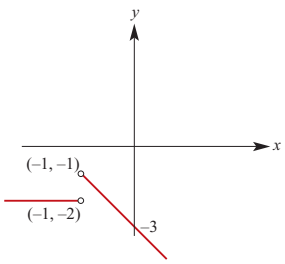
$$f'(x) = \begin{cases} -2x + 3 & \text{if } x \geq 0 \\ 3 & \text{if } x < 0 \end{cases}$$

3



$$f'(x) = \begin{cases} 2x + 2 & \text{if } x \geq 1 \\ -2 & \text{if } x < 1 \end{cases}$$

4



$$f'(x) = \begin{cases} -2x - 3 & \text{if } x \geq -1 \\ -2 & \text{if } x < -1 \end{cases}$$

Multiple-choice questions

1 D **2** B **3** E **4** B **5** C
6 C **7** A **8** E **9** A **10** D

Short-answer questions (technology-free)

1 a $6x - 2$ **b** 0 **c** $4 - 4x$
d $4(20x - 1)$ **e** $6x + 1$ **f** $-6x - 1$

2 a -1 **b** 0 **c** $\frac{4x + 7}{4}$ **d** $\frac{4x - 1}{3}$ **e** x

3 a 1; 2 **b** 3; −4 **c** −5; 4 **d** 28; −36

4 a $\left(\frac{3}{2}, -\frac{5}{4}\right)$ **b** (2, −12)

c $\left(-\frac{1}{3}, \frac{4}{27}\right), (1, 0)$ **d** (−1, 8)(1, 6)

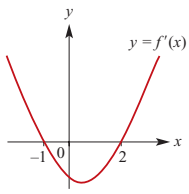
e $(0, 1)\left(\frac{3}{2}, -\frac{11}{16}\right)$ **f** (3, 0)(1, 4)

5 a $x = \frac{1}{2}$ **b** $x = \frac{1}{2}$ **c** $x > \frac{1}{2}$

d $x < \frac{1}{2}$ **e** $R \setminus \left\{\frac{1}{2}\right\}$ **f** $x = \frac{5}{8}$

6 a $a = 2, b = -1$ **b** $\left(\frac{1}{4}, -\frac{1}{8}\right)$

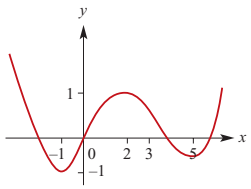
7



8 a $(-1, 4)$ **b** $(-\infty, -1) \cup (4, \infty)$ **c** $\{-1, 4\}$

Extended-response questions

1



2 $y = \frac{7}{36}x^3 + \frac{1}{36}x^2 - \frac{20}{9}x$

3 a **i** 71°34′ **ii** 89°35′ **b** 2 km
4 a 0.12, −0.15
b $x = 2$, $y = 2.16$. The height of the pass is 2.16 km.
5 a $t = \sqrt[3]{250}$, 11.9 cm/s **b** 3.97 cm/s
6 a At $x = 0$, gradient $= -2$; at $x = 2$, gradient $= 2$
Angles of inclination to positive directions of x-axis are supplementary.

Chapter 17

Exercise 17A

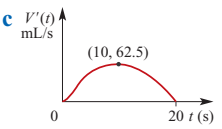
1 a $y = 4x - 4$; $4y + x = 18$
b $y = 12x - 15$; $12y + x = 110$
c $y = -x + 4$; $y = x$
d $y = 6x + 2$; $6y + x = 49$
2 $y = 2x - 10$
3 $y = 2x - 1$; $y = 2x - \frac{8}{3}$;
both have gradient $= 2$;
distance apart $= \frac{\sqrt{5}}{3}$
4 $y = 3x + 2$; $y = 3x + 6$
5 a Tangents both have gradient 2; **b** (0, −3)
6 (3, 12) (1, 4)
7 a $y = 10x - 16$ **b** (−4, −56)
8 a $y = 5x - 1$ **b** (2, 4) (4, −8)

Exercise 17B

1 a 36; $\frac{36}{1} = 36$ **b** $48 - 12h$ **c** 48
2 a $1200t - 200t^2$ **b** 1800 dollars/month
c At $t = 0$ and $t = 6$
3 a −3 cm/s **b** $2\sqrt{3}$ s
4 a $15 - 9.8t$ m/s **b** -9.8 m/s^2
5 a $30 - 4p$ **b** 10; −10
c For $P < 7.5$ revenue is increasing as P increases.
6 a **i** 50 people/year **ii** 0 people/year
iii decreasing by 50 people/year
7 a i 0 mL **ii** $833\frac{1}{3}$ mL
b $V'(t) = \frac{5}{8}(20t - t^2)$

Answers

c



8 a i 64 m/s **ii** 32 m/s **iii** 0 m/s

9 a 0 s, 1 s, 2 s

b 2 m/s, −1 m/s, 2 m/s; -6 m/s^2, 0 m/s^2, 6 m/s^2

c 0 m/s^2

10 a $\frac{1}{2}$ m/s^2 **b** $2\frac{1}{2}$ m/s^2

Exercise 17C

1 a (3, −6) **b** (3, 2) **c** (2, 2) **d** (4, 48)

e (0, 0); (2, −8) **f** (0, −10); (2, 6)

2 $a = 2, b = -8, c = -1$

3 $a = -\frac{1}{2}, b = 1, c = 1\frac{1}{2}$

4 a $a = 2, b = -5$ **b** $\left(\frac{5}{4}, -\frac{25}{8}\right)$

5 $a = -8$ **6** $a = 6$

7 a (2.5, −12.25) **b** $\left(\frac{7}{48}, -\frac{625}{96}\right)$

c (0, 27) (3, 0) **d** (−2, 48) (4, −60)

e (−3, 4) (−1, 0) **f** (−1.5, 0.5)

8 a $a = -1, b = 2$

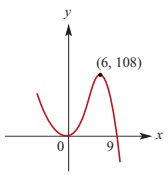
9 $a = -\frac{2}{9}, b = \frac{3}{2}, c = -3, d = 7\frac{1}{2}$

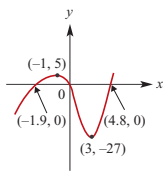
Exercise 17D

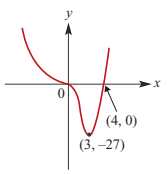
1 a min (0, 0); max (6, 108)

b min (3, −27); max (−1, 5)

c Stationary point of inflexion (0, 0); min (3, −27)







2 a (0, 0) max; $\left(\frac{8}{3}, -\frac{256}{27}\right)$ min

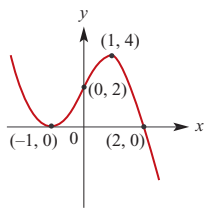
b (0, 0) min; (2, 4) max **c** (0, 0) min

d $\left(\frac{10}{3}, \frac{-200\,000}{729}\right)$ min; (0, 0) inflexion

e (3, −7) min; $\left(\frac{1}{3}, 2\frac{13}{27}\right)$ max

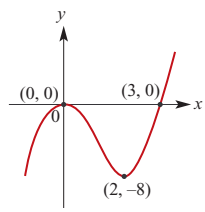
f (6, −36) min; $\left(\frac{4}{3}, \frac{400}{27}\right)$ max

3 a



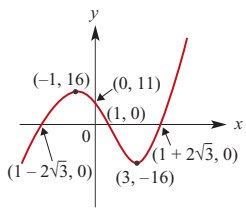
max at (1, 4)
min at (−1, 0)
intercepts (2, 0) and (−1, 0)

b



min at (2, −8)
max at (0, 0)
intercepts (3, 0) and (0, 0)

c



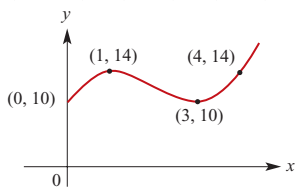
min at (3, −16)
max at (−1, 16)
intercepts (0, 11), $(1 \pm 2\sqrt{3}, 0)$ and (1, 0)

4 a (−2, 10) max

b (−2, 10) stationary point of inflexion

5 a $(-\infty, 1) \cup (3, \infty)$ **b** (1, 14) max; (3, 10) min

c

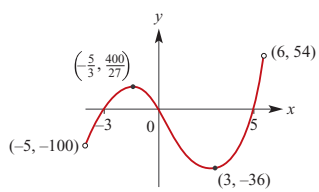


6 25 **7** $\{x : -2 < x < 2\}$

8 a $\{x : -1 < x < 1\}$ **b** $\{x : x < 0\}$

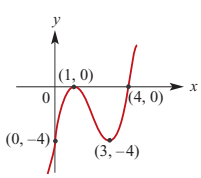
9 a $x = -\frac{5}{3}; x = 3$

b

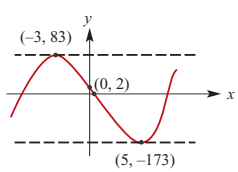


max at $\left(-\frac{5}{3}, \frac{400}{27}\right)$
min at (3, −36)
intercepts (5, 0) (0, 0) (−3, 0)

10



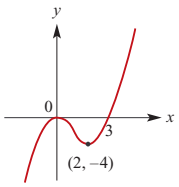
11



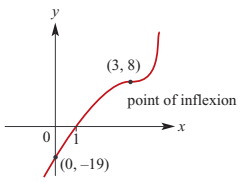
Tangents are parallel to x-axis at (−3, 83) and (5, −173).

12 a i (0, 2)
ii $(-\infty, 0) \cup (2, \infty)$
iii $\{0, 2\}$

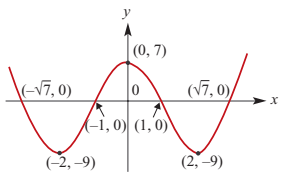
b



13



14

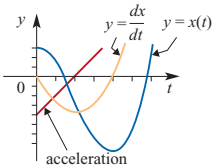


min at $(-2, -9)$; $(2, -9)$
max at $(0, 7)$
intercepts $(\pm\sqrt{7}, 0)$ $(\pm 1, 0)$ $(0, 7)$

Exercise 17E

1 a $y = 4x - 7$ **b** $y = 10x - 15$
c $y = 6x - 7$ **d** $y = -18x + 25$
2 a (0, 1) local max.; (1.33, −0.19) local min.
b No stationary points
c (0, 1) local max.; (0.67, 0.70) local min.
d (−0.22, −0.11) local min.; (1.55, 2.63) local max.
e (0, 0) local min.; (2, 4) local max.
f no stationary points

3



Exercise 17F

1 a 0.6 km^2 **b** 0.7 km^2/h
2 a $t = 1$, $a = 18$ m/s^2; $t = 2$, $a = 54$ m/s^2; $t = 3$, $a = 114$ m/s^2
b 58 m/s^2
3 a i 0.9375 m **ii** 2.5 m **iii** 2.8125 m
b $x = \frac{40}{3}$, $y = \frac{80}{27}$
c i $x = 11.937$, $x = 1.396$ **ii** $x = 14.484$
4 b $V = \frac{75x - x^3}{2}$ **c** 125 cm^3

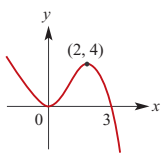
Multiple-choice questions

1 D **2** E **3** E **4** A **5** C
6 D **7** D **8** A **9** A **10** C

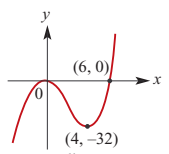
Short-answer questions (technology-free)

1 a $\frac{dy}{dx} = 4 - 2x$ **b** 2 **c** $y = 2x + 1$
2 a $3x^2 - 8x$ **b** −4 **c** $y = -4x$ **d** (0, 0)
3 a $3x^2 - 12$; $x = \pm 2$
b & c minimum when $x = 2$, $y = -14$
maximum when $x = -2$, $y = 18$
4 a $x = 0$ stationary point of inflexion
b $x = 0$ maximum
c minimum when $x = 3$, maximum when $x = 2$
d minimum when $x = 2$, maximum when $x = -2$
e maximum when $x = 2$, minimum when $x = -2$
f maximum when $x = 3$, minimum when $x = 1$
g maximum when $x = 4$, minimum when $x = -3$
h maximum when $x = 3$, minimum when $x = -5$
5 a $\left(-\frac{2}{3}, \frac{16}{9}\right)$ minimum,
$\left(\frac{2}{3}, \frac{16}{9}\right)$ maximum
b (−1, 0) maximum, (2, −27) minimum
c $\left(\frac{2}{3}, \frac{100}{27}\right)$ maximum, (3, −9) minimum

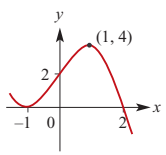
6 a



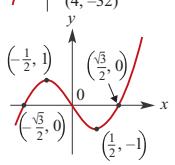
b



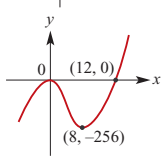
c



d

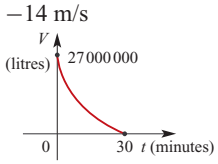


e



Extended-response questions

1 a −14 m/s **b** −8 m/s^2
2 a



b i 17.4 minutes **ii** 2.9 minutes

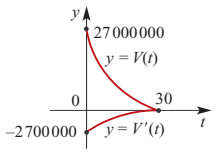
Answers

Answers

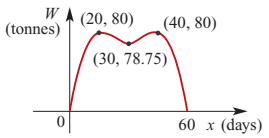
c $\frac{dV}{dt} = -3000(30 - t)^2$

d 30 minutes **e** 28.36 minutes

f



3 a



b after 5.71 days until 54.29 days

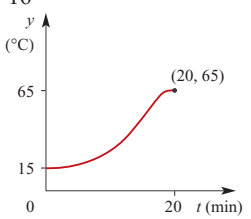
c $x = 20, \frac{dW}{dx} = 0$; $x = 40, \frac{dW}{dx} = 0$; $x = 60, \frac{dW}{dx} = -12$ t/day

d $x = 30, W = 78.75$

4 a 15°C

b 0°C/min, $\frac{45}{16}$ °C/min, $\frac{15}{4}$ °C/min, $\frac{45}{16}$ °C/min, 0°C/min

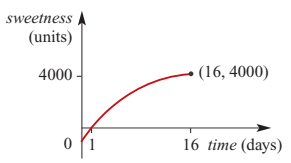
c



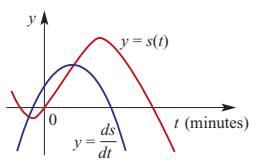
5 a 768 units/day **b** 432, 192, 48, 0

c $t = 16$

d



6 a



b 11.59 am; 12.03 pm **c** $\frac{5}{27}$ km, 1 km

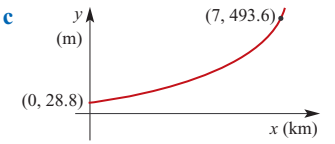
d $\frac{8}{27}$ km/min $= 17\frac{7}{9}$ km/h

e $\frac{1}{3}$ km/min $= 20$ km/h

7 a $0 \leq t \leq 12$ **b i** 27 L/h **ii** 192 L/h

8 a 28.8 m **b** 374.4

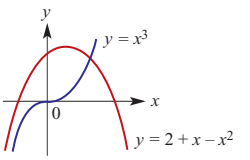
c



d Path gets too steep after 7 km.

e i 0.0384 **ii** 0.0504 **iii** 0.1336

9 a



b For $x < 0$, minimum vertical distance occurs when $x = -1$.
Min distance $= 1$ unit

10 8 mm for maximum and $\frac{4}{3}$ mm for minimum

11 a $y = 5 - x$ **b** $P = x(5 - x)$

c maximum value $= 6.25$, when $x = 2.5$ and $y = 2.5$

12 a $y = 10 - 2x$ **b** $A = x^2(10 - 2x)$

c $\frac{1000}{27}; x = \frac{10}{3}, y = \frac{10}{3}$

13 $20\sqrt{10}$

14 a $y = 8 - x$ **b** $s = x^2 + (8 - x)^2$ **c** 32

15 $\frac{4}{3}; \frac{8}{3}$ **16** 25 m × 25 m = 625 m²

17 $x = 12$ **18** 32 **19** maximum $P = 2500$

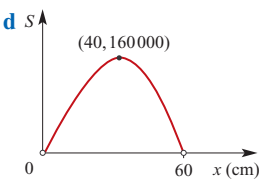
20 2 km × 1 km = maximum of 2 km²

21 $p = \frac{3}{2}, q = \frac{8}{3}$

22 a $y = 60 - x$ **b** $S = 5x^2(60 - x)$

c $0 < x < 60$

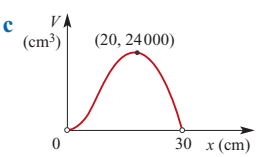
d



e $x = 40, y = 20$

23 12°C

24 b $0 < x < 30$ **c**



d 20 cm, 40 cm, 30 cm

e $x = 14.82$ or $x = 24.4$

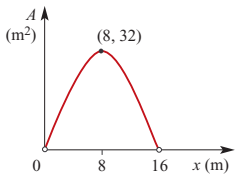
25 b Maximum when $x = 3, y = 18$

26 a 44 cm should be used to form circle, 56 cm to form square

b All the wire should be used to form the circle.

27 Width 4.5 metres, length 7.2 metres

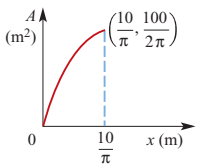
28 a $A = xy$ **b** $A = \left(8 - \frac{x}{2}\right)x$

c $0 < x < 16$ **d** 

e 32 m²

29 $a = 937, h = 1188$

30 a $y = 10 - \pi x$ **b** $0 \le x \le \frac{10}{\pi}$

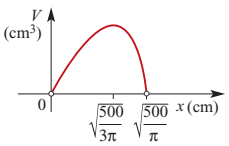
c $A = \frac{x}{2}(20 - \pi x)$ **d** 

e maximum at $x = \frac{10}{\pi}$ **f** a semicircle

31 a $h = \frac{500}{\pi x} - x$ **b** $V = 500x - \pi x^3$

c $\frac{dV}{dx} = 500 - 3\pi x^2$

d $x = 10\sqrt{\frac{5}{3\pi}} \approx 7.28$

e 

f 2427.89 cm³

g $x = 2.05$ and $h = 75.41$ or $x = 11.46$ and $h = 2.42$

32 a $r = 4.3$ cm, $h = 8.6$ cm

Chapter 18

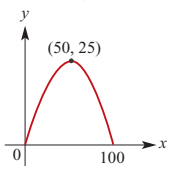
18.1 Multiple-choice questions

1 B	**2** A	**3** B	**4** A	**5** B	**6** A
7 D	**8** B	**9** C	**10** A	**11** C	**12** B
13 A	**14** C	**15** C	**16** A	**17** D	**18** E
19 C	**20** B	**21** E	**22** D	**23** A	**24** A
25 C	**26** D	**27** B	**28** B	**29** B	**30** C
31 D	**32** E	**33** A	**34** A	**35** C	**36** C

18.2 Extended-response questions

1 a 100 **b** $\frac{dy}{dx} = 1 - 0.02x$

c $x = 50, y = 25$

d 

e **i** (25, 18.75) **ii** (75, 18.75)

2 a $\left(66\frac{2}{3}, 14\frac{22}{27}\right)$

b **i** 0.28 **ii** −0.32 **iii** −1

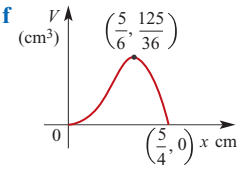
c A gradual rise to the turning point and a descent which becomes increasingly steep (in fact alarmingly steep).

d Smooth out the end of the trip

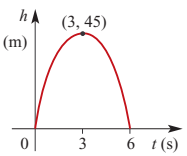
3 a $h = 5 - 4x$ **c** $0 < x < \frac{5}{4}$

d $\frac{dV}{dx} = 30x - 36x^2$

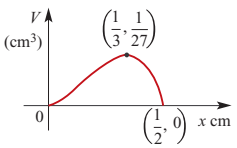
e $\left\{0, \frac{5}{6}\right\}$; maximum volume $= 3\frac{17}{36}$ cm³

f 

4 a $\frac{dh}{dt} = 30 - 10t$ **b** 45 m

c 

5 a $A = 4x - 6x^2$ **b** $V = x^2 - 2x^3$

c 

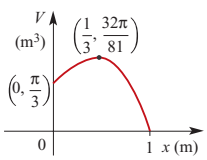
d $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}, \frac{1}{27}$ cm³

6 a **i** $r = \sqrt{1 - x^2}$ **ii** $h = 1 + x$

c $0 < x < 1$

d **i** $\frac{dV}{dx} = \frac{\pi}{3}(1 - 2x - 3x^2)$

ii $\left\{\frac{1}{3}\right\}$ **iii** $\frac{32\pi}{81}$ m³

e 

7 a 1000 insects **b** 1366 insects

c **i** $t = 40$ **ii** $t = 51.70$

d 63.64

e **i** $\frac{1000 \times 2^{\frac{3}{4}}(2^{\frac{h}{20}} - 1)}{h}$

ii Consider h decreasing and approaching zero; instantaneous rate of change = 58.286 insects/day

Answers

Answers

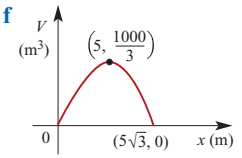
8 a $h = \dfrac{150 - 2x^2}{3x}$

b $V = \dfrac{2}{3}(150x - 2x^3)$

c $\dfrac{dV}{dx} = 2(50 - 2x^2)$ **d** $0 < x < 5\sqrt{3}$

e $\dfrac{1000}{3}$ m^3 when $x = 5$

f



9 a 10

c i $h = 2.5x$

d $V = 40(420x - 135x^2)$

e i $x = \dfrac{14}{9}$, $y = \dfrac{140}{9}$

ii $13\,066\dfrac{2}{3}$ m^3

10 a $a = 200$, $k = 0.000\,01$

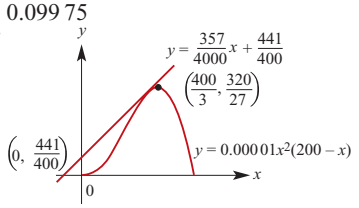
b i $x = \dfrac{400}{3}$ **ii** $y = \dfrac{320}{27}$

c i $\dfrac{8379}{800}$ **ii** $\dfrac{357}{4000}$

d i $y = \dfrac{357}{4000}x + \dfrac{441}{400}$ **ii** $\dfrac{441}{400}$

e 0.099 75

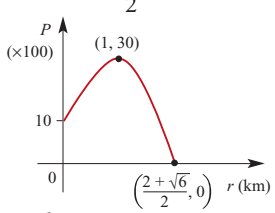
f



11 a 10 000 people/km^2

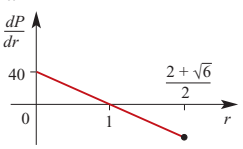
b $0 < r \leq \dfrac{2 + \sqrt{6}}{2}$

c



d i $\dfrac{dP}{dr} = 40 - 40r$ **ii** 20; 0; −40

iii

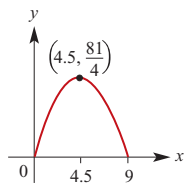


e when $r = 1$

12 a $y = ax - x^2$ **b** $0 < x < a$ **c** $\dfrac{a^2}{4}, \dfrac{a}{2}$

d negative coefficient of x^2 for quadratic function

e i

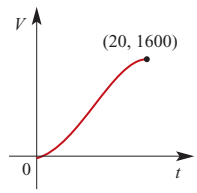


ii $\left(0, \dfrac{81}{4}\right]$

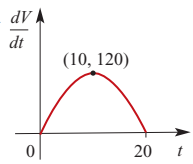
13 a i 0 **ii** 1600

b $\dfrac{dV}{dt} = 0.6(40t - 2t^2)$

c

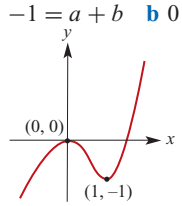


d



14 a $-1 = a + b$ **b** $0 = 3a + 2b$, $a = 2$, $b = -3$

c



15 a i $80 - 2x$ **ii** $h = \dfrac{\sqrt{3}}{2}x$

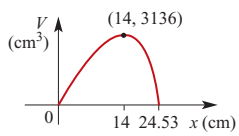
b $A = \dfrac{\sqrt{3}}{4}x(160 - 3x)$ **c** $x = \dfrac{80}{3}$

16 a $y = \dfrac{1400 - 2x^2 - 8x}{4x}$

b $V = -\dfrac{x^3}{2} - 2x^2 + 350x$

c $\dfrac{dV}{dx} = -\dfrac{3}{2}x^2 - 4x + 350$ **d** $x = 14$

e



f maximum volume is 3136 cm^3

g $x = 22.83$ and $y = 1.92$ or $x = 2.94$ and $y = 115.45$

Chapter 19

Exercise 19A

1 a $-6x^{-3} - 5x^{-2}$ **b** $-6x^{-3} + 10x$

c $-15x^{-4} - 8x^{-3}$ **d** $6x - \dfrac{20}{3}x^{-5}$

e $-12x^{-3} + 3$ **f** $3 - 2x^{-2}$

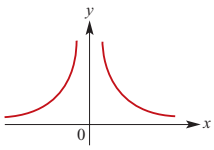
2 a $-2z^{-2} - 8z^{-3}$, $z \neq 0$

b $-9z^{-4} - 2z^{-3}$, $z \neq 0$

c $\dfrac{1}{2}$, $z \neq 0$ **d** $18z + 4 - 18z^{-4}$, $z \neq 0$

e $2z^{-3}$, $z \neq 0$ **f** $-\dfrac{3}{5}$, $z \neq 0$

3 a



b Gradient of chord $PQ = \dfrac{-2-h}{(1+h)^2}$

c -2 **d** $y = \frac{1}{2}x + \frac{1}{2}$

4 a $11\frac{3}{4}$ **b** $\frac{1}{8}$ **c** -1 **d** 5

5 a $-\frac{1}{2}$ **b** $\frac{1}{2}$

Exercise 19B

1 a $30(x-1)^{29}$

b $100(x^4 - 2x^9)(x^5 - x^{10})^{19}$

c $4(1 - 3x^2 - 5x^4)(x - x^3 - x^5)^3$

d $8(x+1)^7$ **e** $-4(x+1)(x^2+2x)^{-3}$

f $-6(x + x^{-2})(x^2 - 2x^{-1})^{-4}$

2 a $24x^2(2x^3+1)^3$ **b** 648

3 a $-\frac{1}{16}$ **b** $-\frac{3}{256}$

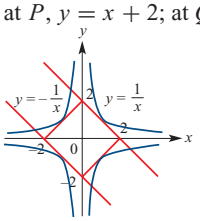
4 a $-\frac{2}{9}$ **b** $\left(-3, -\frac{1}{3}\right)\left(0, \frac{1}{3}\right)$

5 a $-\frac{1}{4}$ **b** $\frac{1}{4}$

c $y = -x + 2$ **d** $y = x - 2$

e at P, $y = x + 2$; at Q, $y = -x - 2$; $(-2, 0)$

f



Exercise 19C

1 a $\frac{1}{3}x^{-\frac{2}{3}}$ **b** $\frac{3}{2}x^{\frac{1}{2}}; x > 0$

c $\frac{5}{2}x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}}; x > 0$ **d** $x^{-\frac{1}{2}} - 5x^{\frac{2}{3}}$

e $-\frac{5}{6}x^{-\frac{11}{6}}$ **f** $-\frac{1}{2}x^{-\frac{3}{2}}; x > 0$

2 a $x(1+x^2)^{-\frac{1}{2}}$ **b** $\frac{1}{3}(1+2x)(x+x^2)^{-\frac{2}{3}}$

c $-x(1+x^2)^{-\frac{3}{2}}$ **d** $\frac{1}{3}(1+x)^{-\frac{2}{3}}$

3 a i $\frac{4}{3}$ **ii** $\frac{4}{3}$ **iii** $\frac{1}{3}$ **iv** $\frac{1}{3}$

4 a $\{x : 0 < x < 1\}$ **b** $\left\{x : x > \left(\frac{2}{3}\right)^6\right\}$

5 a $-5x^{-\frac{1}{2}}\left(2 - 5\sqrt{x}\right)$ **b** $3x^{-\frac{1}{2}}\left(3\sqrt{x} + 2\right)$

c $-4x^{-3} - \frac{3}{2}x^{-\frac{5}{2}}$ **d** $\frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{3}{2}}$

e $\frac{15}{2}x^{\frac{3}{2}} + 3x^{-\frac{1}{2}}$

Exercise 19D

1 a $6x$ **b** 0 **c** $108(3x+1)^2$

d $-\frac{1}{4}x^{-\frac{3}{2}} + 18x$ **e** $306x^{16} + 396x^{10} + 90x^4$

f $10 + 12x^{-3} + \frac{9}{4}x^{-\frac{1}{2}}$

2 a $18x$ **b** 0 **c** 12 **d** $432(6x+1)^2$

e $300(5x+2)^2$ **f** $6x + 4 + 6x^{-3}$

3 -9.8 m/s^2

4 a i -16 **ii** 4 m/s **iii** $\frac{7}{4}$ m/s **iv** -32 m/s

b $t = 0$ **c** -8 m/s

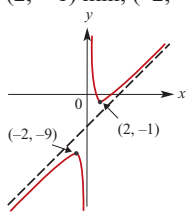
Exercise 19E

1 a $\left(\frac{1}{2}, 4\right)\left(-\frac{1}{2}, -4\right)$ **b** $y = \frac{15}{4}x + 1$

2 $\pm\frac{1}{2}$ **3** $\frac{1}{2}$

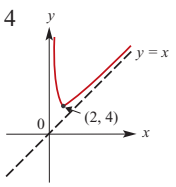
4 a (4, 0) (1, 0) **b** $y = x - 5$; $x = 0$

c (2, −1) min; (−2, −9) max

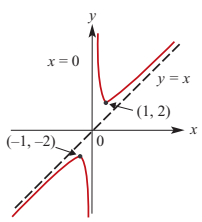


5 3

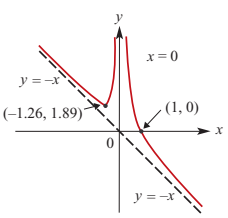
6 4



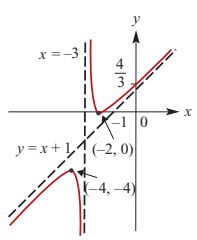
7 a



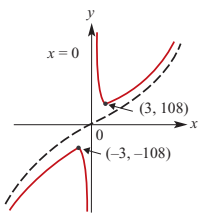
b



c

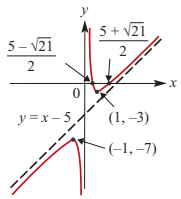


d

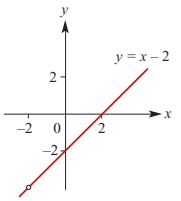


Answers

e



f



Multiple-choice questions

1 B **2** D **3** A **4** A **5** A
6 E **7** A **8** B **9** A **10** D

Short-answer questions (technology-free)

1 a $-4x^{-5}$ **b** $-6x^{-4}$ **c** $\frac{2}{3x^3}$ **d** $\frac{4}{x^5}$

e $\frac{-15}{x^6}$ **f** $\frac{-2}{x^3} - \frac{1}{x^2}$ **g** $\frac{-2}{x^2}$ **h** $10x + \frac{2}{x^2}$

2 a $\frac{1}{2x^{\frac{1}{2}}}$ **b** $\frac{1}{3x^{\frac{2}{3}}}$ **c** $\frac{2}{3x^{\frac{4}{3}}}$

d $\frac{4}{3}x^{\frac{1}{3}}$ **e** $-\frac{1}{3x^{\frac{4}{3}}}$ **f** $-\frac{1}{3x^{\frac{4}{3}}} + \frac{6}{5x^{\frac{2}{5}}}$

3 a $8x + 12$ **b** $24(3x + 4)^3$ **c** $\frac{1}{(3 - 2x)^{\frac{3}{2}}}$

d $\frac{-2}{(3 + 2x)^2}$ **e** $\frac{-4}{3(2x - 1)^{\frac{5}{3}}}$ **f** $\frac{-3x}{(2 + x^2)^{\frac{3}{2}}}$

g $\frac{1}{3}\left(4x + \frac{6}{x^3}\right)\left(2x^2 - \frac{3}{x^2}\right)^{-\frac{2}{3}}$

4 a $\frac{1}{6}$ **b** -2 **c** $-\frac{1}{16}$ **d** -2 **e** $\frac{1}{6}$ **f** 0

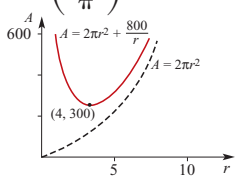
5 $\left(\frac{1}{2}, 2\right)$ and $\left(-\frac{1}{2}, -2\right)$ **6** $\left(\frac{1}{16}, \frac{1}{4}\right)$

Extended-response questions

1 a $h = \frac{400}{\pi r^2}$ **c** $\frac{dA}{dr} = 4\pi r - \frac{800}{r^2}$

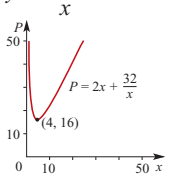
d $r = \left(\frac{200}{\pi}\right)^{\frac{1}{3}} \approx 3.99$ **e** $A = 301$

f



2 a $y = \frac{16}{x}$ **c** $x = 4$, $P = 16$

d



3 a $OA = \frac{120}{x}$ **b** $OX = \frac{120}{x} + 7$

c $OZ = x + 5$ **d** $A = 7x + \frac{600}{x} + 155$

e $x = \frac{10\sqrt{42}}{7} \approx 9.26$ cm

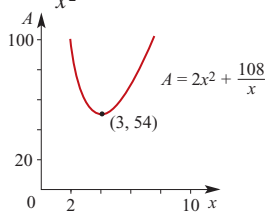
4 a A$(-2, 0)$, B$(0, \sqrt{2})$ **b** $\frac{1}{2\sqrt{x + 2}}$

c i $\frac{1}{2}$ **ii** $2y - x = 3$ **iii** $\frac{3\sqrt{5}}{2}$

d $x > -\frac{7}{4}$

5 a $h = \frac{18}{x^2}$ **c** $x = 3$, $h = 2$

d



6 a $y = \frac{250}{x^2}$

c $\frac{dS}{dx} = 24x - \frac{3000}{x^2}$ **d** S min $= 900\,\text{cm}^2$

Chapter 20

Exercise 20A

1 a $\frac{x^4}{8} + c$ **b** $x^3 - 2x + c$

c $\frac{5x^4}{4} - x^2 + c$ **d** $\frac{x^4}{5} - \frac{2x^3}{3} + c$

e $\frac{x^3}{3} - x^2 + x + c$ **f** $\frac{x^3}{3} + x + c$

g $\frac{z^4}{2} - \frac{2z^3}{3} + c$ **h** $\frac{4t^3}{3} - 6t^2 + 9t + c$

i $\frac{t^4}{4} - t^3 + 3\frac{t^2}{2} - t + c$

2 a $y = x^2 - x$ **b** $y = 3x - \frac{x^2}{2} + 1$

c $y = \frac{x^3}{3} + x^2 + 2$ **d** $y = 3x - \frac{x^3}{3} + 2$

e $y = \frac{2x^5}{5} + \frac{x^2}{2}$

3 a $V = \frac{t^3}{3} - \frac{t^2}{2} + \frac{9}{2}$ **b** $\frac{1727}{6} \approx 287.83$

4 $f(x) = x^3 - x + 2$

5 a B **b** $w = 2000t - 10t^2 + 100\,000$

6 $f(x) = 5x - \frac{x^2}{2} + 4$ **7** $f(x) = \frac{x^4}{4} - x^3 - 2$

8 a $k = 8$ **b** $(0, 7)$ **9** $8\frac{2}{3}$

10 a $k = -4$ **b** $y = x^2 - 4x + 9$

11 a $k = -32$ **b** $f(x) = 201$

12 $y = \frac{1}{3}(x^3 - 5)$

Exercise 20B

1 a $-\frac{3}{x}+c$ **b** $3x^2-\frac{2}{3x^3}+c$

c $\frac{4}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+c$ **d** $\frac{9}{4}x^{\frac{4}{3}}-\frac{20}{9}x^{\frac{9}{4}}+c$

e $\frac{3}{2}z^2-\frac{2}{z}+c$ **f** $\frac{12}{7}x^{\frac{7}{4}}-\frac{14}{3}x^{\frac{3}{2}}+c$

2 a $y=\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2-\frac{22}{3}$

b $y=2-\frac{1}{x}$ **c** $y=\frac{3}{2}x^2-\frac{1}{x}+\frac{9}{2}$

3 $f(x)=x^3+\frac{1}{x}-\frac{17}{2}$

4 $s=\frac{3t^2}{2}+\frac{8}{t}-8$ **5** $y=5$

6 a 2 **b** $y=x^2+1$

7 $y=\frac{x^3}{3}+\frac{7}{3}$

Exercise 20C

1 a $\frac{7}{3}$ **b** 20 **c** $-\frac{1}{4}$ **d** 9 **e** $\frac{15}{4}$

f $\frac{297}{6}=49.5$ **g** $15\frac{1}{3}$ **h** 30

2 a 1 **b** 1 **c** 14 **d** 31 **e** $2\frac{1}{4}$ **f** 0

3 a 8 **b** 16 **c** -4

4 a -12 **b** 36 **c** 20

5 $\frac{26}{3}$ **6** 36 square units

7 3.08 square units

8 a 24, 21, 45 **b** 4, -1, 3

9 4.5 square units **10** $166\frac{2}{3}$ square units

11 $\frac{37}{12}$ square units

12 a $\frac{4}{3}$ square units **b** $\frac{1}{6}$ square units

c $121\frac{1}{2}$ square units **d** $\frac{1}{6}$ square units

e $4\sqrt{3}\approx 6.93$ square units

f 108 square units

Exercise 20D

1 a 13.2 **b** 10.2 **c** 11.7

2 Area $\approx$ 6 square units **3** $\pi\approx 3.13$

4 a 36.8 **b** 36.75

5 a 4.371 **b** 1.128

6 109.5 m^2

Multiple-choice questions

1 C **2** D **3** A **4** D **5** B

6 B **7** D **8** B **9** C **10** A

Short-answer questions (technology-free)

1 a $\frac{x}{2}+c$ **b** $\frac{x^3}{6}+c$ **c** $\frac{x^3}{3}+\frac{3x^2}{2}+c$

d $\frac{4x^3}{3}+6x^2+9x+c$ **e** $\frac{at^2}{2}+c$

f $\frac{t^4}{12}+c$ **g** $\frac{t^3}{3}-\frac{t^2}{2}-2t+c$

h $\frac{-t^3}{3}+\frac{t^2}{2}+2t+c$

2 $f(x)=x^2+5x-25$

3 a $f(x)=x^3-4x^2+3x$ **b** 0, 1, 3

4 a $\frac{-1}{x^2}+c$ **b** $\frac{2x^{\frac{5}{2}}}{5}-\frac{4x^{\frac{3}{2}}}{3}+c$

c $\frac{3x^2}{2}+2x+c$ **d** $\frac{-6x-1}{2x^2}+c$

e $\frac{5x^2}{2}-\frac{4x^{\frac{3}{2}}}{3}+c$ **f** $\frac{20x^{\frac{7}{4}}}{7}-\frac{3x^{\frac{4}{3}}}{2}+c$

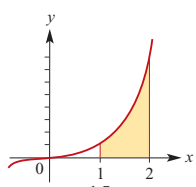
g $2x-\frac{2x^{\frac{3}{2}}}{3}+c$ **h** $-\frac{3x+1}{x^2}+c$

5 $s=\frac{1}{2}t^2+3t+\frac{1}{t}+\frac{3}{2}$

6 a 3 **b** 6 **c** 114 **d** $\frac{196}{3}$ **e** 5

7 a $\frac{14}{3}$ **b** $48\frac{3}{4}$ **c** $\frac{1}{2}$ **d** $\frac{15}{16}$ **e** $\frac{16}{15}$

8



Area $=\frac{15}{4}$ square units

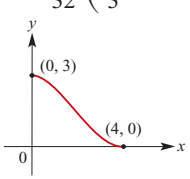
9 $4\frac{1}{2}$ square units **10** $21\frac{1}{12}$ square units

11 a (1, 3) (3, 3) **b** 6 **c** $\frac{4}{3}$

Extended-response questions

1 a $y=\frac{9}{32}\left(\frac{x^3}{3}-2x^2\right)+3$

b



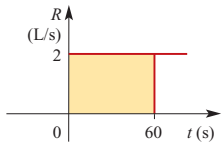
c Yes, for the interval $\left[\frac{4}{3},\frac{8}{3}\right]$

2 a 27 square units **b** $y=\frac{3}{25}(x-4)^2$

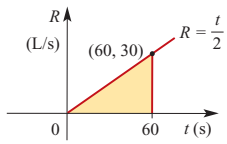
Answers

c $\frac{189}{25}$ square units **d** $\frac{486}{25}$ square units

3 a i 120 L **ii**



b i 900 L **ii**



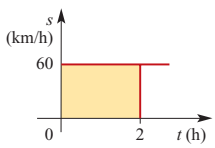
iii $900a^2$ L

c i 7200 square units

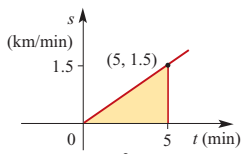
ii volume of water which has flowed in

iii 66.94 s

4 a i & ii



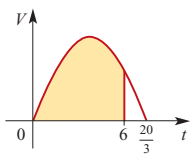
b i



ii 3.75 km

c i $20 - 6t$ m/s^2

ii



iii 144 metres

5 a i 4 m **ii** 16 m **b i** 0.7 **ii** -0.8

c i $\frac{100}{3}$ **ii** $\frac{500}{27}$ **d** $\frac{3125}{6}$ m^2

e i $(15 + 5\sqrt{33}, 12)$

ii $R = 60\sqrt{33} - 60$, $q = 20$, $p = 15 + 5\sqrt{33}$

6 a i 9 **ii** $y = 9x - 3$

iii $y = 3x^2 + 3x$

b i $12 + k$ **ii** $k = -7$

iii $f(x) = 3x^2 - 7x + 12$

7 a 6 m^2

b i $y = x - \frac{1}{2}$ **ii** $\left(x^2 - \frac{1}{4}\right)$ m^2

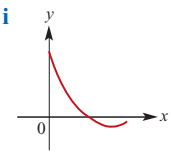
c i $P = (-2, 2)$; $S = (2, 2)$, equation $y = \frac{1}{2}x^2$

ii $\frac{16}{3}$ m^2

8 a $y = 7 \times 10^{-7}x^3 - 0.001\,16x^2 + 0.405x + 60$

b 100 m

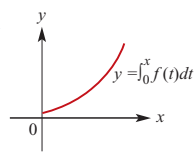
c i



ii (0, 60)

d 51 307 m^2

9 a



b $x = 2.988$

Chapter 21

Multiple-choice questions

1 D	**2** E	**3** C	**4** D	**5** E
6 D	**7** A	**8** A	**9** C	**10** D
11 E	**12** C	**13** B	**14** C	**15** C
16 E				

Glossary

A

Addition rule (for choices) [p. *263*] To determine the number of choices from disjoint alternatives, simply add up the available number for each alternative.

Addition rule for probability [p. *235*] the probability of A or B or both occurring, given by the rule:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Amplitude of circular functions [p. *367*] The distance between the mean position and the maximum position is called the **amplitude**. The graph of $y = \sin x$ has an amplitude of 1.

Antiderivative [p. *529*] To find the general antiderivative F
If $F'(x) = f(x)$, $\int f(x)dx = F(x) + c$, where c is an arbitrary real number.

Arrangements [p. *266*] The number of arrangements of n objects in groups of size r is given by:

$$\frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \ldots (n-r+1)$$

Average speed [p. *430*]

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Average velocity [p. *424*]

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{total time taken}}$$

B

Binomial distribution [p. *290*] The probability of observing X successes in n binomial trials each with probability of success p, given by

$$\Pr(X = x) = \binom{n}{x} (p)^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n,$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$

Binomial experiment [pp. *290*]

- The experiment consists of a number, n, of identical trials.
- Each trial results in one of two outcomes, which are usually designated either a **success**, S or a **failure**, F.
- The probability of success on a single trial, p, is constant for all trials.
- The trials are independent (so that the outcome on any trial is not affected by the outcome of any previous trial).

C

Chain rule [p. *513*] The **chain rule** is often used to differentiate some more complicated functions by transforming the original function into two simpler functions.

E.g. $f(x)$ is transformed to $h(x)$ and $g(u)$, which are 'chained' together as

$$x \xrightarrow{h} u \xrightarrow{g} y$$

Using Leibniz notation, the **chain rule** is stated as

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{du}{dy}.$$

Circle, general equation [p. *114*] The general equation for a circle is $(x-h)^2 + (y-k)^2 = r^2$. The **centre** of the circle is the point (h, k) and the **radius** is r.

Complement, A′ [p. *227*] is the set of points that are in the sample space (ε) but not in A

Compound event [p. *220*] an event for which there is more than one outcome from a random experiment such as observing an even number when a die is rolled

Conditional probability [p. *243*] the probability of an event A occurring when it is known that some event B has occurred; written as $\Pr(A|B)$

Constant function [p. *132*] A function $f: R \to R$, $f(x) = a$ is called a **constant function**.

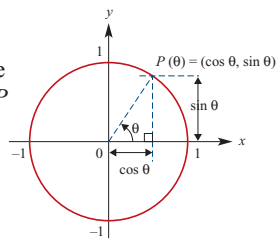
Continuous function [p. 459] A function f is defined as continuous at the point $x = a$ if the following three conditions are met:

- $f(x)$ is defined at $x = a$
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(a)$

Coordinates [p. 24] a unique ordered pair of numbers that identifies a point on the coordinate plane; the first number in the ordered pair identifies the position with regard to the x-axis, while the second number identifies the position with respect to the y-axis

Cosine and sine [p. 357]

- cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive ray of the x-axis



- sine θ is defined as the y-coordinate of the point P on the unit circle where OP forms an angle of θ radian with the positive ray of the x-axis

Cubic function [p. 168] A third degree polynomial is called a cubic and is a function f, with rule

$$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$$

D

Degree of polynomial [p. 59] is given by the value of n, the highest power of x with a non-zero coefficient.

Dependent trials or events [p. 287] the same as sampling without replacement
The probability of one event is influenced by the outcome of another event.

Derived function *see* gradient function

Difference of perfect squares [p. 64]
$(x + a)(x - a) = x^2 - a^2$

Difference of two cubes [p. 179]
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Difference of two squares [p. 64]
$x^2 - y^2 = (x - y)(x + y)$

Differentiation rule [p. 518] The general result for any non-zero real power gives:

For $f(x) = x^a$, $f'(x) = ax^{a-1}$, for $x > 0$ and $a \in R$.

Differentiation rules [p. 445] The general rule of the derived function of $f(x) = x^n, n = 1, 2, 3, \ldots$

- For $f(x) = x^n$, $f'(x) = nx^{n-1}, n = 1, 2, 3, \ldots$
- For $f(x) = 1$, $f'(x) = 0$
- For $f(x) = g(x) + h(x)$, $f'x = g'(x) + h'(x)$
- For $g(x) = kf(x)$, $g'(x) = kf'(x)$

Dilation from the x-axis [p. 150]

- In general, a dilation of a units from the x-axis is described by the rule $(x, y) \to (x, ay)$.
- In general, the curve with equation $y = f(x)$ is mapped to the curve with equation $y = af(x)$ by the transformation with rule $(x, y) \to (x, ay)$.

Dilation from the y-axis [p. 150]

- In general, a dilation of a units from the y-axis is described by the rule $(x, y) \to (ax, y)$
- In general, the curve with equation $y = f(x)$ is mapped to the curve with equation $y = f\left(\frac{x}{a}\right)$ by the transformation with rule $(x, y) \to (ax, y)$.

Discontinuity at a point [p. 458] The function is said to be discontinuous at a point if it is not continuous at that point. We say that a function is continuous everywhere if it is continuous for all real numbers.

Discrete random variable [p. 281] a random variable X which can take only a countable number of values, usually whole numbers

Discriminant, Δ, of a quadratic [p. 84] the expression $b^2 - 4ac$, which is part of the quadratic formula. For the quadratic equation $0 = ax^2 + bx + c$:

- If $b^2 - 4ac > 0$ there are two solutions.
- If $b^2 - 4ac = 0$ there is one solution.
- If $b^2 - 4ac < 0$ there are no real solutions.

Disjoint sets [p. 124] If sets A and B have no elements in common, we say A and B, are **disjoint** and write $A \cap B = \emptyset$.

Distance between given points [p. 46] The distance between the given points $A(x_1, y_1)$ and $B(x_2, y_2)$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Domain [p. 126] the set of all the first elements of the ordered pairs

E

Element [p. 124] a member of or an object in a set

element of a set [p. 124] If x is an element of a set A we write $x \in A$.

not an element of [p. 124] The notation $x \notin A$ means x is **not** an element of A.

Empirical probability [p. 224] the probability assigned to an event on the basis of repeated experimentation; i.e.:

$$\Pr(A) \approx \frac{\text{number of times the event occurs}}{\text{number of trials}} \text{ for large } n$$

Empty set, $\emptyset$ [p. 124] the set that has no members

Exponential function (or index function) [p. 320] the function $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1.

F

Factor [p. 63] a number or expression that divides another number or expression without remainder

Factor theorem [p. 177] If $ax + b$ is a factor of $P(x)$ then $P\left(-\frac{b}{a}\right) = 0$. Conversely, if $P\left(-\frac{b}{a}\right) = 0$ then $ax + b$ is a factor of P.

Factorise [p. 63] to express as a product of factors

Formula [p. 15] an equation containing symbols that states a relationship between two or more quantities

$A = lw$(area = length × width) is an example of a formula. The value of A, called the subject of the formula, can be found by substituting in given values of l and w.

Function [p. 131] a relation for which for each x-value of an ordered pair there is a unique y-value of the ordered pair

This means if (a, b) and (a, c) are ordered pairs of a function then $b = c$.

Function, many-to-one [p. 136] More than one x-value maps onto the same y-value, e.g. $y = x^2$.

Function, one-to-one [p. 136] Each x-value maps onto a unique y-value. e.g. $y = 2x$.

Function, vertical line test [p. 131] used to identify if a relation is a **function**

If a vertical line can be drawn anywhere on the graph and only ever intersects the graph a maximum of once, the relation is a **function**.

Fundamental theorem of integral calculus [p. 537] states that $\int_a^b f(x)dx = G(b) - G(a)$, where G is any antiderivative of f, and $\int_a^b f(x)dx$ is the definite integral from a to b

G

Gradient (m) of a line [p. 25] Gradient: $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are coordinates of points on the line

Gradient function [p. 443] The gradient or derived function is denoted by f', where

$$f' : R \to R \text{ and } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Gradient of a vertical line (parallel to the y-axis) [p. 26] The gradient of a **vertical (parallel to the y-axis)** line is undefined.

H

Horizontal line test [p. 136] If a horizontal line can be drawn anywhere on the graph of a function and only ever intersects the graph a maximum of once, the function is **one-to-one**.

Hybrid functions [p. 139] functions which have different rules for different subsets of the domain

I

Implied domain *see* maximal domain

Independence [p. 250] A and B are independent events if:

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ or $\Pr(A|B) = \Pr(A)$ or $\Pr(B|A) = \Pr(B)$

Independent trials [p. 291] the same as sampling with replacement

Index laws [p. 326]

- To multiply two numbers in exponent form with the same base, **add** the exponents:

  $a^m \times a^n = a^{m+n}$

- To divide two numbers in exponent form with the same base, **subtract** the exponents:

  $a^m \div a^n = a^{m-n}$

- To raise the power of a to another power, **multiply** the exponents:

  $(a^m)^n = a^{m \times n}$

- If $a^x = a^y$ then $x = y$.
- For rational exponents $a^{\frac{1}{n}} = \sqrt[n]{a}$

Inequation [p. 13] a mathematical statement that contains an inequality symbol rather than an equals sign; e.g. $2x + 1 < 4$

Integers [p. 124] the elements of $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Integration, general results [p. 531]

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c, r \in Q \backslash \{-1\}$$
$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$
$$\int kf(x)dx = k\int f(x)dx, \text{ where } k \text{ is a real number}$$

Intersection of two sets [p. 124] The **intersection** of sets A and B and is denoted by $A \cap B$ and $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

Inverse function [p. 143] If f is a one-to-one function a function f, the inverse function, f^{-1} is defined by, $f^{-1}(x) = y$ if $f(y) = x$, for $x \in \text{ran } f$, $y \in \text{dom } f$

Irrational numbers [p. *124*] The real numbers which are not rational, are called **irrational** (e.g. π and $\sqrt{2}$).

K

Karnaugh map [p. *237*] a probability table

L

Law of total probability [p. *246*] In the case of two events, A and B:

$$\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|B')\Pr(B')$$

Limits, properties of [p. *456*]

- $\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$ i.e. the limit of the sum is the sum of the limits
- $\lim_{x \to c}(kf(x)) = k \lim_{x \to c} f(x)$, where k is a given number
- $\lim_{x \to c}(f(x)g(x)) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$ i.e. the limit of the product is the product of the limit
- $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided $\lim_{x \to c} g(x) \neq 0$ i.e. the limit of the quotient is the quotient of the limit

Linear equation [p. *1*] a polynomial equation of degree 1; e.g. $2x + 1 = 0$

Linear function [p. *132*] a function $f: R \to R$, $f(x) = mx + c$; e.g. $f(x) = 3x + 1$

Literal equation [p. *4*] an equation for the variable x in which the coefficients of x, including the constants are pronumerals; e.g. $ax + b = c$

Logarithm [p. *334*] If $a \in R^+ \{1\}$ and $x \in R$, then the statements $a^x = n$ and $\log_a n = x$ are equivalent.

Logarithm laws [p. *335*] Laws of logarithms:

(1) $\log_a(mn) = \log_a m + \log_a n$

(2) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(3) $\log_a\left(\dfrac{1}{n}\right) = -\log_a n$

(4) $\log_a(m^p) = p \log_a m$

M

Many-to-many [p. *136*] More than one x-value maps onto more than one y-value; e.g. $x^2 + y^2 = 4$.

Many-to-one [p. *136*] More than one x-value maps onto the same y-value; e.g. $y = x^2$.

Maximal or implied domain [p. *129*] When the rule for a relation is written and no domain is stipulated then it is understood that the domain taken is the largest for which the rule has meaning.

Midpoint of a line segment [p. *47*] Let $P(x, y)$ be the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$.

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$

Multiplication rule (for choices) [p. *264*] When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

Multiplication rule for probability: [p. *244*] the rule to determine the probability of events A and B occurring:

$$\Pr(A \cap B) = \Pr(A|B) \times \Pr(B)$$

Multi-stage experiment [p. *221*] experiments that could be considered to take place in more than one stage; e.g. tossing two coins

Mutually exclusive [p. *234*] Two sets are said to be **mutually exclusive** if they have no elements in common.

N

n* Factorial, *n! [p. *267*] Denoted $n!$ (and read n factorial) this is an abbreviation for the product of all the integers from n to 1:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 2 \times 1$$

Natural numbers [p. *124*] the elements of $N = \{1, 2, 3, 4, \ldots\}$

${}^nC_r$ (see also selection) [p. *271*] the number of combinations of n objects in groups of size r:

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

Normal, equation of [p. *472*] Let P be the point on the curve $y = f(x)$ with coordinates (x_1, y_1). Then, if f is differentiable for $x = x_1$, the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1)$$

O

One-to-many [p. *136*] One x-value maps onto more than one y-value; e.g. $y = \pm\sqrt{x}$.

Ordered pair [p. *24*] An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first element and y the second.

P

Period of a function [p. *367*] The period of a function f with domain R is the smallest positive number a such that $f(x + a) = f(x)$ for a in R. For example, the period of the sine function is 2π, as

$$\sin(x + 2\pi) = \sin x.$$

Polynomial [p. *167*] A polynomial has a rule of the type

$$y = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \quad (n \in N)$$

where $a_0, a_1, \ldots, a_n$ are numbers called coefficients.

Probability [p. *221*] A numerical value assigned to the likelihood of an event occurring.

If the event A is impossible then $\Pr(A) = 0$, and if event A is certain then $\Pr(A) = 1$, otherwise $0 < \Pr(A) < 1$.

Probability distribution [p. *282*] Denoted $p(x)$ or $\Pr(X = x)$, this is a function which assigns probabilities to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

Probability table [p. *237*] a table used for illustrating a probability problem diagrammatically

Q

Quadratic formula [p. *79*] $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the solution of the quadratic equation $0 = ax^2 + bx + c$

Quadratic relation [p. *59*] A quadratic is defined by the general rule $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Quadratic relation (turning point form) [p. *73*] **turning point** form of a quadratic relation:

$$y = a(x - h)^2 + k$$

Quartic function [p. *168*] a fourth degree polynomial is a function f, with rule

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$$

R

R^+ [p. *125*] $\{x : x > 0\}$

R^- [p. *125*] $\{x : x < 0\}$

$R \setminus \{0\}$ [p. *125*]
= the set of real numbers excluding 0

R^2 [p. *146*] $R^2 = \{(x, y) : x, y \in R\}$; i.e. R^2 is the set of all ordered pairs of real numbers

Radian [p. *356*] One radian (written 1^c) is the angle **subtended** at the centre of the unit circle by an arc of length 1 unit.

Random experiment [p. *220*] an experiment, such as the rolling of a die, in which the outcome of a single trial is uncertain but observable

Random number tables [p. *301*] tables which contain the digits 0, 1, . . . , 9 in random order, and which can be used to generate random sequences of numbers

Random variable [p. *281*] a variable that takes its value from the outcome of a random experiment; e.g. the number of heads observed which a coin is tossed three times

Range [p. *126*] the set of all the second elements of ordered pairs of a relation

Rational numbers [p. *124*] numbers of the form $\dfrac{p}{q}$ with p and q integers, $q \neq 0$

Rectangular hyperbola [p. *108*] The basic rectangular hyperbola has equation $y = \dfrac{1}{x}$.

Reflection in the x-axis [p. *151*] In general

- A reflection in the x-axis is described by the rule $(x, y) \to (x, -y)$.
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = -f(x)$ by the transformation with rule $(x, y) \to (x, -y)$.

Reflection in the y-axis [p. *151*] In general

- A reflection in the y-axis is described by the rule $(x, y) \to (-x, y)$.
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f(-x)$ by the transformation with rule $(x, y) \to (-x, y)$.

Relation [p. *126*] a set of ordered pairs; e.g. $\{(x, y) : y = x^2\}$

Remainder theorem [p. *176*] When the polynomial $P(x)$ is divided by $ax + b$ the remainder is $P\left(-\dfrac{b}{a}\right)$.

S

Sample space [p. *220*] the set of possible outcomes for the experiment, sometimes denoted ε

Sampling with replacement [p. *290*] the process of selecting individual objects sequentially from a group of objects, and replacing the selected object, so that the probability of obtaining a particular object does not change with each successive selection

Sampling without replacement [p. *287*] the process of selecting individual objects sequentially from a group of objects, and not replacing the selected object, so that the probability of obtaining a particular object changes with each successive selection

Selections [p. *270*] The number of combinations of n objects in groups of size r is

$$^nC_r = \frac{n \times (n-1) \times (n-2) \ldots (n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

Set difference [p. *124*] The **set difference** of two sets A and B:

$$A \setminus B = \{x : x \in A, x \notin B\}$$

Simple event [p. *220*] a single outcome from a random experiment; e.g. observing a 2 when a die is rolled

Simulation [p. *298*] the process of finding an approximate solution to a probability problem by repeated trials using a simulation model

Simulation model [p. *298*] A simple model which is analogous to a real-world situation. For example, the outcomes from a toss of a coin (head, tail) could be used as a simulation model for the sex of a child (male, female) under the assumption that in both situations the probabilities are 0.5 for each outcome.

Simultaneous equations [p. *9*] equations of two or more lines or curves in a Cartesian plane, the solution of which is the point of intersection of the lines or curves

Sine *see* Cosine and sine

Stationary point [p. *478*] A point with coordinates $(a, g(a))$ on a curve $y = g(x)$ is said to be a stationary point if $g'(a) = 0$.

Straight line, equation of given two points [p. *30*] $y - y_1 = m(x - x_1)$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$

Straight line, equation of, gradient–intercept form [p. *27*] The general equation of a straight line is $y = mx + c$, where m is the gradient of the line. This form, expressing the relation in terms of y, is called the **gradient–intercept form**.

Straight lines, perpendicular [p. *43*] If two straight lines are perpendicular, the product of their gradients is -1. Conversely, if the product of the gradients of two lines is -1 then the two lines are perpendicular.

Subjective probability [p. *224*] the probability assigned to an event on the basis of prior experience

Subset [p. *124*] A set B is called a subset of set A, **if and only if** $x \in B$ implies $x \in A$.

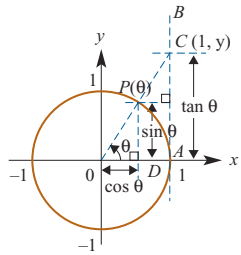
To indicate that B is a subset of A, we write $B \subseteq A$.

Sum of two cubes [p. *179*]

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

T

Tangent function [p. *383*] If a tangent to the unit circle at A is drawn, then the y coordinate of C, the point of intersection of the extension of OP and the tangent is called **tangent** θ (abbreviated to $\tan\theta$).



Tangent, equation of [p. *471*] Let P be the point on the curve $y = f(x)$ with coordinates (x_1, y_1). Then, if f is differentiable for $x = x_1$, the equation of the tangent at (x_1, y_1) is given by

$$(y - y_1) = f'(x_1)(x - x_1).$$

Tree diagram [p. *222*] a diagram representing the outcomes of a multi-stage experiment

Translation [p. *147*] A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule $(x, y) \to (x + h, y + k)$, where h and k are positive numbers. In general, the curve of the image of the curve with equation $y = f(x)$ is $y - k = f(x - h)$.

U

Union of sets [p. *124*] The union of sets A and B, written $A \cup B$, is the set of elements which are either in A or in B. This does not exclude objects which are elements of both A and B.

V

Venn diagram [p. *234*] a diagram showing sets and the relationships between sets

Vertical line test [p. *131*] *see* function